Singular Analysis of the System of ODE Reductions of the Stratified Boussinesq Equations

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Abstract—**We provide here a complete integrability of the system of six coupled ODEs arising as reductions of the stratified Boussinesq equations by using WTC analysis. In this paper we have shown that among the several possible cases of dominant balance the system holds a strong Painlevé property in a single case. The system admits the singular solution in a complex domain with movable pole type singularity.**

Keywords: Stratified Boussinesq equation, WTC analysis.

1 Introduction:

In fluid dynamics, the flow of fluid in the atmosphere and in the ocean is governed by the Boussinesq equations. The authors Majda and Shefter [8] have analyzed certain ODE reductions of the Boussinesq equations in their study of onset of instabilities in the stratified fluids at large Richardson number. The Boussinesq approximation in the literature is also referred to as the Oberbeck-Boussinesq approximation for which, one may consent an interesting article of Rajagopal et al [11] provided the rigorous mathematical justification as perturbances of the Navier-Stokes equations.

In the two papers [5] and [6] S. Kovalevsky has demonstrated the complete integrability of the system of ODEs governing the motion of a spinning top moving under the influence of gravity by seeking analytic solutions whose singularities are movable poles. This was done through substituting a Frobenius series into the system ODEs of spinning top. In the general case, the problem reduces to the integration of the system of differential equations akin to the equations of a spinning top; indeed our system shares many features in common with the latter. Hence, we test our system for complete integrability using Painlevé algorithm. Paul Painlevé [9, 10] classified algebraic differential equations of the first and second order whose solutions in the complex domain are devoid of movable essential singularities or movable branch points. ODEs possessing this property are said to be Painlevé type. Painlevé test in a view of partial differential equations is generally known as WTC (Weiss-Tabor-Carnevale [16]) test which is further modified by S. Kichenassamy and Gopala Srinivasan [4].

In this paper we test the complete integrability of the system of stratified Boussinesq equations in the light of the ARS conjecture. In short, we discuss the algorithm to Painlevé test in the following section and consequently we employ it to our system in section-4 given below.

2 The Painlevé-WTC algorithm:

For the n^{th} order ODEs, the details of the algorithm are described by Ablowitz et al [1]. The analysis seeks a meromorphic series solution to a system of ODEs blowing up at a prescribed time t_0 admitting $n-1$ arbitrary constants in the solution where n is the number of degrees of freedom of a system of first order ODEs. After substituting the Ansatz

$$X_j(t) = (t-t_0)^{\rho_j} \sum_{n=0}^{\infty} X_{jn}(t-t_0)^n,\ j = 1, 2, \ldots, n, \quad (1)$$

into a system of ODEs, where $X(t)$ denotes a vector whose components are the unknown functions, one determines simultaneously the singular exponents ρ_j (at least one of which is negative) and the leading coefficient X_0 through a leading order balance and the successive coefficients X_{jn} ($n \geq 1$) through a recurrence relation of the form

$$M(n)X_{jn} = F_n[X_0, X_1, \ldots, X_{n-1}]. \quad (2)$$

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When the matrix $M(n)$ fails to be invertible at a positive integer n, equation (2) imposes a compatibility condition on the previously determined coefficients $X_0, \dots, X_{n-1}$ which fails in general. It is remarkable that for a large class of integrable systems the compatibility condition holds and the coefficient X_n remains arbitrary in the series expansion of the solution. The roots of $\det(M(n)) = 0$ are called *resonances* and we say that the system passes the Painlevé-WTC test when there are $n-1$ positive integer resonances and the compatibility conditions hold at all resonant levels. The convergence of the series (1) follows from a general result of Kichenassamy and Littman [2, 3].

We now turn to some of the finer details of the method. There are three principle steps in the algorithm:

1. determining the dominant behavior,
2. determining the resonances and
3. examining the compatibility conditions at the resonances.

It is obvious that the algorithm can stop at the first, second or third step. Consider the autonomous system of first order

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{3}$$

where the function $\mathbf{f}$ is a polynomial in $\mathbf{x}$ (the procedure is applicable in a far more general situation (2) but we do not need such a generality here). In the first step we substitute the leading monomial

$$x_i \propto a_{i0}(t-t_0)^{m_i}, \quad i = 1, 2, 3, \dots, n \tag{4}$$

into the system (3). In equation (4) t_0 is the arbitrary position of the singularity which can be assumed to be the origin by performing a time translation $\tau = (t - t_0)$. On simplifying we get the equations

$$\begin{array}{l} m_i a_{i0} \tau^{m_i - 1} = f_i(a_{10}\tau^{m_1}, \dots, a_{n0}\tau^{m_n}), \\ i = 1, 2, \dots, n. \end{array} \tag{5}$$

We solve these equations for m_i's and a_{i0}'s which will balance the leading order terms. It may happen that more than one set of the exponents $m_i\,(i = 1, 2, \dots, n)$ admits of dominant balance. If the only possible m_i's are non-integers, then the algorithm stops. The expansion coefficients $a_{i0}(i = 1, 2, \dots, n)$ satisfy m relations with $m \le n$.

In the second step we determine the resonances as follows:
Definition 1: Let $(m_1, \dots, m_n,\, a_{10}, \dots, a_{n0})$ be a given vector from the first step. We consider the simplified system that retains only the leading terms. Inserting

$$x_i(t) = a_{i0}\tau^{m_i} + \sum_{j=1}^{\infty} a_{ij}\tau^{m_i+j} \tag{6}$$

into the simplified equation yields, for $j \ge 1$, $\mathbf{a} = (a_{1j}, \dots, a_{nj})$,

$$M(j)\mathbf{a} = \mathbf{c},$$

where M is an $n \times n$ matrix whose elements depend on j and $\mathbf{c} = (c_1, \dots, c_n)$ is a vector whose components are expressions in terms of the previously determined coefficients a_{ij}. The nonnegative roots of $det\,M$ are called as the resonances.

For autonomous systems; (-1) always happens to be a resonance when the singular exponent is a negative integer. This is due to arbitrariness of t_0. Kichenassamy and Srinivasan [4] have established this result and have also provided necessary and sufficient conditions for (-1) to be a resonance in case of non autonomous equations. In his thesis, Srinivasan [13] has given a rigorous argument bringing out the connection between the resonance (-1) and arbitrariness of t_0. Unless all the resonances are integers, equation (3) does not pass the Painlevé test and the algorithm stops. While non-integer rational resonances are allowed within the weak extension of the Painlevé test (see Ramani et al [12]), irrational or complex resonances lead to infinite branching, and the system (3) cannot possess the Painlevé property. In the case of rational exponents and resonances, one may introduce an uniform variable and seek Puisseux expansions. A case in point is the Harry-Dym equation

$$u_t = u^3 u_{xxx}$$

which has leading order $2/3$ and resonances $-1, 2/3$ and $4/3$. The equation admits a Puisseux series solution

$$u(x,t) = x^{2/3} \sum_{m=0}^{\infty} u_m(t) x^{m/3}.$$

In the recent paper of Srinivasan and Sharma [14] such Puisseux expansions are employed in connection with problems of imploding shells. The expansions provide precise information regarding the profile of the solution in a neighborhood of the time of implosion.

In the third step we determine whether the Ansatz (1) admits arbitrary coefficients at the resonant levels. We substitute the Laurent expansion into equation (3) and study the resulting system of linear algebraic equations arising at the resonant levels, namely

$$M(j)\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \tag{7}$$

where M is the matrix determined by Definition 1. The property of the solution to the system of linear equations (7) is responsible for the existence of an arbitrary expansion coefficient. If there is no solution to equation (7) then we go for the weak extension of Painlevé test namely logarithmic psi series.

If the resonance has multiplicity k $(1 < k \leq n-1)$, then rank of matrix M is $n-k$ implying thereby that $n-k$ number of expansion coefficients are arbitrary.

3 The ODE Reductions of Stratified Boussinesq Equations:

In certain ranges of scales in the atmosphere and in the ocean where the flow velocities are too slow to account for compressible effects, the fluid dynamics is governed by the following Boussinesq equations that involve the interaction of gravity with density stratification about a reference state.

$$\frac{D\mathbf{v}}{Dt} = -\nabla p + \frac{g\tilde{\rho}}{\rho_b}\hat{\mathbf{e}}_3, \quad \text{Div } \mathbf{v} = 0, \quad \frac{D\tilde{\rho}}{Dt} = 0, \tag{8}$$

Following Majda [7], we look for special solutions of (8) in the following form which are linear functions of $\mathbf{x}$ with coefficients that are functions of time alone, and examine their local structure and then build a larger class of solutions that reflect the local analysis.

$$\begin{aligned} \mathbf{v}(\mathbf{x},t) &= \mathcal{D}(t)\mathbf{x} + \tfrac{1}{2}\mathbf{w}(t)\times\mathbf{x}, \\ \tilde{\rho}(\mathbf{x},t) &= \rho_b + \mathbf{b}(t)\cdot\mathbf{x}, \\ p(\mathbf{x},t) &= \tfrac{1}{2} < \widehat{P}(t)\,\mathbf{x},\ \mathbf{x} >, \end{aligned} \tag{9}$$

where $\mathbf{v}$ denotes the velocity field, $\tilde{\rho}$ the density, p the kinetic pressure that is $p = \tilde{p}/\rho_b$, g the acceleration due to gravity that points in $-\hat{\mathbf{e}}_3$ direction, $\mathcal{D}(t)$ the strain field, an arbitrary traceless symmetric matrix to be chosen, $\mathbf{w} = \text{curl } \mathbf{v}$, the vorticity vector and $\widehat{P}(t)$ the symmetric matrix given by

$$-\widehat{P}(t) = \dot{\mathcal{D}} + \mathcal{D}^2 + \Omega^2 + \frac{g}{2\rho_b}\left\{\mathbf{e}_3\mathbf{b}^T + \mathbf{b}\mathbf{e}_3^T\right\},$$

with Ω being the matrix of the linear transformation $\mathbf{x} \mapsto \frac{1}{2}\mathbf{w}\times\mathbf{x}$ relative to the standard basis and superscript T the transposition. Note that the density stratification about a constant state ρ_b is taken to be of the form $\mathbf{b}\cdot\mathbf{x}$. On substituting the *Ansatz* (9) in (8) one finds that the vectors $\mathbf{w}(t)$ and $\mathbf{b}(t)$ evolve according to the system of ODEs:

$$\begin{aligned} \dot{\mathbf{w}}(t) &= \mathcal{D}(t)\mathbf{w}(t) + \tfrac{g}{\rho_b}\mathbf{e}_3\times\mathbf{b}(\mathbf{t}), \\ \dot{\mathbf{b}}(t) &= -\mathcal{D}(t)\mathbf{b}(t) + \tfrac{1}{2}\mathbf{w}(t)\times\mathbf{b}(\mathbf{t}). \end{aligned} \tag{10}$$

Thus, in the absence of an external strain field the system (10) reduces to the following system of six coupled autonomous ODEs:

$$\dot{\mathbf{w}} = \frac{g}{\rho_b}\mathbf{e}_3\times\mathbf{b}, \quad \dot{\mathbf{b}} = \frac{1}{2}\mathbf{w}\times\mathbf{b}. \tag{11}$$

In there paper Srinivasan, Sharma and Desale [15] has shown that the system of equations (11) is completely integrable. In the following section we look the system (11) for singular analysis in the light of ARS conjecture.

4 Singular solution of the system of ODEs:

We have a system of six coupled ODEs (11). Now we assume that $\mathbf{w} = (\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3})$ and $\mathbf{b} = (\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3})$ with this assumption equations (11) can be written component wise as:

$$\begin{aligned} &\dot{w_1} = -\frac{g}{\rho_b}b_2, \quad \dot{w_2} = \frac{g}{\rho_b}b_1, \quad \dot{w_3} = 0 \\ &\dot{b_1} = \frac{1}{2}(w_2b_3 - w_3b_2), \quad \dot{b_2} = \frac{1}{2}(w_3b_1 - w_1b_3), \\ &\dot{b_3} = \frac{1}{2}(w_1b_2 - w_2b_1). \end{aligned} \tag{12}$$

Since, $\dot{w_3} = 0$ we get $w_3 = \text{constant} = w_{30}$ (arbitrary constant) say and effectively we have a system of five ODEs.

$$\begin{aligned} &\dot{w_1} = -\frac{g}{\rho_b}b_2, \quad \dot{w_2} = \frac{g}{\rho_b}b_1, \\ &\dot{b_1} = \frac{1}{2}(w_2b_3 - w_{30}b_2), \quad \dot{b_2} = \frac{1}{2}(w_{30}b_1 - w_1b_3), \\ &\dot{b_3} = \frac{1}{2}(w_1b_2 - w_2b_1). \end{aligned} \tag{13}$$

We seek solutions to the system (13) in the following form:

$$\begin{aligned} &w_1(t) = \sum_{j=0}^{\infty} w_{1j}\tau^{m_1+j}, \quad w_2(t) = \sum_{j=0}^{\infty} w_{2j}\tau^{m_2+j}, \\ &b_1(t) = \sum_{j=0}^{\infty} b_{1j}\tau^{n_1+j}, \quad b_2(t) = \sum_{j=0}^{\infty} b_{2j}\tau^{n_2+j}, \\ &b_3(t) = \sum_{j=0}^{\infty} b_{3j}\tau^{n_3+j}, \end{aligned} \tag{14}$$

where $\tau = t - t_0$ and t_0 is the arbitrary position of singularity in complex domain. Among the several possible cases for dominant balance, the system of ODEs (11) admits the singular solution in the following case of principle dominant balance.

$$\begin{aligned} \dot{w_1} &= -\frac{g}{\rho_b}b_2, \quad \dot{w_2} = \frac{g}{\rho_b}b_1, \\ \dot{b_1} &= \frac{1}{2}w_2b_3, \quad \dot{b_2} = -\frac{1}{2}w_1b_3, \\ \dot{b_3} &= \frac{1}{2}(w_1b_2 - w_2b_1). \end{aligned} \tag{15}$$

4.1 Determination of leading orders:

To determine the leading orders m_1, m_2, n_1, n_2 and n_3 appearing in the expansion (14), it is sufficient to consider the truncated expansion up to the leading order. Substituting this truncated version of expansions (14) into (15) we obtain

$$\begin{aligned} m_1w_{10}\tau^{m_1-1} &= -\frac{g}{\rho_b}b_{20}\tau^{n_2}, \\ m_2w_{20}\tau^{m_2-1} &= \frac{g}{\rho_b}b_{10}\tau^{n_1}, \\ n_1b_{10}\tau^{n_1-1} &= \frac{1}{2}w_{20}b_{30}\tau^{m_2+n_3}, \\ n_2b_{20}\tau^{n_2-1} &= -\frac{1}{2}w_{10}b_{30}\tau^{m_1+n_2}, \\ n_3b_{30}\tau^{n_3-1} &= \frac{1}{2}\Big(w_{10}b_{20}\tau^{m_1+n_2} - w_{20}b_{10}\tau^{m_2+n_1}\Big). \end{aligned} \tag{16}$$

Equating the like powers of τ on both sides of above equations we obtain the linear equations

$$\begin{aligned} &m_1 - 1 = n_2, \quad m_2 - 1 = n_1, \\ &n_1 - 1 = m_2 + n_3, \quad n_2 - 1 = m_1 + n_3, \\ &n_3 - 1 = m_1 + n_2, \quad n_3 - 1 = m_2 + n_1. \end{aligned} \tag{17}$$

From equations (17) the exponents are uniquely determined as

$$m_1 = m_2 = -1, \quad n_1 = n_2 = n_3 = -2. \tag{18}$$

Substituting the values of exponents from equations (18) into equations (16) and equating the likes powers of τ, we obtain the relations for coefficients in leading order, which are as follows

$$\begin{aligned} &w_{10} = \frac{g}{\rho_b}b_{20}, \quad w_{20} = -\frac{g}{\rho_b}b_{10}, \quad b_{10} = -\frac{1}{4}w_{20}b_{30}, \\ &b_{20} = \frac{1}{4}w_{10}b_{30}, \quad b_{30} = -\frac{1}{4}\Big(w_{10}b_{20} - w_{20}b_{10}\Big). \end{aligned} \tag{19}$$

By solving equations (19), we find two possible branches of leading orders

$$\begin{aligned} &w_{10} = \pm\sqrt{-16 - w_{20}^2}, \\ &w_{20} = \text{arbitrary constant} \\ &b_{10} = -\frac{\rho_b}{g}w_{20}, \quad b_{20} = \pm\frac{\rho_b}{g}\sqrt{-16 - w_{20}^2}, \\ &b_{30} = \frac{4\rho_b}{g}. \end{aligned} \tag{20}$$

Here we have two possible branches of leading orders hence we will get two different singular solutions in complex domain. Our next step is to determine the resonances.

4.2 Determination of resonances:

To determine the resonances first we rewrite the equations (14) by using the exponents given in equations (18)

$$\begin{aligned} w_1(t) &= w_{10}\tau^{-1} + \sum_{j=1}^{\infty} w_{1j}\tau^{j-1}, \\ w_2(t) &= w_{20}\tau^{-1} + \sum_{j=1}^{\infty} w_{2j}\tau^{j-1}, \\ b_1(t) &= b_{10}\tau^{-2} + \sum_{j=1}^{\infty} b_{1j}\tau^{j-2}, \\ b_2(t) &= b_{20}\tau^{-2} + \sum_{j=1}^{\infty} b_{2j}\tau^{j-2}, \\ b_3(t) &= b_{30}\tau^{-2} + \sum_{j=1}^{\infty} b_{3j}\tau^{j-2}. \end{aligned} \tag{21}$$

Substituting the above equations into the system of ODEs (13) we get after some algebraic calculations the following recursion relations for the coefficients which are valid for $j \geq 2$.

$$M(j)\begin{pmatrix} w_{1j} \\ w_{2j} \\ b_{1j} \\ b_{2j} \\ b_{3j} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}A_j \\ \frac{1}{2}B_j \\ \frac{1}{2}C_j \end{pmatrix}, \tag{22}$$

where

$$\begin{aligned} A_j &= -w_{30}b_{2(j-1)} + \sum_{k=1}^{j-1} w_{2k}b_{3(j-k)}, \\ B_j &= w_{30}b_{1(j-1)} - \sum_{k=1}^{j-1} w_{1k}b_{3(j-k)}, \\ C_j &= \sum_{k=1}^{j-1} w_{1k}b_{2(j-k)} - \sum_{k=1}^{j-1} w_{2k}b_{1(j-k)}, \end{aligned} \tag{23}$$

and matrix $M(j)$ is

$$M(j) = \begin{pmatrix} j-1 & 0 & 0 & \frac{g}{\rho_b} & 0 \\ 0 & j-1 & -\frac{g}{\rho_b} & 0 & 0 \\ 0 & -\frac{b_{30}}{2} & j-2 & 0 & -\frac{w_{20}}{2} \\ \frac{b_{30}}{2} & 0 & 0 & j-2 & \frac{w_{10}}{2} \\ -\frac{b_{20}}{2} & \frac{b_{10}}{2} & \frac{w_{20}}{2} & -\frac{w_{10}}{2} & j-2 \end{pmatrix}. \quad (24)$$

The above recursion relations (22) determine the unknown expansion coefficients uniquely unless the determinant of matrix $M(j)$ is zero. Those values of j at which the determinant $det\,(M(j))$ vanishes are called *resonances*. Here we see that for both possible branches of leading orders given in equations (20) the determinant of matrix $M(j)$ is

$$det\,(M(j)) = (j+1)j(j-2)(j-3)(j-4). \quad (25)$$

Hence, the resonances are

$$j = -1,\ 0,\ 2,\ 3,\ 4. \quad (26)$$

Here $j = -1$ is a usual resonance and $j = 0$ is corresponding to the arbitrariness of w_{20} in leading order behavior.

For the next step in the algorithm we check the compatibility conditions at nonnegative resonances given in equation (26).

4.3 Compatibility conditions:

At the positive resonances (26), the recursion relations (22) remain valid if and only if the RHS of (22) lies in the range of M(j). This means that the vector appearing on the right hand side of (22) must be annihilated by every left null vector of $M(j)$ (when j is a resonance) resulting in a set of compatibility conditions to be satisfied by the previously determined coefficients. When these conditions hold, the j-th coefficient vector enters as an arbitrary coefficient vector in the expansion (21). On the other hand if the compatibility condition fails at a resonant level, logarithms need to be introduced in the expansion (see [2, 3] for details). We investigate this in each of the two possible branches given by the leading order analysis and we determine the expansion coefficients in each case up to the last resonant level.

• **Case 1:** Consider the leading order coefficients

$$\begin{aligned} w_{10} &= \sqrt{-16-k_1^2}, \\ w_{20} &= k_1 \text{ (arbitrary constant)}, \\ b_{10} &= -\frac{\rho_b}{g}k_1, \\ b_{20} &= \frac{\rho_b}{g}\sqrt{-16-k_1^2}, \\ b_{30} &= \frac{4\rho_b}{g}. \end{aligned} \quad (27)$$

• **Compatibility condition at $j = 1$.** Since the recursion relations (22) come into force when $j \geq 2$, we directly substitute equations (27), (21) into the equations (13) and equate the like powers of τ on both sides of the resulting expansion thereby obtaining the following system of linear equations for w_{11}, w_{21}, b_{11}, b_{21} and b_{31}.

$$M(1)\begin{pmatrix} w_{11} \\ w_{21} \\ b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{w_{30}\rho_b\sqrt{-16-k_1^2}}{2g} \\ -\frac{k_1\rho_b w_{30}}{2g} \\ 0 \end{pmatrix}, \quad (28)$$

where $M(1)$ is a matrix obtained by substituting (27) and $j = 1$ into the equation (24). The system of linear equations (28) has a unique solution, hence w_{11}, w_{21}, b_{11}, b_{21} and b_{31} are uniquely determined to be

$$\begin{aligned} &w_{11} = -\tfrac{w_{30}k_1}{4}, \quad w_{21} = \tfrac{w_{30}\sqrt{-16-k_1^2}}{4}, \\ &b_{11} = b_{21} = b_{31} = 0 \end{aligned} \quad (29)$$

• **Compatibility condition at the resonance $j = 2$.** Now substituting (29) and (27) into the recursion relations (22) for $j = 2$, we get the following set of linear equations

$$M(2)\begin{pmatrix} w_{12} \\ w_{22} \\ b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (30)$$

In the above homogeneous system of linear equations the rank of coefficient matrix $M(2)$ is 4 and, hence one of the variable is independent. Let b_{32} be independent and assign the arbitrary value $b_{32} = k_2$ so that solutions of system of equations (30) are given in terms of b_{32}, which are as follows.

$$\begin{aligned} &w_{12} = -\frac{gk_2\sqrt{-16-k_1^2}}{4\rho_b},\ w_{22} = -\frac{gk_1k_2}{4\rho_b}, \\ &b_{12} = -\frac{k_1k_2}{4},\ b_{22} = \frac{k_2\sqrt{-16-k_1^2}}{4}, \\ &b_{32} = k_2. \end{aligned} \quad (31)$$

• **Compatibility condition at the resonance** $j = 3$. To determine the arbitrary constant which will be involved with the resonance $j = 3$, we use equations (31), (29) and (27) into the recursion relations obtained from equations (22) corresponding to $j = 3$; we get the following matrix form of linear equations

$$M(3)\begin{pmatrix} w_{13} \\ w_{23} \\ b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (32)$$

The above homogeneous system of linear equations (32) has infinitely many solutions with one independent variable. By using row reduced echelon form we found the variable b_{23} to be independent. Now assign the arbitrary value to b_{23} that is to say $b_{23} = k_3$, the solutions of above system of linear equations (32) are given below.

$$\begin{aligned} &w_{13} = -\frac{gk_3}{2\rho_b},\ w_{23} = \frac{gk_3\sqrt{-16-k_1^2}}{2k_1\rho_b}, \\ &b_{13} = \frac{k_3\sqrt{-16-k_1^2}}{k_1},\ b_{23} = k_3,\ b_{33} = 0\,. \end{aligned} \qquad (33)$$

• **Compatibility condition at the resonance** $j = 4$. To determine the arbitrary constant which is involved with the resonance $j = 4$, we substitute the equations (33), (31), (29) and (27) into the recursion relations given by equations (22) for $j = 4$. We obtain the system of non homogeneous linear equations, which is given below.

$$M(4)\begin{pmatrix} w_{14} \\ w_{24} \\ b_{14} \\ b_{24} \\ b_{34} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}A_4 \\ \frac{1}{2}B_4 \\ \frac{1}{2}C_4 \end{pmatrix}, \qquad (34)$$

where

$$\begin{aligned} A_4 &= -k_3w_{30} - \frac{gk_1k_2^2}{4\rho_b}, \\ B_4 &= \frac{gk_2^2\sqrt{-16-k_1^2}}{4\rho_b} \\ &+ \frac{k_3w_{30}\sqrt{-16-k_1^2}}{k_1}, \\ C_4 &= -\frac{gk_1^2k_2^2}{16\rho_b} + \frac{gk_2^2(16+k_1^2)}{16\rho_b} \\ &- \frac{k_1k_3w_{30}}{4} + \frac{k_3w_{30}(16+k_1^2)}{4k_1}. \end{aligned} \qquad (35)$$

We see that system of equations (34) is consistent and has infinitely many solutions with one independent variable. By row reduced echelon form we see that b_{24} is a free variable. Let $b_{24} = k_4$ be an arbitrary constant so that solutions of equations (34) are given by:

$$\begin{aligned} &w_{14} = -\frac{gk_4}{3\rho_b},\quad w_{24} = -\frac{gk_1k_4}{3\rho_b\sqrt{-16-k_1^2}}, \\ &b_{14} = -\frac{k_1k_4}{\sqrt{-16-k_1^2}},\ b_{24} = k_4, \\ &b_{34} = -\frac{8k_4}{3\sqrt{-16-k_1^2}} + \frac{gk_1k_2^2 + 4\rho_bk_3w_{30}}{4\rho_bk_1}. \end{aligned} \qquad (36)$$

• **Compatibility condition for** $j \geq 5$. We see from equation (25) that the determinant of matrix $M(j)$ is non zero for $j \geq 5$; hence the equations (22) have a unique solution. To determine the expansion coefficients w_{ij} and b_{ij} where $i = 1, 2, 3$ and $j \geq 5$, we substitute (27), (29), (31), (34) and (36) into the recursion relations (22) and determine all the expansion coefficients uniquely.

In the present case of leading orders given in equations (27), we substitute (27), (29), (31), (33) and (36) into the equations (21), we get the general solution to the system of equation (11) in terms of Laurent series given below

$$\begin{aligned} w_1(t) &= \sqrt{-16-k_1^2}\,\tau^{-1} - \frac{w_{30}k_1}{4} \\ &- \frac{gk_2\sqrt{-16-k_1^2}}{4\rho_b}\tau - \frac{gk_3}{2\rho_b}\tau^2 \\ &- \frac{gk_4}{3\rho_b}\tau^3 + \sum_{j=5}^{\infty} w_{1j}\tau^{j-1}, \\ w_2(t) &= k_1\tau^{-1} + \frac{w_{30}\sqrt{-16-k_1^2}}{4} \\ &- \frac{gk_1k_2}{4\rho_b}\tau + \frac{gk_3\sqrt{-16-k_1^2}}{2k_1\rho_b}\tau^2 \\ &- \frac{gk_1k_4}{3\rho_b\sqrt{-16-k_1^2}}\tau^3 + \sum_{j=5}^{\infty} w_{2j}\tau^{j-1}, \\ w_3(t) &= w_{30}, \end{aligned} \qquad (37a)$$

$$
\begin{aligned}
b_1(t) &= -\frac{k_1\rho_b}{g}\tau^{-2} - \frac{k_1k_2}{4} \\
&+ \frac{k_3\sqrt{-16-k_1^2}}{k_1}\tau - \frac{k_1k_4}{\sqrt{-16-k_1^2}}\tau^2 \\
&+ \sum_{j=5}^{\infty} b_{1j}\tau^{j-2}, \\
b_2(t) &= \frac{\rho_b\sqrt{-16-k_1^2}}{g}\tau^{-2} + \frac{k_2\sqrt{-16-k_1^2}}{4} \\
&+ k_3\tau + k_4\tau^2 + \sum_{j=5}^{\infty} b_{2j}\tau^{j-2}, \\
b_3(t) &= \frac{4\rho_b}{g}\tau^{-2} + k_2 \\
&+ \Big(\frac{-8k_4}{3\sqrt{-16-k_1^2}} + \frac{gk_1k_2^2 + 4w_{30}k_3\rho_b}{4k_1\rho_b}\Big)\tau^2 \\
&+ \sum_{j=5}^{\infty} b_{3j}\tau^{j-2},
\end{aligned}
\tag{37b}
$$

where w_{ij}, b_{ij} (for $i = 1, 2, 3$ and $j \geq 5$) are uniquely determined by the recursion relations (22) and (23).

The Laurent series in (37a, 37b) contains the six arbitrary constant k_1, k_2, k_3, k_4, w_{30}, arbitrary position of singularity t_0 and satisfying the system of ODEs (11). Thus, in the present case of leading orders the system of reduced ODEs of Stratified Boussinesq equations passes the Painlevé test and has a movable pole type singularity.

- **Case 2:** Consider the leading order coefficients

$$
\begin{aligned}
&w_{10} = -\sqrt{-16-k_1^2},\ w_{20} = k_1 \text{ (arbitrary constant)} \\
&b_{10} = -\frac{\rho_b}{g}k_1,\ b_{20} = -\frac{\rho_b}{g}\sqrt{-16-k_1^2},\ b_{30} = \frac{4\rho_b}{g}.
\end{aligned}
\tag{38}
$$

Using the same approach as in the previous case we have determined the expansion coefficients of (21) for $j = 1$, $j = 2$, $j = 3$ and $j = 4$. Plugging these coefficient into the recursion relations (22) and (23), we can uniquely determine the expansion coefficients w_{ij} and b_{ij} for $j \geq 5$. Hence, in this case of leading order coefficients the system of ODEs (11) also passes the Painlevé test and general solution is given below.

$$
\begin{aligned}
w_1(t) &= -\sqrt{-16-k_1^2}\,\tau^{-1} - \frac{w_{30}k_1}{4} \\
&+ \frac{gk_2\sqrt{-16-k_1^2}}{4\rho_b}\tau - \frac{gk_3}{2\rho_b}\tau^2 \\
&- \frac{gk_4}{3\rho_b}\tau^3 + \sum_{j=5}^{\infty} w_{1j}\tau^{j-1}, \\
w_2(t) &= k_1\tau^{-1} - \frac{w_{30}\sqrt{-16-k_1^2}}{4} \\
&- \frac{gk_1k_2}{4\rho_b}\tau - \frac{gk_3\sqrt{-16-k_1^2}}{2k_1\rho_b}\tau^2 \\
&+ \frac{gk_1k_4}{3\rho_b\sqrt{-16-k_1^2}}\tau^3 + \sum_{j=5}^{\infty} w_{2j}\tau^{j-1}, \\
w_3(t) &= w_{30}, \\
b_1(t) &= -\frac{k_1\rho_b}{g}\tau^{-2} - \frac{k_1k_2}{4} - \frac{k_3\sqrt{-16-k_1^2}}{k_1}\tau \\
&+ \frac{k_1k_4}{\sqrt{-16-k_1^2}}\tau^2 + \sum_{j=5}^{\infty} b_{1j}\tau^{j-2}, \\
b_2(t) &= -\frac{\rho_b\sqrt{-16-k_1^2}}{g}\tau^{-2} - \frac{k_2\sqrt{-16-k_1^2}}{4} \\
&+ k_3\tau + k_4\tau^2 + \sum_{j=5}^{\infty} b_{2j}\tau^{j-2}, \\
b_3(t) &= \frac{4\rho_b}{g}\tau^{-2} + k_2 \\
&+ \Big(\frac{8k_4}{3\sqrt{-16-k_1^2}} + \frac{gk_1k_2^2 + 4w_{30}k_3\rho_b}{4k_1\rho_b}\Big)\tau^2 \\
&+ \sum_{j=5}^{\infty} b_{3j}\tau^{j-2}.
\end{aligned}
\tag{39}
$$

5 Conclusion:

Now we conclude that among the several possible cases of principle dominant balance the system of ODE reduction of stratified Boussinesq equations (11) is completely integrable (in the light of ARS conjecture) in the cases of principle dominant balance (15). In both possible branches of leading orders the system of ODEs (11) passes the strong Painlevé test and general solutions are given by equations (37a, 37b), (39). From equations (37a, 37b), (39) we see that solutions of the system of ODEs (11) are singular in a complex domain and it has a movable pole type singularity at $t = t_0$.

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