Rendezvous On A Discrete Line

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Abstract–**In a rendezvous problem on a discrete line two players are placed at points on the line. At each moment of time each player can move to an adjacent point or remain at the point at which it stands. The goal is for both players to reach the same point in the least expected time time. This paper contains results that refer to both general and specific situations. First we show how this problem relates to the continuous problem. Next we prove there are optimal strategy pairs for which both players tend toward the center. Third, we derive matrix equations that apply to movement along the line. Using these previous result we employ a symbolic program (Maxima) to determine all possible solutions to searches on lines having four or five points, the cases of 1, 2, or 3 points being trivial. Finally we consider the special problem in which one player begins with certainty at a specified point of the line.**

Index Term: cooperative games,rendezvous, search,tactics

Introduction

We consider the problem in which two teams called Player I and Player II are placed at locations i and j respectively with probability $p_{i,j}$ on a discrete line. Thereafter the two players move to adjacent locations until they finally meet by arriving at the same location. The goal is for the players to meet in the shortest time. Thus if Player I starting at i chooses a path f_i and Player II starting at j chooses a path g_j the goal is to minimize the quantity

$$E\left(\{f_i\},\{g_j\}\right)=\sum_{i,j}p_{i,j}\left[f_i,g_j\right] \quad (1)$$

where $[f_i, g_j]$ denotes the time before the two paths are at the same location.

The problem described above is known as the rendezvous problem on the discrete line. A description of results for this problem on the line and other lattices is described in [1], and some results for lines of arbitrary length appear in [2] and [3]. First we show how the continuous problem can help to handle the continuous problem. Next we prove the Restriction Theorem that asserts there are optimal strategy pairs for which both players tend toward the center. Third, we derive matrix equations that apply to movement along the line. Using these previous result we employ a symbolic program (Maxima) to determine all possible solutions to searches on lines having four or five points, the cases of 1, 2, or 3 points being trivial. The result for five points indicates that the solution for the discrete rendezvous problem in its most general setting must be relatively complicated. Finally we consider the special problem, that we call the one sided problem, in which one player begins with certainty at a specified point of the line.

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Relation between Continuous and Discrete Search

If both players can adequately measure space and time and have a prior knowledge of the region in which they operate then they can reduce continuous search to discrete search. Although we are focusing on linear problems most of what we say in this section applies to any dimension. To transform a continuous problem into a discrete problem the players should previously agree upon an array of points, i. e. *bases* in the search region. They should also agree on a common time interval that is adequate to travel between any pair of adjacent bases. At the end of an interval each player should be at some base, either because it arrived there from an adjacent base or because it remained there during the previous interval. When the players are at the same base at the end of an interval then they will have met and the total time required will be the total of all intervals elapsed. In this way we can use the discrete problem to approximate the solution of the continuous problem.

On the line this approximation is quite efficient, but in higher dimensions we run into problems. For example, if we divide a two dimensional region into a rectangular lattice we encounter the question of whether the interval chosen will be long enough to travel between bases that are diagonal to one another. If the answer is "yes" then it will be necessary for a team to wait at an interval if it came there along a side. If the answer is "no" then motion along a diagonal line will take a greater total time.

The Restriction Theorem

The main result in this section is that in every rendezvous game on the line there is always a pair of optimal strategies that are within increasingly shorter lines as the search proceeds. A more general theorem of this type is found in [4], but the present result is not a special case since there we defined a meeting to be in the same or *adjacent* locations at the same time. As a matter of fact, the result given in this section does not generalize to arbitrary rectangular lattices unless we allow diagonals to be adjacent as we did in [4].

In this section it is convenient to represent the set L of locations on a line by

$$L = \{-n, -(n-1), ..., -1, 0, 1, ..., n\} \tag{2}$$

if the line has an odd number $(2n+1)$ of locations and by

$$L = \{-n, -(n-1), ..., -1, 1, ..., n\}, \tag{3}$$

omitting 0, if the line has an even number $(2n)$ of locations.

A **path** f_i beginning at location i is a function from the set of positive integers into L such that (1) $f_i(1) = i$ and (2) $f_i(t+1) \in \{f_i(t) - 1, f_i(t), f_i(t) + 1\}$ for each positive integer k. We express this briefly by saying that $f_i(t+1)$ is **adjacent** to $f_i(t)$. If f_i and g_j are two paths then $[f_i, g_j]$ is the smallest integer k for which $f_i(k) = g_j(k)$ or ∞ if the paths never meet. A path f_i is said to be k **restricted** where $0 \leq k \leq n$ after time T if $-k \leq f_i(t) \leq k$ for $t > T$. If f_i is a k restricted path after time T we define $P_{k-1}(f_i)$ to be the function g from the positive integers into L defined by

$$g(k) = \begin{cases} f_i(t) \text{ if } t \leq T+1 \\ f_i(t) \text{ if } t > T+1 \text{ and } -(k-1) \leq f_i(t) \leq k-1 \\ k-1 \text{ if } t > T+1 \text{ and } f_i(t) = k \\ -(k-1) \text{ if } t > T+1 \text{ and } f_i(t) = -k \end{cases} \tag{4}$$

Thus g coincides with f_i until time $T+1$ coincides with f_i except it stays at $-(k-1)$ when f_i goes to $-k$ or at $(k-1)$ when f_i goes to k.

Proposition 1 *If f_i is a k restricted path after time T then $g = P_{k-1}(f_i)$ is a path that begins at i and is $(k-1)$ restricted after time $T+1$ as well as k restricted after time T.*

Proof. Since $1 \leq T+1$ it follows that $g(1) = f_i(1) = i$. If $t \leq T+1$ or $-(k-1) < f_i(t) < k-1$ then $g(t+1)$ is adjacent to $g(t)$ because it coincides with f_i. If $t > T+1$ and $f_i(t) = k$ then $g(t)$ can be k (if $t = T+1$) or $k-1$ and $f_i(t+1)$ can be k or $k-1$ so $g(t+1)$ has to be $k-1$ which is adjacent to $g(t)$. If $t > T+1$ and $f_i(t) = k-1$ then $g(t)$ is $k-1$ and $f_i(t+1)$ can be k or $k-1$ so $g(t+1)$ has to be $k-1$ which is adjacent to $g(t)$. We omit the similar argument for $f_i(t)$ equal $-k$ or $-(k-1)$. That g is k restricted after time T follows since $g(T+1) = f_i(T+1)$. □

Proposition 2 *If f_i and g_j are both k restricted paths after time T then*

$$[P_{k-1}f_i, P_{k-1}g_j] \leq [f_i, g_j]. \tag{5}$$

Proof. Denote $P_{k-1}f_i$ by f_i^* and $P_{k-1}g_j$ by g_j^*, and let $[f_i, g_j] = t_0$. If $t_0 \leq T+1$ or $f_i(t_0) \notin \{k, -k\}$ then $f_i^*(t_0) = f_i(t_0) = g_j(t_0) = g_j^*(t_0)$. If $t_0 > T+1$ and $f_i(t_0) = k$ then $f_i^*(t_0)$ and $g_i^*(t_0)$ are both $k-1$. If $t_0 > T+1$ and $f_i(t_0) = -k$ then $f_i^*(t_0)$ and $g_i^*(t_0)$ are both $-(k-1)$. Thus f_i^* and g_i^* both meet at time t_0 and possibly before. □

Definition 3 *A path f_i on L is called* **restricted** *if it is $n-T$ restricted after time T for $T = 0, 1, 2, .., n-1$ and $f_i(n) = 0$ when L has an odd number $(2n+1)$ locations or $f_i(n) = 1$ when L has an even number $(2n)$ locations.*

Proposition 4 *If f_i and g_j is any pair of paths on L, there are restricted paths f_i^* and g_j^* such that $\left[f_i^*, g_j^*\right] \leq [f_i, g_j]$.*

Proof. Since f_i and g_i are paths on L, they are n restricted so by Proposition $P_{n-1}f_i$ and $P_{n-1}g_j$ are n restricted after time 0 and $n-1$ restricted after time 1 with $[P_{n-1}f_i, P_{n-1}g_j] \leq [f_i, g_j]$. We can iterate this process $n-1$ times to obtain the desired f_i^* and g_i^*. □

Proposition 5 *Suppose Players I and II begin at locations i and j respectively with probability $p_{i,j}$. If $\{f_i : i \in L\}$ is any set of paths for Player I and $\{g_j : j \in L\}$ is any set of paths for Player II then there are sets of restricted paths If $\{f_i^* : i \in L\}$ and $\left\{g_j^* : j \in L\right\}$ such that*

$$\sum_{i,j} p_{i,j}\left[f_i^*, g_j^*\right] \leq \sum_{i,j} p_{i,j}[f_i, g_j]. \tag{6}$$

Proof. For each i, j let f_i^*, g_j^* satisfy the conclusion of Proposition 4. □

Theorem 6 *Suppose Players I and II begin at locations i and j respectively with probability $p_{i,j}$. There are restricted paths $\{f_i^* : i \in L\}$ and $\left\{g_j^* : j \in L\right\}$ such that*

$$\sum_{i,j} p_{i,j}\left[f_i^*, g_j^*\right] \leq \sum_{i,j} p_{i,j}[f_i, g_j] \tag{7}$$

for any pair of sets of paths $\{f_i : i \in L\}$ and $\{g_j : j \in L\}$.

Proof. Since the set of restricted paths is finite so is the set of pairs of restricted paths. Thus there is a pair $\{f_i^* : i \in L\}$ and $\left\{g_j^* : j \in L\right\}$ of restricted paths for which

$$\sum_{i,j} p_{i,j}\left[f_i^*, g_j^*\right] \tag{8}$$

is a minimum. If $\{f_i : i \in L\}$ and $\{g_j : j \in L\}$ is any pair of paths, by Proposition 5 there is a pair of restricted paths $\left\{\hat{f_i} : i \in L\right\}$ and $\left\{\hat{g_j} : j \in L\right\}$ such that

$$\sum_{i,j} p_{i,j}\left[\hat{f_i}, \hat{g_j}\right] \leq \sum_{i,j} p_{i,j}[f_i, g_j] \tag{9}$$

but we also have

$$\sum_{i,j} p_{i,j}\left[f_i^*, g_j^*\right] \leq \sum_{i,j} p_{i,j}\left[\hat{f_i}, \hat{g_j}\right] \tag{10}$$

because $\sum_{i,j} p_{i,j}\left[f_i^*, g_j^*\right]$ is minimal over restricted paths. □

Matrix Representation

In this section it is convenient to represent the locations on the line L by $\{1,2,...,n\}$ where n can be odd or even. A collection of n motions to other locations can be represented by an $n\times n$ matrix D where the j^{th} column (d_{ij}) has $d_{kj}=1$ to represent a motion from j to k and 0's elsewhere. The transpose $D^{\top}$ of such a matrix also represents such a motion.

Proposition 7 *If $Q=(q_{i,j})$ is a matrix for which $q_{i,j}$ denotes the probability that Player I is at i and Player II is at j then $DQE^{\top}=(r_{i,j})$ is a matrix in which $r_{i,j}$ is the probability that Player I is at i and Player II is at j given that Player I performs the motions represented by D and Player II performs the motions represented by $E^{\top}$.*

Proof. If $DQ=(s_{i,j})$ then for each i we have

$$s_{i,j}=\sum_{h\in A}p_{h,j} \tag{11}$$

where A is the set of all h that Player I moved to i from h. Thus $s_{i,j}$ represents the probability that Player I is at i and Player II is at j after the move. A similar argument applies for $DQE^{\top}$. □

In the situation we are studying moves are restricted to adjacent locations so we shall take d_{kj} to be 1 for $k\in\{j-1,j,j+1\}$ and 0 elsewhere. We denote by $\mathbf{e}_i$ the column matrix that has 1 in the i^{th} row and 0's elsewhere. A path for Player I is represented by a sequence of matrices $D_t:t=1,2,....$

Proposition 8 *For $n=2m$, or $n=2m+1$ a sequence of matrices (D_t) represents a restricted path for Player I if and only if for each $h=1,2,...,m-1$ (1) the h^{th} column of D_h is $\mathbf{e}_{h+1}$, (2) the $h+1^{th}$ column of D_h is $\mathbf{e}_{h+1}$ or $\mathbf{e}_{h+2}$, (3) the $n-h^{th}$ column of D_h is $\mathbf{e}_{n-(h+1)}$, (4) the $n-(h+1)^{th}$ column of D_h is $\mathbf{e}_{n-(h+1)}$ or $\mathbf{e}_{n-(h+2)}$, and (5) For n even, the m^{th} and $(m+1)^{th}$ columns of D_m are both $\mathbf{e}_{m+1}$, and for n odd the $m^{th},(m+1)^{th}$ and $(m+2)^{th}$ are all $\mathbf{e}_{m+1}$.*

Proof. Conditions (1) and (2) hold if and only if on the h^{th} turn Player I moves toward the center from location h. Conditions (3) and (4) hold if and only if on the h^{th} turn Player I moves toward the center from location $n-h$. If these conditions are satisfied by matrices D_t for $t<h$ the probability that Player I is outside of the interval $\{h,h+1,...,n-h\}$ on turn h is zero. That is the first and last h rows of the matrix

$$D_1D_2...D_hP \tag{12}$$

are zero. The last condition holds if and only if Player I moves to $m+1$ on move m. □

For a matrix $A=[a_{i,j}]$ we denote by $\Delta(A)$ the matrix $D=[d_{i,j}]$ for which $d_{i,i}=a_{i,i}$ and $d_{i,j}=0$ for $i\neq j$; we denote by $Tr(A)$ the sum $\sum_i a_{i,i}$.

Proposition 9 *Suppose $P=[p_{i,j}]$ is the matrix for which $p_{i,j}$ is the probability that Player I begins at location i and Player II begins at location j. Suppose the number of locations is either $n=2m$ or $n=2m+1$ and Player I uses the restricted paths $\{f_i\}$ described by the matrices $\{D_t:t=1,...,m\}$ while Player II uses the restricted paths $\{g_j\}$ described by the matrices $\left\{E_t^{\top}:t=1,...,m\right\}$. Let*

$$P_1=D_1\left(P-\Delta\left(P\right)\right)E_1^{\top} \tag{13}$$

and for $t=2,...,m$ let

$$P_t=D_t\left(P_{t-1}-\Delta\left(P_{t-1}\right)\right)E_1^{\top} \tag{14}$$

Then the probability that Player I and II meet immediately after turn t is $Tr\left(P_t\right)$.

Proof. Each element $p_{i,j}$ of $P-\Delta(P)$ is the probability that I is at i and II is at j and they did not meet at the start. Thus the diagonal elements of P_1 are the probabilities that I and II meet immediately after the first move (Proposition 7). If $P_{t-1}=[s_{i,j}]$ then $s_{i,j}$ is the probability that after turn $t-1$ I is at i and II is at j and they have not previously met. Thus $Tr\left(P_{t-1}\right)$ is the probability that they meet immediately after turn $t-1$, $P_{t-1}-\Delta\left(P_{t-1}\right)$ is the matrix of probabilities that they have not yet met and are at different locations after turn $t-1$ and $D_t\left(P_{t-1}-\Delta\left(P_{t-1}\right)\right)E_1^{\top}$ is the matrix of probabilities that they are at their various locations after I and II make their moves. □

If we write $b_0=Tr\left(P\right)$ and $b_t=Tr\left(P_t\right)$ then assuming Player I uses the restricted paths $\{f_i\}$ described by the matrices $\{D_t:t=1,...,m\}$ while Player II uses the restricted paths $\{g_j\}$ described by the matrices $\left\{E_t^{\top}:t=1,...,m\right\}$ we have

$$E\left(\{f_i\},\{g_j\}\right)=\sum_{t=1}^{m}tb_t \tag{15}$$

since the players will certainly meet at the end of turn m it follows that $\sum_{t=0}^{m}b_t=\sum_{i,j}p_{i,j}=1$ so that

$$E\left(\{f_i\},\{g_j\}\right)=\sum_{t=1}^{m-1}tb_t+m\left(1-\sum_{t=0}^{m-1}b_t\right) \tag{16}$$

$$=m\left(1-b_0\right)-\sum_{t=1}^{m-1}\left(m-t\right)b_t \tag{17}$$

The quantity $m\left(1-b_0\right)$ is fixed so minimizing $E\left(\{f_i\},\{g_j\}\right)$ is equivalent to maximizing

$$\sum_{t=1}^{m-1}\left(m-t\right)b_t. \tag{18}$$

Solutions for Four and Five Locations

We have used the matrix method described in the previous section to completely solve the rendezvous problem for a line of four or five locations. To do this we have employed the symbolic computational program MAXIMA, which is a descendent of the Macsyma program maintained at the U. S. Department of Energy, and now available without charge on the internet.

First observe that the solutions for the one, two or three point line are obvious. In the case of one point the players meet at time 0. For two points the players decide before on a point to end at if they do not meet at time 0 and both go (or remain) there. For three points both players go to the center if they do not meet at time 0.

When there are four or five locations the players can meet after no more than two moves using restricted strategies. The terminal point in the four point case on the line $[1, 2, 3, 4]$ being 2 or 3 (chosen beforehand by the players or their controller) and the terminal point in the five point case on the line $[1, 2, 3, 4, 5]$ being 3. If the players have not met after the first move they both move to the terminal point on the second.

Four Locations

When there are four locations each player has only two possible tactics on the first move. They are described by the vectors

$$\mathbf{e}_2 = [0, 1, 0, 0] \text{ and } \mathbf{e}_3 = [0, 0, 1, 0] \tag{19}$$

The strategy described by $\mathbf{e}_2$ in row i is to move to 2 from i and the strategy described by $\mathbf{e}_3$ in row i is to move to 3 from i. The first column must be $\mathbf{e}_2$ (if the strategy is restricted) and the fourth column must be $\mathbf{e}_3$ while the two middle columns can be either. Thus on the first move there are four strategy matrices for each player resulting in a total of sixteen strategy pairs for the two teams. In formula the sum that has to be maximized is simply b_1. We have calculated the quantities b_1 for all sixteen strategy pairs and have found that each pair can be optimal in the appropriate situation.

Example 10 *The pair of matrices*

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{20}$$

describes the pair of strategies for which Player I moves to 2 if it begins at 1, to 3 if it begins at 2, to 2 if it begins at 3 and to 3 if it begins at 4 while Player II moves to 2 if it begins at 1 to remains at 2 if it begins at 2 moves to 2 if it begins at 3 and to 3 if it begins at 4. Since

$$A \begin{bmatrix} 0 & p_{1,2} & p_{1,3} & p_{1,4} \\ p_{2,1} & 0 & p_{2,3} & p_{2,4} \\ p_{3,1} & p_{3,2} & 0 & p_{3,4} \\ p_{4,1} & p_{4,2} & p_{4,3} & 0 \end{bmatrix} B \tag{21}$$

has trace equal to

$$b_1 = p_{3,2} + p_{3,1} + p_{1,3} + p_{1,2} + p_{2,4} \tag{22}$$

it follows that if the players use this strategy the expected time will be

$$b_1 + 2\left(1 - b_1 - \sum_{j=1}^{4} p_{j,j}\right). \tag{23}$$

Since no other strategy results in all of these terms in b_1 it follows that if $p_{3,2} = p_{3,1} = p_{1,3} = p_{1,2} = p_{2,4} = \frac{1}{5}$ then the expected time will be 1 and this strategy and no other will be optimal.

The results of the calculations are displayed in the following table. The first four columns are interpreted as follows: first column - Player I strategy at location 2, 0 means stay, 1 means move to 3; second column - Player I strategy at location 3, 0 means stay, -1 means move to location 2; third column - Player II strategy at location 2 , 0 means stay, 1 means move to 3; fourth column - Player II strategy at location 3, 0 means stay, -1 means move to 2. Since we are dealing with restricted paths if a player is at an end point it will move to the adjacent point. The last column denotes quantity b_1. If each of the quantities appearing in the last column are equal and have sum 1 then no strategy will do as well as that described in the previous row. For example if $p_{4,3} = p_{4,2} = p_{3,4} = p_{3,2} = p_{2,1} = \frac{1}{5}$ then no strategy will do as well as that depicted in row 2: Player I remains in place at location 2 or 3 while player moves to 3 if at location 2 and remains in place if at location 3.

$$\begin{array}{cccll} 0 & 0 & 0 & 0 & p_{4,3} + p_{3,4} + p_{2,1} + p_{1,2} \\ 0 & 0 & 1 & 0 & p_{4,3} + p_{4,2} + p_{3,4} + p_{3,2} + p_{2,1} \\ 0 & 0 & 0 & -1 & p_{3,4} + p_{2,3} + p_{2,1} + p_{1,3} + p_{1,2} \\ 0 & 0 & 1 & -1 & p_{4,2} + p_{3,4} + p_{3,2} + p_{2,3} + p_{2,1} + p_{1,3} \\ 1 & 0 & 0 & 0 & p_{4,3} + p_{3,4} + p_{2,4} + p_{2,3} + p_{1,2} \\ 1 & 0 & 1 & 0 & p_{4,3} + p_{4,2} + p_{3,4} + p_{3,2} + p_{2,4} + p_{2,3} \\ 1 & 0 & 0 & -1 & p_{3,4} + p_{2,4} + p_{1,3} + p_{1,2} \\ 1 & 0 & 1 & -1 & p_{4,2} + p_{3,4} + p_{3,2} + p_{2,4} + p_{1,3} \\ 0 & -1 & 0 & 0 & p_{4,3} + p_{3,2} + p_{3,1} + p_{2,1} + p_{1,2} \\ 0 & -1 & 1 & 0 & p_{4,3} + p_{4,2} + p_{3,1} + p_{2,1} \\ 0 & -1 & 0 & -1 & p_{3,2} + p_{3,1} + p_{2,3} + p_{2,1} + p_{1,3} + p_{1,2} \\ 0 & -1 & 1 & -1 & p_{4,2} + p_{3,1} + p_{2,3} + p_{2,1} + p_{1,3} \\ 1 & -1 & 0 & 0 & p_{4,3} + p_{3,2} + p_{3,1} + p_{2,4} + p_{2,3} + p_{1,2} \\ 1 & -1 & 1 & 0 & p_{4,3} + p_{4,2} + p_{3,1} + p_{2,4} + p_{2,3} \\ 1 & -1 & 0 & -1 & p_{3,2} + p_{3,1} + p_{2,4} + p_{1,3} + p_{1,2} \\ 1 & -1 & 1 & -1 & p_{4,2} + p_{3,1} + p_{2,4} + p_{1,3} \end{array} \tag{24}$$

Five Locations

When there are five locations, there are 12 matrices describing restricted strategies resulting in 144 strategy pairs. The following matrix describes these strategies. The first column is a number used to name the strategy. The action of the strategy at location 2,3,4 are given in the columns marked 2,3,4 respectively. For example, Strategy 6 is that of staying in place at location 2 moving to 4 from location 3 and staying in place at location 4.

$$\begin{array}{cccc} & 2 & 3 & 4 \\ 1 & 0 & -1 & -1 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \\ 4 & 0 & 0 & 0 \\ 5 & 0 & 1 & -1 \\ 6 & 0 & 1 & 0 \\ 7 & 1 & -1 & -1 \\ 8 & 1 & -1 & 0 \\ 9 & 1 & 0 & -1 \\ 10 & 1 & 0 & 0 \\ 11 & 1 & 1 & -1 \\ 12 & 1 & 1 & 0 \end{array} \tag{25}$$

Of the 144 possible values of b_i 97 result in values that are dominated by other values so there are 47 non dominated strategy pairs. We have listed below the 47 non dominated strategy pairs using the designations described in the matrix. The first number is the strategy used by I, the second by II and the third column is the resulting value of b_1.

1	8	$p_{5,4} + p_{4,2} + p_{3,1} + p_{2,3} + p_{2,1} + p_{1,3}$
1	10	$p_{5,4} + p_{4,3} + p_{4,2} + p_{3,1} + p_{2,1}$
1	12	$p_{5,4} + p_{5,3} + p_{4,2} + p_{3,1} + p_{2,1}$
2	2	$p_{5,4} + p_{4,5} + p_{3,2} + p_{3,1} + p_{2,3} + p_{2,1} + p_{1,3} + p_{1,2}$
2	6	$p_{5,4} + p_{5,3} + p_{4,5} + p_{4,3} + p_{3,2} + p_{3,1} + p_{2,1} + p_{1,2}$
3	7	$p_{4,2} + p_{3,4} + p_{3,2} + p_{2,3} + p_{2,1} + p_{1,3}$
3	8	$p_{5,4} + p_{4,2} + p_{3,2} + p_{2,3} + p_{2,1} + p_{1,3}$
3	9	$p_{4,3} + p_{4,2} + p_{3,4} + p_{3,2} + p_{2,1}$
3	10	$p_{5,4} + p_{4,3} + p_{4,2} + p_{3,2} + p_{2,1}$
3	11	$p_{5,3} + p_{4,2} + p_{3,4} + p_{3,2} + p_{2,1}$
3	12	$p_{5,4} + p_{5,3} + p_{4,2} + p_{3,2} + p_{2,1}$
4	7	$p_{4,5} + p_{3,4} + p_{3,2} + p_{2,3} + p_{2,1} + p_{1,3}$
4	11	$p_{5,3} + p_{4,5} + p_{4,3} + p_{3,4} + p_{3,2} + p_{2,1}$
5	8	$p_{5,4} + p_{4,2} + p_{3,5} + p_{3,4} + p_{2,3} + p_{2,1} + p_{1,3}$
5	10	$p_{5,4} + p_{4,3} + p_{4,2} + p_{3,5} + p_{3,4} + p_{2,1}$
5	12	$p_{5,4} + p_{5,3} + p_{4,2} + p_{3,5} + p_{3,4} + p_{2,1}$
6	2	$p_{5,4} + p_{4,5} + p_{3,5} + p_{3,4} + p_{2,3} + p_{2,1} + p_{1,3} + p_{1,2}$
6	6	$p_{5,4} + p_{5,3} + p_{4,5} + p_{4,3} + p_{3,5} + p_{3,4} + p_{2,1} + p_{1,2}$
7	3	$p_{4,3} + p_{3,2} + p_{3,1} + p_{2,4} + p_{2,3} + p_{1,2}$
7	4	$p_{5,4} + p_{4,3} + p_{3,2} + p_{3,1} + p_{2,3} + p_{1,2}$
7	7	$p_{4,2} + p_{3,1} + p_{2,4} + p_{1,3}$
7	9	$p_{4,3} + p_{4,2} + p_{3,1} + p_{2,4} + p_{2,3}$
7	10	$p_{5,4} + p_{4,3} + p_{4,2} + p_{3,1} + p_{2,3}$
7	11	$p_{5,3} + p_{4,2} + p_{3,1} + p_{2,4}$
8	1	$p_{4,5} + p_{3,2} + p_{3,1} + p_{2,4} + p_{1,3} + p_{1,2}$
8	3	$p_{4,5} + p_{3,2} + p_{3,1} + p_{2,4} + p_{2,3} + p_{1,2}$
8	5	$p_{5,3} + p_{4,5} + p_{4,3} + p_{3,2} + p_{3,1} + p_{2,4} + p_{1,2}$
9	3	$p_{4,3} + p_{3,4} + p_{2,4} + p_{2,3} + p_{1,2}$
9	7	$p_{4,2} + p_{3,4} + p_{3,2} + p_{2,4} + p_{1,3}$
9	9	$p_{4,3} + p_{4,2} + p_{3,4} + p_{3,2} + p_{2,4} + p_{2,3}$
9	10	$p_{5,4} + p_{4,3} + p_{4,2} + p_{3,2} + p_{2,3}$
9	11	$p_{5,3} + p_{4,2} + p_{3,4} + p_{3,2} + p_{2,4}$
10	1	$p_{4,5} + p_{3,4} + p_{2,4} + p_{1,3} + p_{1,2}$
10	3	$p_{4,5} + p_{3,4} + p_{2,4} + p_{2,3} + p_{1,2}$
10	5	$p_{5,3} + p_{4,5} + p_{4,3} + p_{3,4} + p_{2,4} + p_{1,2}$
10	7	$p_{4,5} + p_{3,4} + p_{3,2} + p_{2,4} + p_{1,3}$
10	9	$p_{4,5} + p_{3,4} + p_{3,2} + p_{2,4} + p_{2,3}$
10	11	$p_{5,3} + p_{4,5} + p_{4,3} + p_{3,4} + p_{3,2} + p_{2,4}$
11	3	$p_{4,3} + p_{3,5} + p_{2,4} + p_{2,3} + p_{1,2}$
11	4	$p_{5,4} + p_{4,3} + p_{3,5} + p_{3,4} + p_{2,3} + p_{1,2}$
11	7	$p_{4,2} + p_{3,5} + p_{2,4} + p_{1,3}$
11	9	$p_{4,3} + p_{4,2} + p_{3,5} + p_{2,4} + p_{2,3}$
11	10	$p_{5,4} + p_{4,3} + p_{4,2} + p_{3,5} + p_{3,4} + p_{2,3}$
11	11	$p_{5,3} + p_{4,2} + p_{3,5} + p_{2,4}$
12	1	$p_{4,5} + p_{3,5} + p_{2,4} + p_{1,3} + p_{1,2}$
12	3	$p_{4,5} + p_{3,5} + p_{2,4} + p_{2,3} + p_{1,2}$
12	5	$p_{5,3} + p_{4,5} + p_{4,3} + p_{3,5} + p_{2,4} + p_{1,2}$

Example 11 *The most obvious situation occurs when both players begin at a location with equal and independent probabilities so that for each* i, j, $p_{i,j} = \frac{1}{25}$. *Any strategy in which there is a maximal number of terms in the third column will then be optimal. These are* $(2,2), (2,6), (6,2), (6,6)$, *having 8 terms. Each of these strategies will give an expected time of* $\frac{8}{25} + 2\left(1 - \frac{8}{25} - \frac{5}{25}\right) = \frac{32}{25}$. *A similar situation is when both players are placed with equal probability at pairs of different locations. The same strategy pairs are optimal and the expected time is then* $\frac{8}{20} + 2\left(1 - \frac{8}{20}\right) = \frac{8}{5}$.

You may wonder why the number of non-dominated strategy pairs is odd rather than even in view of the observation that every strategy has a reflection due to symmetry. The answer to this conundrum is that the strategy number 9 is its own reflection and the pair $(9,9)$ is a non-dominated pair that is its own partner.

One Sided Rendezvous

In this section we shall consider the special case when one player whom we designate Player 1 begins on the line with certainty at a given location that we designate d and the other player (Player 2) begins at location i with probability p_i. We continue to denote the points on the line with integers from 1 to n. The problem is to design a path f for Player 1 to follow and paths $\{g_i : i \neq d\}$ for Player 2 for which the quantity

$$E\left(f, \{g_i\}\right) = \sum_{i \neq d} p_i \left[f, g_i\right] \tag{26}$$

where as in Section 3, $[f, g_i]$ denotes the number of turns until $f(j) = g_i(j)$. It is unnecessary to determine g_d because if Player 2 begins at d it will meet Player 1 at time 0.

The following result is an immediate consequence of the Restriction Theorem.

Proposition 12 *If $d = 1$ then there is an optimal strategy pair of the form $(f, \{g_i\})$ where the path for Player 1 is $f(i) = i, i = 1, 2, ..., n$. If $d = n$ there is such a pair for which $f(i) = i + 1 - n, i = 1, 2, ..., n$.*

If $d = 1$ and Player 1 uses the path described in the Proposition then if Player 2 starts at location 2 it should wait for Player 1 rather than moving to location 1 (and missing Player 1). If Player 2 starts at location 3 it will meet Player 1 upon entering location 2. If we think of the probability of Player 2 starting at location i as a potential player then we can envision Player 1 encountering two potential players at each location visited before turn $[n/2]$ provided both players travelled without stopping in the direction of Player 1 before that time. At turn $i < [n/2]$ these are the potential player that started from $2i$ and is waiting for Player 1 to arrive from location $i-1$ and the potential player that started from $2i+1$ and will meet Player 1 when the two of them arrive. Since each potential player will meet Player 1 in the minimum possible number of turns we have proved the following result.

Proposition 13 *When $d = 1$ an optimal strategy pair is $(f, \{g_i : i = 2, ..., n\})$ where $f(j) = j$ for $j = 1, 2, ..., [n/2]$ and for each i, g_i is described inductively by $g_i(1) = i$*

$$g_i(j+1) = \begin{cases} g_i(j) - 1 \text{ if } g_i(j) > [i/2] \\ [i/2] \ \ otherwise \end{cases} \tag{27}$$

The minimal time thus obtained is

$$m = \begin{cases} (p_2 + p_3) + ... + \left(\frac{n}{2} - 1\right)(p_{n-2} + p_{n-1}) + \left(\frac{n}{2}\right)(p_n) \ n \ even \\ (p_2 + p_3) + ... + \left(\frac{n-1}{2}\right)(p_{n-1} + p_n) \ n \ odd. \end{cases} \tag{28}$$

Consideration of the case $d = 1$ shows the appropriate strategy for Player 2 in all cases given that it is aware of the path to be taken by Player 1. If the distance from Player 2 is more than one from that of the path then Player 2 should move in the direction of that path. If the distance is 1 then Player 2 should move in that direction if it knows Player 1 intends to remain in place or to move in the opposite direction; otherwise, Player 2 should remain in place and meet Player 1 on the next turn.

The path for Player 1 can be determined by an iterative process often called "dynamic programming" if we assume the players use restricted strategies. For with such strategies the line in which the movement occurs is reduced by the two end points at each stage and the resulting situation is still a one sided rendezvous in which Player 1 may or may not be relocated by 1. In fact, all such rendezvous problems on the line are subject to dynamic programming, but only with the one sided situation is the growth of the number of subcases sufficiently restricted to make it feasible. At each turn Player 1 being at location d can remain in place and receive potential players from the $d-1$ and $d+1$, can move right (except at a right end point) and receive the potential players from $d+1$ which has waited for it and from $d+2$ which has moved to meet it, or move left and receive potential players from $d-1$ and $d-2$. A path for Player 1 consists of a sequence of such choices.

A central tool in the study of one sided rendezvous is the following exercise in algebra

Proposition 14 *Suppose A and B are two sets of N positive real numbers. The minimum of the set*

$$\left\{ \sum_{(a.b) \in A \times B} ab \right\} \tag{29}$$

occurs for the sum of the form

$$\sum_{i=1}^{N} a_i b_j \tag{30}$$

where $A = \{a_1 \leq a_2 \leq ... \leq a_N\}$ and $B = \{b_1 \geq b_2 \geq ... \geq b_N\}$, that is to say, where the sequence of one set of multiplicands is non-decreasing and the set of the other set of multiplicands is non-increasing.

Definition 15 *Player 1 will be said to be following the* **opportunistic** *strategy if at each turn it chooses the action for which the sum of the two potential players is the largest. That is it compares the quantities*

$$x_{-1} = p_{d-1} + p_{d-2}, x_0 = p_{d-1} + p_{d+1}, x_1 = p_{d+1} + p_{d+2} \tag{31}$$

and moves left if x_{-1} is maximal, remains in place if x_0 is maximal and moves right if x_1 is maximal. In case a subscript i

is beyond an end point we let $p_i = 0$. We shall write $X(k)$ to denote the maximum of $\{x_{-1}, x_0, x_1\}$ for turn k.

The opportunistic strategy for Player 1 is not necessarily unique since the maximum may be assumed for two or even three values of x_i. That the opportunistic strategy need not be optimal is easily shown by the following example.

Example 16 *Let $p_1 = 3/4, p_2 = p_3 = 0, p_4 = 1/4$, and let Player 1 start from 3. If Player 1 uses the opportunistic strategy it waits at 3 and the expected number of turns is $1 \cdot (1/4) + 2 \cdot (3/4) = 7/4$ whereas if it meets the potential player from 1 by moving to 2 on the first turn the expected number of turns is $1 \cdot (3/4) + 2 \cdot (1/4) = 5/4$.*

A situation in which the opportunistic strategy is optimal is described in the following proposition.

Proposition 17 *Suppose Player 1 begins at location d with $1 < d < n$, and suppose $p_1 \leq p_2 \leq ... \leq p_{d-1}$ while $p_{d+1} \geq p_{d+2} \geq ... \geq p_n$. Then the opportunistic strategy will obtain the smallest expected search time.*

Proof. We carry out the proof assuming n is even; the proof when n is odd follows the same plan. Let $T = (n-2)/2$, let A be the set $\{p_i : i \neq d\}$ and let $B = \{1, 1, ..., T, T, T+1\}$. Then A and B are sets of positive numbers having the same cardinality so the minimum of sums of their products occurs when one sequence of multiplicands is non-decreasing and the other is non-increasing. A number in set B represents the time at which Player 1 meets a potential player from set A. Because of the ordering of the members of Set A, Player 1 can select an ordering of the p_i that is non-increasing by using the opportunistic strategy. □

When d is different from $n-d-1$ Proposition 17 still applies but Player 1 exhausts the locations on one side it commences a sequential search on the remaining locations. A special case of the Proposition occurs when $d = n-d+i$ the probabilities are symmetric that is when $p_{d-i+1} = p_{d+i}$ for $i = 1, 2, .., d-1$. If $p_i = p_{i+1}$ for some values of i then the Opportunistic strategy allows more than one possible move, but if $p_{i+1} < p_i$ for all $i < d$ the only Opportunistic strategy is for Player 1 to remain at d and let the potential players come to it.

In the situation when $p_i = 1/n$ for each $i = 1, 2, ..., n$, i. e. the case of the uniform distribution the Opportunistic strategy allows many solutions. Player 1 will encounter two potential players if it remains in place, and will encounter one or two potential players if it moves to the right or the left. It will encounter one potential player if it moves right from $n-1$ or left from 1. Thus the expected meeting time will be

$$E = x + (1-x)\left(E'(h) + 1\right) \tag{32}$$

where x is $1/n$ or $2/n$ and $E'(h)$ is the expected meeting time in the problem with $n-2$ location and with Player 1 beginning at $h = d-1, d$, or $d+1$. If $n-2$ is 1 then $E = 0$ and if $n-2 = 2$ then $E = \frac{1}{2}$ because in both of these situations Player 1 starts at an end point. We can then apply induction to solve one sided problems with uniform distribution of any length. The following Proposition shows that in this case the opportunistic strategy is only one of many possible strategies.

Proposition 18 *In the one sided problem with uniform with a uniform distribution for player 2 with $n = 3$, and $d \neq 1, n$, Player 1 should not move to an end point and the resulting minimal value is the same as when $d = 1$ namely*

$$m = \begin{cases} \frac{n^2+2n+2}{4n} & n \text{ even} \\ \frac{n^2-1}{4n} & n \text{ odd.} \end{cases} \tag{33}$$

If $n = 3$ then $d = 2$. If Player 1 remains in place then the remaining two potential players move to it at time 1 so $E = 2/3$. Player 1 moves to 1 or 3 then the potential player awaits it there and the other moves to 2 and is met by Player 1 at time 2 so that $E = 1/3 + 2(1/3) = 1$. Thus the minimal expected time will occur when Player 1 remains in the middle.

If $n = 4$ and $d = 2$ then if Player 1 remains in place the expected meeting time is $\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{5}{4}$ while if it moves to 1 then the expected meeting time is $\frac{1}{4} + \frac{3}{4}\left(\frac{1}{2} + 1\right) = \frac{11}{8}$ which is larger.

This establishes the proposition for $n = 3$ and 4. Assume the proposition is true for $3 \leq k < n$. If n is even then $n-2$ is also even so if Player 1 remains in place the expected meeting time will be

$$E = \frac{2}{n} + \left(1 - \frac{2}{n}\right)\left(\frac{(n-2)^2 + 2(n-2) + 2}{4(n-2)}\right) = \frac{n^2 + 2n + 2}{4n} \tag{34}$$

If Player 1 moves to an end point the expected meeting time will be $1/n\left(1 + (n-1)(E'+1)\right) \geq 1/n\left(2 + (n-2)(E'+1)\right)$ because E' the expected meeting time in the $n-2$ point problem is greater than 1. □

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