Stationary and Nonstationary Problems of Active Shielding

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Abstract—**The problem of active shielding of some domains from the effect of the sources distributed in other domains is considered. The problem can be formulated either in a bounded domain or in an unbounded domain. The active shielding is realized via the implementation of additional sources in such a way that the total contribution of all sources leads to the desirable effect. Mathematically the problem is reduced to the search of the source terms satisfying some *a priori* described requirements to the solution and belongs to the class of inverse source problems. From the application standpoint, this problem can closely be related to the active shielding of noise, active vibration control and active scattering. It is important to note that along with undesirable field (noise) to be shielded the presence of a desirable component is accepted in the analysis. The solution of the problem requires only the knowledge of the total field on the perimeter of the shielded domain. This is the first publication where the solution of the problem is proved in general nonstationary linear and stationary nonlinear formulations. The examples of acoustic and Maxwell equations are considered.**

Keywords: inverse source problem, active noise shielding, active sound control, distribution.

1 Introduction

The problem of active shielding (AS) of some domains from the effect of the field (noise) generated in other domains is solved in a quite general formulation. Its solution is realized via the implementation of additional sources in such a way that the total contribution of all sources leads to the desirable effect. Mathematically the problem is reduced to the search of the source terms satisfying some *a priori* described requirements to the solution of an appropriate boundary value problem (BVP). Thus, it can be formulated as an inverse source problem [1]. From the application standpoint, this problem can closely be related to the active noise shielding, active vibration control and active scattering. Some comprehensive reviews of the theoretical and experimental methods related to these subjects can be found in books [2], [3], [4] and report [5]. Most theoretical approaches assume some quite detailed information about the undesirable sources and the properties of the medium. The JMC method [6], [7], [5], based on the Huygens' construction, requires only the value of the undesirable field on the perimeter of the shielded domain. Yet this method is not used in the case if a desirable field ("friendly sound"), generated in the shielded domain, has to be taken into account. Moreover, the JMC method can only be used for the problems formulated in unbounded domains.

Substantially new opportunities are provided by the Difference Potential Method (DPM) in [8], [9]. The solution obtained in a finite–difference formulation requires only the knowledge of the total field (both desirable and undesirable) at the grid boundary of the shielded domain. Any other information on the sources and medium is not required. It is possible to say that the solution demands, in some sense, minimal information which is *a priori* available. The same result can be achieved via the implementation of Green's function [10], [11] if it is known for the BVP studied. A comprehensive study of the general solution [9] in the application to the Helmholtz equation including its optimization can be found in [11], [12], [13], [14]. In [15] the problem of AS in composite domains is formulated for the first time and its general solution is provided in a general finite–difference formulation. The principal novelty of the problem considered in [15] is that it allows a selective communication between different subdomains. The solution of the problem is realized via a predictor–corrector algorithm. The counterpart of the problem in continuous spaces is considered in [16].

For the acoustic Euler equations in continuous spaces, the AS solution is first obtained in [19]. The solution is derived via the apparatus of distributions for time–harmonic waves under quite general assumptions. It is shown that there exists equivalence between the DPM–based discrete solution, if the space step vanishes, and the obtained solution. The DPM–based solution is extended to arbitrary hyperbolic systems of equations including acoustic equations with constant and variable coefficients in [17]. It can be demonstrated, [18], that the control (additional) sources do not disturb even the echo of the "friendly" sound component if the AS problem is considered in bounded domains. In the current paper, for the first time the approach [19] is generalized to sub-

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stantially nonstationary problems (non time–harmonic waves). The solution of the stationary nonlinear AS problem is also obtained in a general formulation. The examples of the acoustic equations (Helmholtz, Euler and wave equations), the Maxwell equations and the Euler nonlinear equations are considered. In the case of the Maxwell equation the well known problems of a bounded conductor in an electrostatic field and a superconductor in a magnetic field are interpreted as AS problems. The AS solution for the Euler equations and wave equation are obtained. In the examples it is demonstrated that the known solutions can be derived as a particular case from the general solution of the general AS inverse source problem provided.

2 General formulation of the AS problem

The AS problem can be formulated as follows. Let us assume that some field (sound) U is described by the following BVP in a domain $D \subseteq \mathbb{R}^m$:

$$L(U) = f, \tag{1}$$
$$U \in \Xi_D. \tag{2}$$

Here, the operator L is a differential operator, Ξ_D is some functional space specified further. It is supposed the boundary conditions to be set at the boundary ∂D of the domain D and they are implicitly included into the definition of the space Ξ_D. In particular, the operator L can correspond to the acoustic equations.

In (1), $f \in F_D$ where F_D is some linear space of functions f. We assume that the spaces Ξ_D and F_D are specified in such a way that BVP (1), (2) is correct. Thus, there exists the inverse operator $L^{-1} : L^{-1}(f) = U$. It is supposed that the solution of the homogeneous BVP (1), (2) exists and it is only trivial: $L^{-1}(0) = 0$.

Consider some bounded domain D^+: $\overline{D^+} \subset D$. It is worth noting that the domain D^+ can be composite. It is assumed that the domain D^+ has the smooth boundary Γ. The sources on the right–hand side can be distributed both in D^+ and outside D^+:

$$\begin{aligned} &f = f^+ + f^-, \\ &\operatorname{supp} f^+ \subset D^+, \\ &\operatorname{supp} f^- \subset D^- \stackrel{def}{=} D \diagdown \overline{D^+}. \end{aligned} \tag{3}$$

Here, $f^+ \in F_D$ is the source of a "friendly" field (sound), while f^- is the source of an "adverse" field (noise).

Suppose that we know the trace of the function U on the boundary Γ of the domain D^+: $U_\Gamma = U(\Gamma)$. It is to be noted that only this information is assumed to be available. In particular, the distribution of the sources f is unknown. The AS problem is reduced to the search of additional sources G in $\overline{D^-}$ such that the solution of the following BVP

$$\begin{aligned} &L(U') = f + G, \\ &\operatorname{supp} G \subset \overline{D^-}, \\ &U' \in \Xi_D \end{aligned} \tag{4}$$

coincides on the domain D^+ with the solution of BVP (1), (2) if $f^- \equiv 0$:

$$\begin{aligned} &L(U^+) = f^+, \\ &U^+ \in \Xi_D. \end{aligned} \tag{5}$$

Thus, we seek a source term G such that

$$U'_{D^+} = U^+_{D^+}. \tag{6}$$

Here and further, V_Ω ($\Omega \subset D$) means the restriction of some function V on a domain Ω.

One can note that an "obvious" solution $f = -f^-$ is not appropriate here because the function f^- is unknown.

3 Solution of the stationary linear AS problem

First, we consider the stationary formulation of BVP (1), (2). Assume that the operator L is linear and given by:

$$L \stackrel{def}{=} \sum_1^m A^i \frac{\partial}{\partial x^i}, \tag{7}$$

where $\{x^i\}$ $(i = 1, ..., m)$ is a Cartesian coordinate system, where U and f are vector–functions with the dimension of n; A^i are $n \times n$ matrices: $A^i(\mathbf{x}) \in C^1(D)$ $(i = 1, ..., m)$. We also suppose that some linear boundary conditions are set on the boundary of D.

Thus, BVP (1), (2) reduces to the following:

$$\begin{aligned} &\sum_1^m A^i \frac{\partial U}{\partial x^i} = f, \\ &U \in \Xi_D, \\ &f = f^- + f^+, \\ &\operatorname{supp} f^+ \subset D^+, \quad \operatorname{supp} f^- \subset D^-. \end{aligned} \tag{8}$$

Let us consider the solution of BVP (1), (2), (7), (8) in the generalized sense [20], [21]. For this purpose we introduce the space of basic functions $\Phi \in C_0^\infty(D)$. Equality (1), (2) is then considered in the weak sense: $< LU, \Phi > = < f, \Phi >$ for any $\Phi \in C_0^\infty(D)$ where $< f, \Phi >$ means a distribution determined on the space of the basic functions $C_0^\infty(D)$.

We define the functional space Ξ_D in such a way that the weak solution of BVP (1) satisfies the governing equation in the classical sense almost everywhere, and it is bounded. Thus, we require that if $U \in \Xi_D$ then it can be represented as follows:

$$U = \theta(D^+)U_+ + \theta(D^-)U_-,$$

where $U_+ \in H^s(D^+)$, $U_- \in H_0^s(D^-)$, $s > 1/2$, $s \neq$ integer $+ 1/2$, H_0^s and H^s are Sobolev spaces, see, e.g., [21], $\theta(\Omega)$ $(\Omega \subset D)$ is the Heaviside–type characteristic function equaled to 1 on Ω and 0 outside.

The formulated AS problem is an inverse source problem and, hence, its solution is not unique [19]. From the application point of view the most interesting solution is represented by a single–layer source term and provided by the next Proposition.

Proposition 1 *A solution of the AS problem* (1), (2), (4), (8) *is given by the following distribution:*

$$G = G_0 \stackrel{def}{=} A_n U_\Gamma \delta(\Gamma), \tag{9}$$

where $A_n \stackrel{def}{=} \sum_1^m n_i A^i$, n_i *are the coordinates of the unit vector of the external normal* $\boldsymbol{n}$ *to the boundary* Γ, $\delta(\Gamma)$ *is the Dirac delta–function assigned to the boundary* Γ.

Proof. Thus, it is required to prove that the solution of BVP (4) coincides with the solution of BVP (5) on D^+: $U' = U^+$ if $\mathbf{x} \in D^+$. For this purpose, let us consider the following four auxiliary BVPs.

The first BVP corresponds to BVP (5) generally formulated.

BVP 1^0:

$$\begin{aligned} &LU^+ = f^+, \\ &U^+ \in \Xi_D. \end{aligned} \tag{10}$$

The solution of this BVP exists and it is unique since $f^+ \in F_D$.

The next BVP is the counterpart of BVP (10), formulated for the field to be shielded.

BVP 2^0:

$$\begin{aligned} &LU^- = f^-, \\ &U^- \in \Xi_D. \end{aligned} \tag{11}$$

The solution of this BVP also exists and it is unique.

BVP 3^0:

$$\begin{aligned} &L\overline{U} = A_n U_\Gamma \delta(\Gamma), \\ &\overline{U} \in \Xi_D. \end{aligned} \tag{12}$$

The solution of the formulated BVP is the following:

$$\overline{U}(\mathbf{x}) = \begin{cases} -U^-, & \text{if } \mathbf{x} \in D^+ \\ U^+ & \text{if } \mathbf{x} \in D^- \end{cases} \tag{13}$$

Indeed,

$$< L\overline{U}, \Phi >= -\sum_1^m < A^i\overline{U}, \nabla_i \Phi > - < \nabla A\overline{U}, \Phi >=$$

$$-\sum_1^m \int_D (A^i\overline{U}, \nabla_i\Phi) d\mathbf{x} - < \nabla A\overline{U}, \Phi >=$$

$$-\sum_1^m [\int_{D^-} (A^i\overline{U}, \nabla_i\Phi) d\mathbf{x} + \int_{D^+} (A^i\overline{U}, \nabla_i\Phi) d\mathbf{x}] -$$

$$< \nabla A\overline{U}, \Phi >=< \left\{L\overline{U}\right\}, \Phi > + \int_\Gamma A_n \left[\overline{U}\right]_\Gamma \Phi d\mathbf{x} =$$

$$< \left\{L\overline{U}\right\}, \Phi > + < A_n U_\Gamma \delta(\Gamma), \Phi >=< A_n U_\Gamma \delta(\Gamma), \Phi >$$

Here, (a, b) denotes the scalar product of vectors a and b, $\left\{L\overline{U}\right\}$ is the part of $L\overline{U}$ supported on $D\backslash\Gamma$, $\nabla A \stackrel{def}{=} \sum_1^m \nabla_i A^i$, $[.]_\Gamma$ means the discontinuity across the boundary Γ:

$$[V]_\Gamma \stackrel{def}{=} \lim_{\mathbf{x}\to\Gamma\cap\mathbf{x}\in D^-} V(\mathbf{x}) - \lim_{\mathbf{x}\to\Gamma\cap\mathbf{x}\in D^+} V(\mathbf{x}).$$

BVP 4^0:

$$\begin{aligned} &LU' = f + A_n U_\Gamma \delta(\Gamma), \\ &U' \in \Xi_D. \end{aligned} \tag{14}$$

The solution of this BVP exists and it is unique because of the linearity of the problem.

Then, from BVPs $1^0 - 3^0$ it follows that:

$$U'(\mathbf{x}) = \begin{cases} U^+, & \text{if } \mathbf{x} \in D^+ \\ U + U^+ & \text{if } \mathbf{x} \in D^-. \end{cases}$$

□

Thus, the source term provides noise cancelation in the domain to be shielded and doubles the field propagating from this domain outside.

It is worth noting that AS solution (9) provided by Proposition1 does not explicitly depend on the boundary conditions. Although the boundary conditions are not explicitly specified, we are able to obtain the AS source term if the solution of the considered IBVP is correct.

As written above, the solution of the AS problem as an inverse source problem is not unique. Indeed, the AS source term G can be represented in the following form:

$$G = G_0 + LW, \tag{15}$$

where $W \in \Xi_D$ and $\operatorname{supp} W \subset D^-$. It is clear that any additional source term LW does not affect the field in the domain D^+.

4 Nonstationary linear AS problem

The single–layer AS solution (9) can be generalized on a nonstationary formulation in $\mathbb{R}^{m+1}$ under some additional requirements.

Suppose that the field U is the solution of a correct initial–boundary value problem (IBVP) in the cylinder $K_T = D \times (0,T)$ $(T > 0)$:

$$LU \stackrel{def}{=} L_t^{(p)} U + \sum_1^m A^i \frac{\partial U}{\partial x^i} = f, \tag{16}$$

$$U \in \Xi_D, \tag{17}$$

where $L_t^{(p)}$ is a linear differential operator of an order p with respect to the time variable t. In (17), in addition to the stationary formulation, we assume that the space Ξ_D consist of the functions smooth enough with respect to the time variable: $\Xi_D \subset C^p(K_T)$ and satisfying homogeneous initial conditions. Thus, if $U \in \Xi_D$, then

$$\frac{d^k}{dt^k} U(\mathbf{x}, 0) = 0, \quad (k = 0, ..., p-1). \tag{18}$$

We consider the generalized solution of IBVP (16):

$$\int_0^T \int_D (LU - f, \Phi) d\mathbf{x} dt = 0 \tag{19}$$

for any $\Phi \in C_0^\infty(K_T)$.

It is to be noted here that without the violation of generality we can suppose that initial data (18) are homogeneous. Indeed, we can always represent the solution of IBVP (16), (17), (18) as: $U = U^{(f)} + U^{(t)}$, where $U^{(f)}$ is the solution of IBVP problem with the homogeneous initial data (18), while $U^{(t)}$ is the solution of IBVP with the homogeneous right–hand side. It is clear that the function $U^{(f)}$ has nothing to do with the unwanted component of the total field U; it can only represent "residual" noise.

Then, the AS solution is represented by a source term, similar to the source term (9), with a time–dependent density $U_\Gamma = U(\Gamma, t)$.

Proposition 2 *A solution of the AS problem* (1)*,* (2)*,* (4)*,* (16) *is given by the following one–layer distribution:*

$$G = G_0 \stackrel{def}{=} A_n U_\Gamma \delta(\Gamma). \tag{20}$$

Proof. For the sake of simplicity, we assume that $A^i(\mathbf{x}) \in C^\infty(D)$ $(i = 1, ..., m)$. The case of only smooth matrices $A^i(\mathbf{x})$ can be considered identically to the stationary problem.

Thus, it is required to prove that the solution of problem (4) coincides with the solution of IBVP (5) in D^+: $U' = U^+$ if $\mathbf{x} \in D^+$. Again, we introduce four auxiliary IBVPs.

IBVP 1^0:

$$LU^+ = f^+, \tag{21}$$
$$U^+ \in \Xi_D.$$

IBVP 2^0:

$$LU^- = f^-, \tag{22}$$
$$U^- \in \Xi_D.$$

It is clear that the solution of each of the two BVPs exists and it is unique.

IBVP 3^0:

$$L\overline{U} = A_n U_\Gamma \delta(\Gamma), \tag{23}$$

$$\overline{U} \in \Xi_D. \tag{24}$$

The solution of this problem is the following:

$$\overline{U}(\mathbf{x}) = \begin{cases} -U^-, & \text{if } \mathbf{x} \in D^+ \\ U^+ & \text{if } \mathbf{x} \in D^-. \end{cases} \tag{25}$$

Indeed,

$$\begin{aligned} &< L\overline{U}, \Phi > = < L_t^{(p)} \overline{U}, \Phi > - \sum_1^m < \overline{U}, \nabla_i (A^i \Phi) > = \\ &< L_t^{(p)} \overline{U}, \Phi > - \sum_1^m \int_0^T \int_D (\overline{U}, \nabla_i(A^i\Phi)) d\mathbf{x} dt = \\ &\int_0^T \int_{D^+} L_t^{(p)} \overline{U} d\mathbf{x} dt + \int_0^T \int_{D^-} L_t^{(p)} \overline{U} d\mathbf{x} dt - \\ &\sum_1^m \int_0^T \int_{D^-} (\overline{U}, \nabla_i(A^i\Phi)) d\mathbf{x} dt - \\ &\sum_1^m \int_0^T \int_{D^+} (\overline{U}, \nabla_i(A^i\Phi)) d\mathbf{x} dt = < \left\{ L\overline{U} \right\}, \Phi > + \\ &\int_0^T \int_\Gamma A_n \left[\overline{U} \right]_\Gamma \Phi d\mathbf{x} dt = < A_n U_\Gamma \delta(\Gamma), \Phi > \end{aligned}$$

IBVP 4^0:

$$LU' = f + A_n U_\Gamma \delta(\Gamma), \tag{26}$$
$$U' \in \Xi_D.$$

The solution of this IBVP is provided by:

$$U'(\mathbf{x}) = \begin{cases} U^+, & \text{if } \mathbf{x} \in D^+ \\ U + U^+ & \text{if } \mathbf{x} \in D^-. \end{cases}$$

□

It is important to note that in this solution we do not take into account the influence of the AS source term (feedback) on the value of $U_\Gamma(t)$.

It appears that the inverse AS source problem can be solved in a nonlinear formulation.

5 Nonlinear stationary problem

Assume that the operator L is nonlinear and it is as follows:

$$L(U) \stackrel{def}{=} \sum_1^m \frac{\partial F^i}{\partial x^i}, \tag{27}$$

where $F^i = F^i(U) \in C^1(D)$, $F^i(0) = 0 \ \ (i = 1, ..., m)$.

Thus, BVP (1), (2) reduces to the following problem:

$$\begin{aligned} &\sum_1^m \frac{\partial}{\partial x^i} F^i(U) = f, \\ &U \in \Xi_D, \\ &f = f^- + f^+, \\ &\operatorname{supp} f^+ \subset D^+, \quad \operatorname{supp} f^- \subset D^-. \end{aligned} \tag{28}$$

As in the linear case, the solution of BVP (28) is considered in the generalized sense. Thus, equality (28) means

$$< L(U), \Phi > = < f, \Phi >$$

for any $\Phi \in C_0^\infty(D)$.

We assume that if $U \in \Xi_D$, then $L(U) \in F_D$. Then, we arrive at the following important result:

if $U \in \Xi_D$, $V \in \Xi_D$, then

$$L(U) = L(V) \quad \Rightarrow U = V. \tag{29}$$

Then, the following Proposition is valid.

Proposition 3 *If Assumption 1 is valid, then a solution of the AS problem* (1), (2), (4), (28) *is represented by the distribution of a single layer:*

$$G = G_0 \stackrel{def}{=} F_n(U_\Gamma)\delta(\Gamma), \tag{30}$$

where $F_n \stackrel{def}{=} \sum_1^m n_i F^i$.

Proof. Let us introduce the following decomposition of the right–hand side f in (28):

$$f = \overline{f}^{\,+} + \overline{f}^{\,-}, \tag{31}$$

where

$$\begin{aligned} \overline{f}^{\,+} &= L(\theta(D^+)U), \\ \overline{f}^{\,-} &= L(\theta(D^-)U). \end{aligned}$$

Thus,

$$\begin{aligned} &\operatorname{supp} \overline{f}^{\,+} \subset D^+ \cup \Gamma, \\ &\operatorname{supp} \overline{f}^{\,-} \subset D^- \cup \Gamma. \end{aligned}$$

The solution on D^+ is fully determined by $\overline{f}^{\,+}$. If we replace $\overline{f}^{\,-}$ by some function $\widetilde{\overline{f}}^{\,-} = L(\theta(D^-)\widetilde{U})$, where $\widetilde{U} \in \Xi_D$, then the solution on D^+ obviously remains:

$$L^{-1}(\widetilde{\overline{f}}^{\,-} + f^{\,+})_{|D^+} = U_{D^+}.$$

It is possible to represent $\overline{f}^{\,+}$ as follows:

$$L(\theta(D^+)U) = f^+ - F_n(U_\Gamma)\delta(\Gamma).$$

It follows from

$$(L(\theta(D^+)U), \Phi) = -\sum_1^m (F^i, \nabla_i \Phi) =$$

$$-\sum_1^m \int_D F^i \nabla_i \Phi d\mathbf{x} = -\sum_1^m \int_{D^+} F^i \nabla_i \Phi d\mathbf{x} =$$

$$(\{L(U)\}, \Phi) - \int_\Gamma F_n(U_\Gamma)\Phi d\mathbf{x} = f^+ - \int_\Gamma F_n(U_\Gamma)\Phi d\mathbf{x}.$$

Similar, it is possible to obtain that

$$L(\theta(D^-)U) = f^- + F_n(U_\Gamma)\delta(\Gamma).$$

Thus, the influence of f^- on the domain D^+ is realized only via the single–layer source: $-F_n(U_\Gamma)\delta(\Gamma)$. It fully corresponds to Huygens' construction in waves propagation.

Thus, the AS source term

$$g_0 = F_n(U_\Gamma)\delta(\Gamma) \tag{32}$$

is capable to provide the noise cancelation.

Indeed, let us consider the contribution of $f^- + g_0$ to the field on D^+. It is provided by V_{D^+} where V is the solution of the following BVP:

$$\begin{aligned} &L(V) = f^- + F_n(U_\Gamma)\delta(\Gamma), \\ &V \in \Xi_D. \end{aligned}$$

In turn,

$$f^- + F_n(U_\Gamma)\delta(\Gamma) = L(\theta(D^-)U).$$

Hence, $V = \theta(D^-)U$ and $V_{D^+} = 0_{D^+}$. Thus, the field on D^+ is fully determined by the sources situated on D^+ as long as the total field at the boundary is equal to U_Γ.

□

6 Examples

Let us now consider a few examples of AS sources terms for stationary and nonstationary problems. We assume that the boundary conditions set for each of the problem considered such that the appropriate BVP is correct.

1^0. *A bounded metallic body in electrostatic field.*

Consider a metallic bounded body in an external electrostatic field $\mathbf{E}_{out}$. It is well known that if the problem is static, then the field inside the body must equal zero. Charges in the body are redistributed on its surface in such a way that the internal electric field equals zero. Thus, the contribution of the surface charges is similar to shielding the body from the external field $\mathbf{E}_{out}$. From the Maxwell equations it follows that

$$div\mathbf{E} = 4\pi\rho + g_d, \tag{33}$$
$$curl\mathbf{E} = g_c, \tag{34}$$

where $\mathbf{E}$ is the electric field, ρ is the density of charges, g_d and g_c are the AS source terms.

Assume that $f^- = 4\pi\rho$. Let us now represent the AS source term g_d in the following form: $g_d = 4\pi\sigma_\rho\delta(\Gamma)$.

For equations (33), (34) the appropriate matrix A_n from (9) is not square and given by

$$A_n = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}, \tag{35}$$

provided that $U = (E_1, E_2, E_3)^T$ where E_i $(i = 1, 2, 3)$ are the coordinates of the vector $\mathbf{E}$.

Then, we obtain that

$$g_d = \mathbf{E}_{out|\Gamma} \cdot \mathbf{n}\delta(\Gamma), \tag{36}$$
$$g_c = \mathbf{n} \times \mathbf{E}_{out|\Gamma}\delta(\Gamma), \tag{37}$$

where $\mathbf{E}_{out|\Gamma}$ is the external field on the perimeter of the body.

Hence,

$$\sigma_\rho = \frac{1}{4\pi}E_{out|\Gamma}. \tag{38}$$

From (37) it follows that the external field $\mathbf{E}_{out|\Gamma}$ must be orthogonal to the boundary otherwise the field is not potential since the right–hand side in (34) is not zero. It fully corresponds to the results known in electrostatics (see, e.g., [22]) since there is no current in the body.

2^0. *A superconductor in magnetostatic field.*

Let us now consider a magnetic field around a superconductor. It is well known that the magnetic field inside a superconductor equals zero. The magnetic external field induces a bound current with a density $\mathbf{j}$ which plays the shielding role.

Consider the Maxwell equations for a static magnetic field:

$$div\mathbf{H} = g_d, \tag{39}$$
$$curl\mathbf{H} = \frac{4\pi}{c}\sigma\mathbf{E} + \mathbf{g}_c, \tag{40}$$

where $\mathbf{H}$ is the magnetic field, σ is the conductivity. Let us set that $\mathbf{g}_c = \frac{4\pi}{c}\mathbf{j}_b\delta(\Gamma)$.

Similar to the previous example, from Proposition 1 we obtain that $\mathbf{g}_c = \mathbf{n} \times \mathbf{H}_\Gamma\delta(\Gamma)$. Hence,

$$\frac{4\pi}{c}\mathbf{j}_b = \mathbf{n} \times \mathbf{H}_\Gamma. \tag{41}$$

From equation (39) we have:

$$g_d = H_{n|\Gamma}\delta(\Gamma), \tag{42}$$

where $H_n = \mathbf{H} \cdot \mathbf{n}$.

Since the magnetic field is to be solenoidal ($g_d = 0$), the field must be either orthogonal to the boundary or equals zero:

$$H_{n|\Gamma} = 0.$$

This result coincides with the well known result about the bound current [22] on the surface of a superconductor.

The next four examples are related to active noise shielding in acoustics.

3^0. *Helmholtz–type equation.*

Let us consider the following Helmholtz–type equation with variable coefficients:

$$\nabla(p\nabla\phi) + k^2\phi = s, \tag{43}$$

where $p \in C^1(\overline{D}),\ p > 0$.

If $p \equiv 1$, then equation (43) coincides with the Helmholtz equation describing the propagation of monochromatic waves.

We can rewrite (43) as the system of first-order equations:

$$\nabla\mathbf{a} + k^2\phi = s, \tag{44}$$
$$\nabla\phi - \mathbf{a}/p = 0.$$

In $\mathbb{R}^3$, we have:

$$U = (a_1, a_2, a_3, \phi)^T, \tag{45}$$

where a_i $(i = 1, 2, 3)$ are the coordinates of the vector $\mathbf{a}$. Hence,

$$A_n = \begin{pmatrix} n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & n_1 \\ 0 & 0 & 0 & n_2 \\ 0 & 0 & 0 & n_3 \end{pmatrix}, \tag{46}$$

and

$$G_0(\Gamma) = (a_n, \phi n_1, \phi n_2, \phi n_3)^T_\Gamma\delta(\Gamma), \tag{47}$$

where $a_n = \mathbf{a} \cdot \mathbf{n}$.

Then, we arrive at the following set of equations:

$$\begin{aligned} \nabla \mathbf{a} + k^2\phi &= s + p\frac{\partial \phi}{\partial \mathbf{n}}_{|\Gamma}\delta(\Gamma), \\ p\nabla\phi - \mathbf{a} &= p\phi_\Gamma \mathbf{n}\delta(\Gamma). \end{aligned} \tag{48}$$

Having eliminated the auxiliary vector **a** and turned back to the Helmholtz equation for the variable ϕ, we obtain

$$\Delta\phi + k^2\phi = s + g_0, \tag{49}$$

where the shielding function g_0 is as follows:

$$g_0 = \delta(\Gamma)p\frac{\partial \phi}{\partial \mathbf{n}}_{|\Gamma} + \nabla(\delta(\Gamma)p\phi_\Gamma \mathbf{n})$$

or

$$g_0 = \delta(\Gamma)p\frac{\partial \phi}{\partial \mathbf{n}}_{|\Gamma} + \frac{\partial \delta(\Gamma)p\phi_\Gamma}{\partial \mathbf{n}}. \tag{50}$$

The AS term is represented via the sum of single–layer and double–layer additional source terms. Here, the densities of the potentials include the values ϕ_Γ and $\frac{\partial \phi}{\partial \mathbf{n}}_{|\Gamma}$ to be known. If $p \equiv 1$, this solution fully coincides with the solution obtained in [11]. The solution is applicable to the linear analogue of the Helmholtz equation with variable coefficients.

Now, we consider three nonstationary linear problems.

4^0. *Wave equation.*

In the case of the wave equation

$$\phi_{tt} - a^2\Delta\phi = s \tag{51}$$

the operator L is the following: $L := \frac{\partial^2}{\partial t^2} - a^2\Delta$.

Similar to the Helmholtz equation, we can rewrite it as the system of first-order equations with respect to the space variables:

$$\begin{aligned} \phi_{tt} - \nabla \mathbf{a} &= s, \\ -\nabla\phi &= -\mathbf{a}. \end{aligned} \tag{52}$$

From Proposition 2 we obtain:

$$G_0(\Gamma) = -(a_n, \phi n_1, \phi n_2, \phi n_3)^T\delta(\Gamma). \tag{53}$$

Having eliminated the auxiliary vector **a**, we arrive at the following source term:

$$g_0 = -\delta(\Gamma)\frac{\partial \phi}{\partial \mathbf{n}} - \frac{\partial \delta(\Gamma)\phi}{\partial \mathbf{n}}. \tag{54}$$

Similar to the previous example, this result can be generalized to the case of variable coefficients. Thus, AS source term (50), in fact, is applicable to quite arbitrary nonstationary fields.

5^0. *Acoustic equations.*

Next, let us consider the acoustic equations:

$$\begin{aligned} \frac{1}{\rho_0 c_0^2}p'_t + \nabla \mathbf{u}' &= \frac{1}{\rho_0 c_0^2}f^{(p)} + q_{vol}, \\ \rho_0\mathbf{u}'_t + \nabla p' &= \mathbf{f}^{(u)} + \mathbf{f}_{vol}, \end{aligned} \tag{55}$$

where u'_j $(j = 1, 2, 3)$ are the components of the particle velocity $\mathbf{u}'$, p' is the sound pressure, c_0 is the sound speed, the functions marked by 0 correspond to some main flow, q_{vol} is the volume velocity per a unit volume and $\mathbf{f}_{vol}$ is the force per a unit volume [2]. In this case, we have

$$U = (u'_1, u'_2, u'_3, p')^T. \tag{56}$$

Then, the matrix A_n appears to coincide with the appropriate matrix (46) of the Helmholtz equation.

As the result, we obtain the following AS source terms in the form of a single layer:

$$\begin{aligned} q_{vol} &= \mathbf{u}' \cdot \mathbf{n}_{|\Gamma}\delta(\Gamma), \\ \mathbf{f}_{vol} &= p'_{|\Gamma}\mathbf{n}\delta(\Gamma). \end{aligned} \tag{57}$$

Thus, the AS solution depends on the normal component of the particle velocity $u'_{n|\Gamma}$ and the sound pressure p'_Γ on the boundary of the shielded domain. In applications, these values can to be taken from measurements and based on the contribution of both desirable and undesirable sources without their factorization. It should be noted that the AS solution (60) was obtained in [19] for a continuous space and in [17] for a finite–difference formulation.

Let us now consider the Linearize Euler equations (LEE) describing acoustic wave propagation in nonhomogeneous media.

6^0. *Linearized Euler equations*

$$\begin{aligned} &\frac{1}{\rho_0 c_0^2}(p'_t + (\mathbf{u}_0, \nabla)p') + (\mathbf{u}', \nabla)p_0 + \nabla \cdot \mathbf{u}' + \\ &\nabla \cdot \mathbf{u}_0 = \frac{1}{\rho_0 c_0^2}f^{(p)} + q_{vol}, \\ &\rho_0(\mathbf{u}'_t + (\mathbf{u}_0, \nabla)\mathbf{u}' + (\mathbf{u}', \nabla)\mathbf{u}_0) + \nabla p' = \mathbf{f}^{(u)} + \mathbf{f}_{vol}, \end{aligned} \tag{58}$$

where, as in the previous example, the functions marked by 0 correspond to the main flow.

Then, the matrix A_n is given by

$$A_n = \begin{pmatrix} n_1 & n_2 & n_3 & \frac{u_n}{\rho_0 c_0^2} \\ \rho_0 u_n & 0 & 0 & n_1 \\ 0 & \rho_0 u_n & 0 & n_2 \\ 0 & 0 & \rho_0 u_n & n_3 \end{pmatrix}, \tag{59}$$

where $u_n = \mathbf{u}_0 \cdot \mathbf{n}$.

Thus, we obtain the following AS source terms:

$$q_{vol} = (\mathbf{u}' \cdot \mathbf{n}_{|\Gamma} + \frac{u_n}{\rho_0 c_0^2} p'_{|\Gamma})\delta(\Gamma), \qquad (60)$$
$$\mathbf{f}_{vol} = (p'_{|\Gamma}\mathbf{n} + \rho_0 u_n \mathbf{u}'_{|\Gamma})\delta(\Gamma).$$

It can be seen that in the case of flux through the boundary Γ some corrections of AS terms (60) are required.

The next example demonstrates the application of the general AS solution (9) to a nonlinear problem.

7^0. *Euler equations*:

Let us now consider the Euler equations for gas dynamics:

$$L(U) = U_t + \sum_1^3 F^i(U)_{x_i}, \qquad (61)$$

where

$$U = (\rho, \rho u_1, \rho u_2, \rho u_3, E)^T, \qquad (62)$$

$$F^i(U) = u_i U + p(0, \delta_{1,i}, \delta_{2,i}, \delta_{3,i}, u_i)^T, \qquad (63)$$

where: ρ is the density; u_1,u_2,u_3 are the velocity coordinates in some Cartesian coordinate system $\{x_i\}$ $(i = 1, 2, 3)$; E is the total energy density; p is the pressure; $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$.

The AS solution is then as follows:

$$G_0 = \delta(\Gamma)\times \qquad (64)$$
$$(\rho V_n, \rho u_1 V_n + pn_1, \rho u_2 V_n + pn_2, \rho u_3 V_n + pn_3, HV_n)^T_\Gamma\ ,$$

where $H = E + p$, $V_n = \mathbf{u} \cdot \mathbf{n}$, is the component of the velocity normal to the boundary Γ .

In contrast to the acoustic equations, solution (64) depends on the all components of the velocity, not only on the normal one, on the perimeter of the shielded domain.

7 Conclusion

The solution of the AS inverse source problem has been obtained in general nonstationary linear and stationary nonlinear formulations. The solution only requires the knowledge of the total field (desirable and undesirable) on the perimeter of the shielded domain. It does not use any additional information on either the characteristics of the sources or the surrounding medium. The knowledge of Green's function of the problem is not required either. A single–layer AS solution has been obtained for a stationary nonlinear problem. The application of the general AS solution to the Maxwell equations, Helmholtz–type equation with variable coefficients, wave equation, linearized Euler equations and nonlinear Euler equations provide us the appropriate AS source terms.

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