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Existence and Multiplicity of Positive Solutions for a m-Point Boundary Value Problem

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Abstract—**We study a second order differential equation with nonlinear multi-point boundary conditions. The existence and multiplicity of positive solutions is proved through Krasnoselskii fixed point theorem and Avery-Peterson theorem.**

Keywords: m-point, Krasnoselskii, Avery-Peterson, Multiple Solutions.

1 Introduction

Historically multi-point boundary problems have had as pioneer works the Il'in and Moiseev 's papers (see, [4, 5]) in which they consider the following problem

$$\begin{cases} u''(x) + f(x, u(x), u'(x)) = 0 \\ u(0) = 0 \ , \ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)) \end{cases} \tag{1}$$

where $f : [0, 1] \times R \times R \to R$, $\eta_1, ..., \eta_{m-2} \in (0, 1)$ and $\alpha_1, ..., \alpha_{m-2} \in R$.

Since then several authors have studied variations of this class problem, sometimes involving three points other times involving multiple points, for example [6, 1, 2, 3, 8, 9, 10]. This category of problems describe many phenomena in applied mathematical science (see, [13]). On account of this the relevance of its study. In this paper we consider the problem in a more generalized and more comprehend form:

$$\begin{cases} u''(x) + f(x, u(x), u'(x)) = 0 \\ u(0) = 0 \ , \ u(1) = g(u(\eta_1), ..., u(\eta_{m-2})) \end{cases} \tag{2}$$

where $f : [0, 1] \times R \times R \to R$ and $g : R^{m-2} \to R$ are continuos functions possibly nonlinear and $\eta_1, ..., \eta_{m-2} \in (0, 1)$.

This work is organized in the following way: in section 2 we present a result of a positive solution existence using the Krasnoselskii's theorem in cones [11] and in section 3 we present a result of multiple positive solution existence using the Avery-Peterson's theorem [7]. In the literature we find techniques using the Krasnoselskii's theorem to solve multi-point boundary problems with the following characterization:

$$u''(x) + f(x, u(x)) = 0 \tag{3}$$

and these techniques has been very much used by several authors. However in (2) we notice the existence of the "u' " term in the argument of function f. This term makes the use of Krasnoselskii's theorem difficult and consequently makes the use of Avery-Peterson's theorem difficult too. We demonstrate in this work the use of these theorems keeping the "u' " term and keeping general conditions in the boundary and on account of this we have the relevance of the technique exposed in this paper. For the purpose of this work we need to introduce the mains tools.

Theorem 1.1 *Let $X = (X, \|\cdot\|)$ be a Banach space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $S : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator such that, either*

(a)$\|Su\| \le \|u\|, u \in P \cap \partial\Omega_1$, and $\|Su\| \ge \|u\|, u \in P \cap \partial\Omega_2$, or

(b)$\|Su\| \ge \|u\|, u \in P \cap \partial\Omega_1$, and $\|Su\| \le \|u\|, u \in P \cap \partial\Omega_2$.

Then S has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Now, we need to consider the convex sets

$$P(\gamma, d) = \{x \in P | \gamma < d\}$$

$$P(\gamma, \alpha, b, d) = \{x \in P | b \le \alpha(x) \text{ and } \gamma(x) < d\}$$

$$P(\gamma, \theta, \alpha, b, c, d) = \{x \in P | b \le \alpha(x), \theta(x) \le c \text{ and } \gamma(x) < d\}$$

and the closed set

$$R(\gamma, \psi, a, d) = \{x \in P | a \le \psi(x) \text{ and } \gamma(x) < d\}.$$

Theorem 1.2 *Let P be a cone in a real Banach space X. Let γ and θ nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P*

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satisfying $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers μ and d,

$$\alpha(x) \leq \psi(x) \text{ and } \|x\| \leq \mu\gamma(x),$$

for all $x \in \overline{P(\gamma, d)}$. Suppose

$$T : \overline{P(\gamma, d)} \rightarrow \overline{P(\gamma, d)}$$

is completely continuous and there exist positive numbers a, b, c with $a < b$, such that

$$\{u \in P(\gamma, \theta, \alpha, b, c, d) | \alpha(u) > b\} \neq \emptyset \text{ and}$$

$$u \in P(\gamma, \theta, \alpha, b, c, d) \Rightarrow \alpha(Tu) > b \tag{4}$$

$$\alpha(Tu) > b \text{ for } u \in P(\gamma, \alpha, b, d) \text{ with } \theta(Tu) > c, \tag{5}$$

$$0 \notin R(\gamma, \psi, a, d) \text{ and } \psi(Tu) < \alpha \text{ for} \tag{6}$$

$$u \in R(\gamma, \psi, a, d) \text{ with } \psi(u) = a.$$

Then T has at least three distinct fixed points in $\overline{P(\gamma, d)}$.

The first theorem is a fixed point theorem in cone due to Krasnoselskii. The second result is the Avery-Peterson theorem.

2 Existence of Positive Solutions

Let $C^1[0, 1]$ be the Banach space of all continuous functions at $[0, 1]$ equipped with the norm

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$$

We begin this section by pointing out that the solutions of (2) can be written as

$$u(x) = \int_0^1 G(x, t) f(t, u(t), u'(t))\, dt$$
$$+ g(u(\eta_1), ..., u(\eta_{m-2}))x,$$

where G is the Green's function for $u''(x) + f(x) = 0$ with $u(0) = u(1) = 0$, namely,

$$G(x, t) = \begin{cases} t(1 - x) & t \leq x \\ x(1 - t) & x \leq t. \end{cases}$$

We can note that $G(x, t)$ has the following properties:

$$\partial_x G(x, t) = \begin{cases} -t & t \leq x \\ (1 - t) & x \leq t \end{cases}, \tag{7}$$

$$G(x, t) = |G(x, t)| \leq |\partial_x G(x, t)|, \tag{8}$$

$$\int_0^1 |\partial_x G(x, t)| dt \leq \frac{1}{2}, \tag{9}$$

$$G(x, t) \geq mG(t, t), \forall x \in [m, 1 - m] \text{ and } 0 \leq m \leq \frac{1}{2}, \tag{10}$$

and

$$\int_0^1 G(x, t) dt \leq \frac{1}{8}. \tag{11}$$

We know that u is solution of (2) if and only if it is a fixed point of the operator $T : C^1[0, 1] \rightarrow C^1[0, 1]$ defined by

$$(Tu)(x) = \int_0^1 G(x, t) f(t, u(t), u'(t))\, dt + g(u(\eta_1), ..., u(\eta_{m-2}))x. \tag{12}$$

In this sense we have the following result.

Theorem 2.1 *Suppose that there exist $t \in [0, 1]$ with $f(t, 0, 0) > 0$ and there exist $d > 0$ such that the following conditions hold:*

(A1) $0 \leq f(t, u, v) \leq d$, $\forall (t, u, v) \in [0, 1] \times [0, d] \times [-d, d]$,

(A2) $0 \leq g(y) \leq \frac{d}{2}$, $\forall y \in [0, d]^{m-2}$.

Then (2) has a positive solution $u^* \in C^1[0, 1]$.

Proof. We shall employ Krasnoselskii theorem. To begin, note that the operator $T : C^1[0, 1] \rightarrow C^1[0, 1]$ is completely continuous by Arzela-Ascoli's theorem. Next, we define a cone P by

$$P = \{u \in C^1[0, 1]; u \geq 0, u(0) = 0\}.$$

We see that $T : P \cap B(0, d) \rightarrow C$ where $B(0, d) = \{u \in C^1[0, 1]; \|u\| \leq d\}$. In fact, $Tu(0) = 0$ because $G(0, t) = 0$. Furthermore, by $(A1)$ and $(A2)$ we conclude $Tu \geq 0$. Therefore, $T : C \cap B(0, d) \rightarrow C$. Now, we shall show that $\|Tu\| \leq \|u\|$ for all $u \in C \cap \partial B(0, d)$. Thus let $u \in C \cap \partial B(0, d)$. By to use (8) and $(A1)$ we have

$$\left| \int_0^1 G(x, t) f(t, u(t), u'(t)) dt + g(u(\eta_1), ..., u(\eta_{m-2}))x \right|$$

$$\leq \int_0^1 |\partial_x G(x, t)| f(t, u(t), u'(t)) dt + |g(u(\eta_1), ..., u(\eta_{m-2}))|.$$

Then

$$\|Tu\| \leq \int_0^1 |\partial_x G(x, t)| f(t, u(t), u'(t)) dt$$
$$+ |g(u(\eta_1), ..., u(\eta_{m-2}))|,$$

that is,

$$\|Tu\| \leq d \int_0^1 |\partial_x G(x, t)| dt + |g(u(\eta_1), ..., u(\eta_{m-2}))|$$

Using $(A3)$ and (9) we can get

$$\|Tu\| \leq d \int_0^1 |\partial_x G(x, t)| dt + \frac{d}{2} \leq d.$$

Thus we conclude that $\|Tu\| \leq \|u\|$, $\forall u \in C \cap \partial B(0, d)$.

Next we need to show that there exist $\bar{d} > 0$ such that $\|Tu\| \geq \|u\|$, $\forall u \in P \cap \partial B(0, \bar{d})$, where $B(0, \bar{d}) = \{u \in C^1[0,1]; \|u\| \leq \bar{d}\}$. In fact, let us suppose the inequality is false. Thus for each $n > d$ we can get $u_n \in P$ such that

$$\|u_n\| = \frac{1}{n}$$

and

$$\|Tu_n\| < \frac{1}{n}.$$

Then $u_n \to 0$ and $Tu_n \to 0$. Since T is continuous we have $T0 = 0$. But, from the hypothesi of theorem there exist $t \in [0,1]$ such that $f(t, 0, 0) > 0$. Therefore $T0 > 0$ which is a contradiction. Thus there exist $\bar{d} > 0$ such that $\|Tu\| \geq \|u\|$, $\forall u \in P \cap \partial B(0, \bar{d})$. If we considerer $\Omega_1 = B(0, \bar{d})$ and $\Omega_2 = B(0, d)$ we have from item (b) of Krasnoselskii theorem that (2) has a positive solution $u^* \in C^1[0,1]$.

Example 2.2 *Let us consider*

$$f(t, u, v) = 2t + \frac{3}{10}u + \frac{v^2}{20}$$

and

$$g(z) = \frac{z}{2}.$$

Taking $d = 10$ we get

- $0 \leq f(t, u, v) \leq 10 = d$ *if* $(t, u, v) \in [0,1] \times [0,10] \times [-10, 10]$,
- $\max_{t \in [0,1]} f(t, 0, 0) = 2 > 0$,
- $0 \leq g(z) \leq 5 = \frac{d}{2}, \forall z \in [0, 10]$.

Thus we can obtain from theorem 2.1 that the problem (2) has a positive solution.

3 Multiple Solutions

In this section we are presenting our multiplicity result.

Theorem 3.1 *Suppose that the hypothesis of Theorem 2.1 hold. Suppose in addition that there exist $a > 0$ such that f and g satisfies the following conditions:*

(A3) $f(t, u, v) < 6a$, $\forall (t, u, v) \in [0,1] \times [0, a] \times [-d, d]$ and $g(u) < \frac{a}{4}$ with $\|u\| \leq a$,

(A4) $f(t, u, v) > 70a$, $\forall (t, u, v) \in [\frac{1}{4}, \frac{3}{4}] \times [2a, 8a] \times [-d, d]$.

Then the problem (2) has at least three positive solutions.

Proof. We apply Avery-Peterson theorem. Thus, we consider T and P defined as before. Furthermore, we need define the following functionals motivated by Bai-Wang-Ge [12],

$$\gamma(u) = \|u\|,$$

$$\psi(u) = \theta(u) = \max_{t \in [0,1]} |u(t)|,$$

$$\alpha(u) = \min_{t \in [\frac{1}{4}, \frac{3}{4}]} |u(t)|.$$

From $(A1)$ we have $Tu \geq 0$ if $\gamma(u) = \|u\| \leq d$. Using $(A2)$ we get $\gamma(Tu) \leq d$ if $\gamma \leq d$. Therefore, $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$.

Now, we considerer

$$b = 2a$$

and

$$c = 8a.$$

Clearly, we have $\{u \in P(\gamma, \theta, \alpha, b, c, d) | \alpha(u) > b\} \neq \emptyset$. Let us verify (4).

$$\begin{aligned}
\alpha(Tu) &= \min_{x \in [\frac{1}{4}, \frac{3}{4}]} (Tu)x \\
&\geq \min_{x \in [\frac{1}{4}, \frac{3}{4}]} \int_0^1 G(x,t) f(t, u(t), u'(t)) dt \\
&\quad + \min_{x \in [\frac{1}{4}, \frac{3}{4}]} g(u(\eta_1), ..., u(\eta_{m-2}))x \\
&\geq \frac{1}{4} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,t) f(t, u(t), u'(t)) dt \\
&\quad + \frac{1}{4} g(u(\eta_1), ..., u(\eta_{m-2})) \\
&\geq \frac{1}{4} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,t) f(t, u(t), u'(t)) dt
\end{aligned}$$

Using $(A4)$ we can get

$$\begin{aligned}
\alpha(Tu) \geq (70a) \frac{1}{4} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,t) dt \geq (70a) \frac{11}{384} \\
> 2a = b.
\end{aligned}$$

Let us show (5). Let $u \in P(\gamma, \alpha, b, d)$ with $\theta(Tu) > c$. Then

$$\begin{aligned}
\alpha(Tu) &= \min_{x \in [\frac{1}{4}, \frac{3}{4}]} \Big(\int_0^1 G(x,t) f(t, u(t), u'(t)) dt \\
&\quad + g(u(\eta_1), ..., u(\eta_{m-2})) \Big) \\
&\geq \min_{x \in [\frac{1}{4}, \frac{3}{4}]} \Big(\int_0^1 G(x,t) f(t, u(t), u'(t)) dt \Big) \\
&\quad + \min_{x \in [\frac{1}{4}, \frac{3}{4}]} g(u(\eta_1), ..., u(\eta_{m-2}))x \\
&\geq \frac{1}{4} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,t) f(t, u(t), u'(t)) dt \\
&\quad + \frac{1}{4} \max_{x \in [0,1]} g(u(\eta_1), ..., u(\eta_{m-2}))x \\
&\geq \frac{1}{4} \Big\{ \max_{x \in [0,1]} \int_0^1 G(x,t) f(t, u(t), u'(t)) dt \\
&\quad + \max_{x \in [0,1]} g(u(\eta_1), ..., u(\eta_{m-2})) \Big\} \\
&= \frac{1}{4} \theta(Tu) > \frac{c}{4} = \frac{8a}{4} = 2a = b.
\end{aligned}$$

Now, let us show (6). Thus, let $u \in R(\gamma, \psi, a, d)$ with $\psi(u) = a$. From $(A1) - (A2)$ we have,

$$\begin{aligned}
\psi(Tu) &= \max_{x \in [0,1]} |Tu(x)| \\
&\leq \max_{x \in [0,1]} \int_0^1 G(x,t) f(t, u(t), u'(t)) dt \\
&\quad + \max_{x \in [0,1]} g(u(\eta_1), ..., u(\eta_{m-2})).
\end{aligned}$$

Then, from $(A3)$ we get

$$\psi(Tu) \leq 6a \max_{x\in[0,1]} \int_0^1 G(x,t)dt + \frac{a}{4}.$$

Applying Avery-Peterson theorem we have the result.

Example 3.2 *Suppose that*

$$f(t,u,v) = \begin{cases} t + \frac{70}{32}u^5 + (\frac{v}{80})^2, & 0 \leq u \leq 2 \\ t + 70 + \frac{u-2}{10} + (\frac{v}{80})^2, & 2 \leq u \leq 80 \end{cases}$$

and

$$g(z) = \frac{z}{4}.$$

Taking $a = 1$ and $d = 80$ we get

- $0 \leq f(t,u,v) \leq 80 \ (t,u,v) \in [0,1] \times [0,80] \times [-80,80]$,
- $f(t,u,v) < 6 \ (t,u,v) \in [0,1] \times [0,1] \times [-80,80]$,
- $f(t,u,v) > 70 \ (t,u,v) \in [\frac{1}{4} \frac{3}{4}] \times [2,8] \times [-80,80]$,
- $0 \leq g(z) \leq \frac{1}{4} \ z \in [0,1]$,
- $0 \leq g(z) \leq 20 \ z \in [0,80]$.

Thus, all assumptions of theorem 3.1 are satisfied. Therefore, the present problem has at least three positive solutions.

References

[1] Castelani, E.V., Ma, T. F., “Numerical solutions for a three-point boundary value problem,” *Communications in Applied Analysis*, V11, N1, pp. 87-96, 2007.

[2] Gupta, C. P., “Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation,” *J. Math. Anal. Appl*, V168, N2, pp. 540-551, 08/92.

[3] Gupta, C. P., Trofimchuk, S. I., “A sharper condition for the solvability of a three-point second order boundary value problem,” *J. Math. Anal. Appl*, V205, N2, pp. 586-597, 1/97.

[4] Il’in, V. A., Moiseev, E. I., “Nonlocal boundary-value problem of the secod kind for a Sturm-Liouville operator,” *Differential Equations*, V23, pp.979-987, 1987.

[5] Il’in, V. A., Moiseev, E. I., “A nonlocal boundary value problem for the Sturm-Liouville operator in the differential and difference treatments,” *Sov. Math. Dokl*, V34, pp. 507-511, 1987.

[6] Bitsadze, A.V., Samarskii, A.A., “Some elementary generalizations of linear elliptic boundary value problems,” *Dokl. Akad. Nauk SSSR*, V185, pp. 739-740, 1969.

[7] Avery, R. I., Peterson, A. C., “Three positive fixed points of nonlinear operators in ordered Banach spaces,” *Computers and Mathematics with Applications*, V42, N3, pp. 313-322, 8/2001.

[8] Ma, R., “Existence theorems for a second order m-point boundary value problem,” *J. Math. Anal. Appl*, V211, pp. 545-555, 1997.

[9] Ma, R., “Multiplicity of positive solutions for second-order three-point boundary value problems,” *Comput. and Math. with Appl*, V40. N2, pp. 193-204, 7/2000.

[10] Wong, P. J. Y., Agarwal, R. P., “Existence and uniqueness of solutions for three-point boundary value problems for second order difference equations”, *Proceedings of Dynamic Systems and Applications*, Atlanta, United States of America, pp. 553-560, 1995.

[11] Krasnoselskii, M. A., *Positive solutions of operator equations*, Noordhoff, Groningen, 1964.

[12] Bai, Z., Wang, H., Ge, W., “Triple positive solutions for a class of two-point boundary value problems,” *Eletronic Journal of Differential Equations*, V6, pp. 1-8, 1/2004.

[13] Moshinsky, M., “Sobre los problemas de condiciones a la frontiera en una dimension de caracteristicas discontinuas,” *Bol. Soc. Mat. Mexicana*, V7, pp. 1-25, 1950.

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