Associated Rational Functions based on a Three-term Recurrence Relation for Orthogonal Rational Functions*

Karl Deckers† and Adhemar Bultheel‡

Abstract— **Consider the sequence of poles $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\}$, and suppose the rational functions φ_n with poles in $\mathcal{A}$ form an orthonormal system with respect to an Hermitian positive-definite inner product. Further, assume the φ_n satisfy a three-term recurrence relation. Let the rational function $\varphi^{(1)}_{n\setminus 1}$ with poles in $\{\alpha_2, \alpha_3, \ldots\}$ represent the associated rational function of φ_n of order 1; i.e. the $\varphi^{(1)}_{n\setminus 1}$ satisfy the same three-term recurrence relation as the φ_n. In this paper we then give a relation between φ_n and $\varphi^{(1)}_{n\setminus 1}$ in terms of the so-called rational functions of the second kind. Next, under certain conditions on the poles in $\mathcal{A}$, we prove that the $\varphi^{(1)}_{n\setminus 1}$ form an orthonormal system of rational functions with respect to an Hermitian positive-definite inner product. Finally, we give a relation between associated rational functions of different order, independent of whether they form an orthonormal system.**

Keywords: ***Orthogonal rational functions, associated rational functions, rational functions of the second kind, three-term recurrence relation, Favard theorem.***

1 Introduction

Let ϕ_n denote the polynomial of degree n that is orthonormal with respect to a positive measure μ on a subset S of the real line. Further, assume the measure μ is normalized (i.e. $\int_S d\mu = 1$) and suppose the orthonormal polynomials (OPs) ϕ_n satisfy a three-term recurrence relation of the form

$$\begin{aligned} &\phi_{-1}(x) \equiv 0, \quad \phi_0(x) \equiv 1, \\ &\alpha_n\phi_n(x) = (x-\beta_n)\phi_{n-1}(x) - \alpha_{n-1}\phi_{n-2}(x), \quad n \geq 1, \end{aligned}$$

where the recurrence coefficients α_n and β_n are real, and $\alpha_n \neq 0$ for every n.

Let the polynomial $\phi^{(k)}_{n-k}$ of degree $n-k$ denote the associated polynomial (AP) of order $k \geq 0$, with $n \geq k$. By definition, these APs are the polynomials generated by the three-term recurrence relation given by

$$\begin{aligned} &\phi^{(k)}_{-1}(x) \equiv 0, \quad \phi^{(k)}_0(x) \equiv 1, \\ &\alpha_n\phi^{(k)}_{n-k}(x) = (x-\beta_n)\phi^{(k)}_{(n-1)-k}(x) - \alpha_{n-1}\phi^{(k)}_{(n-2)-k}(x), \\ &\hfill n \geq k+1. \end{aligned}$$

Note that this way the APs of order 0 and the OPs are in fact the same.

The following relation exists between APs of different order

$$\begin{aligned} \alpha_{m+1}\left[\phi^{(k)}_{m-k}(x)\phi^{(j)}_{n-j}(x) - \phi^{(j)}_{m-j}(x)\phi^{(k)}_{n-k}(x)\right] \\ = \alpha_j\phi^{(m+1)}_{n-(m+1)}(x)\phi^{(k)}_{(j-1)-k}(x), \end{aligned} \quad (1)$$

where $n+1 \geq m+1 \geq j \geq k \geq 0$ (see e.g. [10, Eqns (2.5)–(2.6)] for the special case in which $m = j = k+1$, respectively $m = n-1$).

From the Favard theorem it follows that the APs of order k form an orthonormal system with respect to a positive normalized measure $\mu^{(k)}$ on S. Therefore, another relation exists between the APs of order j and k in terms of polynomials of the second kind:

$$\phi^{(k)}_{n-k}(x) = \alpha_k\int_S \frac{\phi^{(j)}_{n-j}(t) - \phi^{(j)}_{n-j}(x)}{t-x}\phi^{(j)}_{(k-1)-j}(t)d\mu^{(j)}(t),$$
$$0 \leq j \leq k-1 \leq n, \quad (2)$$

and hence,

$$\frac{\phi^{(j)}_{n-j}(t) - \phi^{(j)}_{n-j}(x)}{t-x} = \sum_{k=j+1}^{n}\frac{1}{\alpha_k}\phi^{(k)}_{n-k}(x)\phi^{(j)}_{(k-1)-j}(t) \quad (3)$$

(see e.g. [10, Eqns (2.9) and (2.13)] for the special case in which $j = 0$). For $t = x$, relation (3) can be rewritten as

$$\frac{d}{dx}\left[\phi^{(j)}_{n-j}(x)\right] = \sum_{k=j+1}^{n}\frac{1}{\alpha_k}\phi^{(k)}_{n-k}(x)\phi^{(j)}_{(k-1)-j}(x). \quad (4)$$

Orthonormal rational functions (ORFs) on a subset S of the real line (see e.g. [2, 8, 9] and [1, Chapt. 11]) are a generalization of OPs on S in such a way that they are of increasing degree with a given sequence of complex poles, and the OPs result if all the poles are at infinity. Let φ_n denote the rational

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function with n poles outside S that is orthonormal with respect to a positive normalized measure μ on S. Under certain conditions on the poles, these ORFs satisfy a three-term recurrence relation too. Consequently, associated rational functions (ARFs) can be defined based on this three-term recurrence relation. Furthermore, in [1, Chapt. 11.2], the rational function of the second kind $\varphi_n^{[1]}$ of φ_n is defined similarly as in (2); i.e.

$$\varphi_n^{[1]}(x) = \int_S \frac{\varphi_n(t) - \varphi_n(x)}{t - x} d\mu(t), \quad n > 0. \tag{5}$$

The aim of this paper is to generalize the relations for APs, given by (1)–(4), to the case of ARFs. The outline of the paper is as follows. After giving the necessary theoretical background in Section 2, in Section 3 we deal with the generalization of relation (1). Next, we give a relation between ARFs of order $k-1$ and k in terms of rational functions of the second kind in Section 4. We conclude the article with the generalization of relation (3) and (4) in Section 5.

This paper is an updated and extended version of the conference paper [5]. First, we have proved a more general relation between ARFs of different order in Theorem 3.3. Secondly, in Section 5 we have given a generalization of relation (3) and (4) to the case of ARFs. Whereas in [5], the generalization of relation (2) has only been proved for $k = j + 1$.

2 Preliminaries

The field of complex numbers will be denoted by $\mathbb{C}$ and the Riemann sphere by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For the real line we use the symbol $\mathbb{R}$, while the extended real line will be denoted by $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Further, we represent the positive real line by $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. If the value $a \in X$ is omitted in the set X, this will be represented by X_a; e.g.

$$\mathbb{C}_0 = \mathbb{C} \setminus \{0\}.$$

Let $c = a + \mathbf{i}b$, where $a, b \in \mathbb{R}$, then we represent the real part of $c \in \mathbb{C}$ by $\Re\{c\} = a$ and the imaginary part by $\Im\{c\} = b$.

Given a sequence $\mathcal{A}_n = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \overline{\mathbb{C}}_0$, we define the factors

$$Z_l(x) = \frac{x}{1 - x/\alpha_l}, \qquad l = 1, 2, \ldots, n,$$

and products

$$b_0(x) \equiv 1, \qquad b_l(x) = Z_l(x) b_{l-1}(x), \qquad l = 1, 2, \ldots, n,$$

or equivalently,

$$b_l(x) = \frac{x^l}{\pi_l(x)}, \quad \pi_l(x) = \prod_{i=1}^{l} (1 - x/\alpha_i), \quad \pi_0(x) \equiv 1.$$

The space of rational functions with poles in $\mathcal{A}_n$ is then given by

$$\mathcal{L}_n = \operatorname{span}\{b_0(x), b_1(x), \ldots, b_n(x)\}.$$

We will also need the reduced sequence of poles $\mathcal{A}_{n\setminus k} = \{\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n\}$, where $0 \leq k \leq n$, and the reduced space of rational functions with poles in $\mathcal{A}_{n\setminus k}$ given by

$$\mathcal{L}_{n\setminus k} = \operatorname{span}\{b_{k\setminus k}(x), b_{(k+1)\setminus k}(x), \ldots, b_{n\setminus k}(x)\},$$

where

$$b_{l\setminus k}(x) = \frac{b_l(x)}{b_k(x)} = \frac{x^{l-k}}{\pi_{l\setminus k}(x)},$$

for $l \geq k$ and

$$\pi_{l\setminus k}(x) = \prod_{i=k+1}^{l} (1 - x/\alpha_i), \quad \pi_{l\setminus l}(x) \equiv 1.$$

In the special case in which $k = 0$ or $k = n$, we have that $\mathcal{A}_{n\setminus 0} = \mathcal{A}_n$ and $\mathcal{L}_{n\setminus 0} = \mathcal{L}_n$, respectively $\mathcal{A}_{n\setminus n} = \emptyset$ and $\mathcal{L}_{n\setminus n} = \mathcal{L}_0 = \overline{\mathbb{C}}$. We will assume that the poles in $\mathcal{A}_n$ are arbitrary complex or infinite; hence, they do not have to appear in pairs of complex conjugates.

We define the substar conjugate of a function $f(x) \in \mathcal{L}_\infty$ by

$$f_*(x) = \overline{f(\overline{x})}.$$

Consider an inner product that is defined by the linear functional M:

$$\langle f, g \rangle = M\{f g_*\}, \quad f, g \in \mathcal{L}_\infty.$$

We say that M is an Hermitian positive-definite linear functional (HPDLF) if for every $f, g \in \mathcal{L}_\infty$ it holds that

$$f \neq 0 \Leftrightarrow M\{f f_*\} > 0 \quad \text{and} \quad M\{f g_*\} = \overline{M\{f_* g\}}.$$

Further, assume M is normalized ($M\{1\} = 1$) and suppose there exists a sequence of rational functions $\{\varphi_n\}_{n=1}^{\infty}$, with $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, so that the φ_n form an orthonormal system with respect to the HPDLF M.

Let $\alpha_0 \in \overline{\mathbb{C}}_0$ be arbitrary but fixed in advance. Then the orthonormal rational functions (ORFs) $\varphi_n = \frac{p_n}{\pi_n}$ are said to be regular for $n \geq 1$ if $p_n(\alpha_{n-1}) \neq 0$ and $p_n(\overline{\alpha}_{n-1}) \neq 0$. A zero of p_n at ∞ means that the degree of p_n is less than n. We now have the following recurrence relation for ORFs. For the proof, we refer to [8, Sec. 2] and [3, Sec. 3].

Theorem 2.1. *Let $E_0 \in \mathbb{C}_0$, $\alpha_{-1} \in \overline{\mathbb{R}}_0$ and $\alpha_0 \in \overline{\mathbb{C}}_0$ be arbitrary but fixed in advance. Then the ORFs φ_l, $l = n-2, n-1, n$, with $n \geq 1$, are regular iff there exists a three-term recurrence relation of the form*

$$\varphi_n(x) = Z_n(x) \left\{ E_n \left[1 + \frac{F_n}{Z_{n-1}(x)} \right] \varphi_{n-1}(x) - \frac{C_n}{Z_{n-2*}(x)} \varphi_{n-2}(x) \right\}, \quad E_n, C_n \in \mathbb{C}_0, \quad F_n \in \mathbb{C}, \tag{6}$$

with

$$|E_n|^2 - 4 \frac{\Im\{\alpha_n\}}{|\alpha_n|^2} \cdot \frac{\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2} =: \Delta_n \in \mathbb{R}_0^+, \tag{7}$$

$$C_n = \frac{E_n\left[1 + F_n/Z_{n-1}(\overline{\alpha}_{n-1})\right]}{\overline{E}_{n-1}}, \tag{8}$$

and

$$\Im\{F_n\} = \frac{\Im\{\alpha_n\}}{|\alpha_n|^2} \cdot \frac{1}{|E_n|^2} - \frac{\Im\{\alpha_{n-2}\}}{|\alpha_{n-2}|^2} \cdot \frac{1}{|E_{n-1}|^2} \tag{9}$$

whenever $\alpha_{n-1} \in \overline{\mathbb{R}}_0$*, respectively*

$$[\Re\{F_n\}]^2 + [\Im\{F_n\} - \mathbf{i}Z_{n-1}(\overline{\alpha}_{n-1})]^2 = [\mathbf{i}Z_{n-1}(\overline{\alpha}_{n-1})]^2 \frac{|E_{n-1}|^2}{|E_n|^2} \cdot \frac{\Delta_n}{\Delta_{n-1}} \tag{10}$$

whenever $\alpha_{n-1} \notin \overline{\mathbb{R}}$*. The initial conditions are* $\varphi_{-1}(x) \equiv 0$ *and* $\varphi_0(x) \equiv 1$*.*

In the remainder we will assume that the system of ORFs $\{\varphi_n\}_{n=1}^{\infty}$ is regular.

Let $\varphi_{n\backslash k}^{(k)} = \frac{p_{n-k}^{(k)}(x)}{\pi_{n\backslash k}(x)} \in \mathcal{L}_{n\backslash k}$ denote the associated rational function (ARF) of φ_n of order k; i.e. $\varphi_{n\backslash k}^{(k)}$, $n = k+1, k+2, \ldots$, is generated by the three-term recurrence relation

$$\begin{aligned} &\varphi_{(k-1)\backslash k}^{(k)}(x) \equiv 0, \quad \varphi_{k\backslash k}^{(k)}(x) \equiv 1, \\ &\varphi_{n\backslash k}^{(k)}(x) = Z_n(x)\left\{E_n\left[1 + \tfrac{F_n}{Z_{n-1}(x)}\right]\varphi_{(n-1)\backslash k}^{(k)}(x) \right. \\ &\qquad \left. - \tfrac{C_n}{Z_{n-2*}(x)}\varphi_{(n-2)\backslash k}^{(k)}(x)\right\}, \ n \geq k+1. \end{aligned}$$

Note that in the special case in which $k = 0$, we have that $\varphi_{n\backslash 0}^{(0)} = \varphi_n$.

As a consequence of the Favard theorem for rational functions with complex poles (see [4, Thm. 4.1]) we then have the following theorem.

Theorem 2.2. *Let* $\{\varphi_{n\backslash k}^{(k)}\}_{n=k+1}^{\infty}$ *be a sequence of rational functions generated by the three-term recurrence relation* (6)–(10) *for* $n > k \geq 0$*, with initial conditions* $\varphi_{(k-1)\backslash k}^{(k)}(x) \equiv 0$ *and* $\varphi_{k\backslash k}^{(k)}(x) \equiv 1$*. Furthermore, assume that*

1. $\alpha_{k-1} \in \overline{\mathbb{R}}_0$,
2. $\varphi_{n\backslash k}^{(k)} \in \mathcal{L}_{n\backslash k} \setminus \mathcal{L}_{(n-1)\backslash k}$, $n = k+1, k+2, \ldots$.

Then there exists a normalized HPDLF $M^{(k)}$ *so that*

$$\langle f, g \rangle = M^{(k)}\{f g_*\}$$

defines an Hermitian positive-definite inner product for which the rational functions $\varphi_{n\backslash k}^{(k)}$ *form an orthonormal system.*

3 ARFs of different order

The aim of this section is to generalize relation (1) to the case of ARFs. First we need the following two lemmas.

Lemma 3.1. *The ARFs* $\varphi_{n\backslash s}^{(s)}$*, with* $s = k, k+1, k+2$ *and* $n \geq k+1$*, satisfy the relation given by*

$$\varphi_{n\backslash k}^{(k)}(x) = Z_{k+1}(x)E_{k+1}\left[1 + \frac{F_{k+1}}{Z_k(x)}\right]\varphi_{n\backslash(k+1)}^{(k+1)}(x) - C_{k+2}\frac{Z_{k+2}(x)}{Z_{k*}(x)}\varphi_{n\backslash(k+2)}^{(k+2)}(x). \tag{11}$$

Proof. First, consider the case in which $n = k+1$. From the three-term recurrence relation we deduce that

$$\varphi_{(k+1)\backslash k}^{(k)}(x) = Z_{k+1}(x)E_{k+1}\left[1 + \frac{F_{k+1}}{Z_k(x)}\right].$$

We also have that $\varphi_{(k+1)\backslash(k+1)}^{(k+1)}(x) \equiv 1$, while $\varphi_{(k+1)\backslash(k+2)}^{(k+2)}(x) \equiv 0$. Hence, the statement clearly holds for $n = k+1$.

Next, consider the case in which $n = k+2$. From the three-term recurrence relation we now deduce that

$$\varphi_{(k+2)\backslash k}^{(k)}(x) = Z_{k+2}(x)E_{k+2}\left[1 + \frac{F_{k+2}}{Z_{k+1}(x)}\right]\varphi_{(k+1)\backslash k}^{(k)}(x) - C_{k+2}\frac{Z_{k+2}(x)}{Z_{k*}(x)}.$$

Also now we have that $\varphi_{(k+2)\backslash(k+2)}^{(k+2)}(x) \equiv 1$. Moreover,

$$\begin{aligned} &Z_{k+2}(x)E_{k+2}\left[1 + \frac{F_{k+2}}{Z_{k+1}(x)}\right]\varphi_{(k+1)\backslash k}^{(k)}(x) \\ &= Z_{k+2}(x)E_{k+2}\left[1 + \frac{F_{k+2}}{Z_{k+1}(x)}\right] \times \\ &\quad Z_{k+1}(x)E_{k+1}\left[1 + \frac{F_{k+1}}{Z_k(x)}\right] \\ &= Z_{k+1}(x)E_{k+1}\left[1 + \frac{F_{k+1}}{Z_k(x)}\right]\varphi_{(k+2)\backslash(k+1)}^{(k+1)}(x). \end{aligned}$$

Consequently, the statement clearly holds for $n = k+2$ as well.

Finally, assume that the statement holds for $n-2$ and $n-1$. By induction, the statement is then easily verified for $n \geq k+3$ by applying the three-term recurrence relation to the left hand side of (11) for $\varphi_{n\backslash k}^{(k)}$, as well as to the right hand side of (11) for $\varphi_{n\backslash(k+1)}^{(k+1)}$ and $\varphi_{n\backslash(k+2)}^{(k+2)}$. □

Lemma 3.2. *The ARFs* $\varphi_{n\backslash s}^{(s)}$*, with* $s = k, j, j+1$ *and* $k \leq j \leq n$*, are related by*

$$\varphi_{n\backslash k}^{(k)}(x) = \varphi_{n\backslash j}^{(j)}(x)\varphi_{j\backslash k}^{(k)}(x) - C_{j+1}\frac{Z_{j+1}(x)}{Z_{j-1*}(x)}\varphi_{n\backslash(j+1)}^{(j+1)}(x)\varphi_{(j-1)\backslash k}^{(k)}(x). \tag{12}$$

Proof. For every $l \geq 0$ we have that

$$\varphi^{(l)}_{(l-1)\backslash l}(x) \equiv 0, \quad \varphi^{(l)}_{l\backslash l}(x) \equiv 1$$
$$\text{and} \quad \varphi^{(l)}_{(l+1)\backslash l}(x) = Z_{l+1}(x)E_{l+1}\left[1 + \frac{F_{l+1}}{Z_l(x)}\right].$$

Thus, the relation given by (12) clearly holds for $j = n$ or $j = k$. While for $j = n-1$ or $j = k+1$, (12) is nothing more than the three-term recurrence relation, respectively the relation given by (11).

So, suppose that the statement holds for j. From the three-term recurrence relation it follows that

$$\varphi^{(k)}_{(j+1)\backslash k}(x) = \varphi^{(j)}_{(j+1)\backslash j}(x)\varphi^{(k)}_{j\backslash k}(x) - C_{j+1}\frac{Z_{j+1}(x)}{Z_{j-1*}(x)}\varphi^{(k)}_{(j-1)\backslash k}(x),$$

while for $j \leq n-1$ it follows from Lemma 3.1 that

$$-C_{j+2}\frac{Z_{j+2}(x)}{Z_{j*}(x)}\varphi^{(j+2)}_{n\backslash(j+2)}(x) = \varphi^{(j)}_{n\backslash j}(x) - \varphi^{(j+1)}_{n\backslash(j+1)}(x)\varphi^{(j)}_{(j+1)\backslash j}(x).$$

Consequently, by induction we then find for $j+1$ that

$$\begin{aligned}
&\varphi^{(j+1)}_{n\backslash(j+1)}(x)\varphi^{(k)}_{(j+1)\backslash k}(x) \\
&\quad - C_{j+2}\frac{Z_{j+2}(x)}{Z_{j*}(x)}\varphi^{(j+2)}_{n\backslash(j+2)}(x)\varphi^{(k)}_{j\backslash k}(x) \\
&= -C_{j+1}\frac{Z_{j+1}(x)}{Z_{j-1*}(x)}\varphi^{(j+1)}_{n\backslash(j+1)}(x)\varphi^{(k)}_{(j-1)\backslash k}(x) \\
&\quad + \varphi^{(j)}_{n\backslash j}(x)\varphi^{(k)}_{j\backslash k}(x) = \varphi^{(k)}_{n\backslash k}(x),
\end{aligned}$$

which ends the proof. □

We are now able to prove our first main result.

Theorem 3.3. *Let $P^{(j)}_{m+1}$ and $Q^{(k,j)}_{m,n}$, with $n+1 \geq m+1 \geq j \geq k \geq 0$, be given by*

$$\begin{aligned}
P^{(j)}_{m+1}(x) &= \prod_{i=j+1}^{m+1} C_i\frac{Z_i(x)}{Z_{i-2*}(x)}, \quad P^{(j)}_j(x) \equiv 1, \quad (13)\\
Q^{(k,j)}_{m,n}(x) &= \varphi^{(k)}_{m\backslash k}(x)\varphi^{(j)}_{n\backslash j}(x) - \varphi^{(j)}_{m\backslash j}(x)\varphi^{(k)}_{n\backslash k}(x).
\end{aligned}$$

Then it holds that

$$Q^{(k,j)}_{m,n}(x) = P^{(j)}_{m+1}(x)\varphi^{(m+1)}_{n\backslash(m+1)}(x)\varphi^{(k)}_{(j-1)\backslash k}(x). \quad (14)$$

Proof. Since for every $l \geq 0$ it holds that $Q^{(l,l)}_{m,n}(x) \equiv 0 \equiv \varphi^{(l)}_{(l-1)\backslash l}(x)$, the statement clearly holds for $k = j$. Similarly, for every $l \geq j-1$ it holds that $Q^{(k,j)}_{l,l}(x) \equiv 0 \equiv \varphi^{(l+1)}_{l\backslash(l+1)}(x)$, so that the statement clearly holds for $m = n$ as well. Thus, it remains to prove the statement for $n+1 > m+1 \geq j > k \geq 0$.

Let k and j be fixed. For $m = j$, (14) reduces to the relation given by (12). While for $m = j-1$ we have that

$$\begin{aligned}
Q^{(k,j)}_{j-1,n}(x) &= \varphi^{(k)}_{(j-1)\backslash k}(x)\varphi^{(j)}_{n\backslash j}(x) \\
&= P^{(j)}_j(x)\varphi^{(j)}_{n\backslash j}(x)\varphi^{(k)}_{(j-1)\backslash k}(x).
\end{aligned}$$

So, suppose that the statement holds for $Q^{(k,j)}_{m-2,n-2}$, $Q^{(k,j)}_{m-1,n-2}$, $Q^{(k,j)}_{m-2,n-1}$ and $Q^{(k,j)}_{m-1,n-1}$, with $n > m \geq j+1$. By induction, we then find for $Q^{(k,j)}_{m,n}$ that (see also Figure 1 for a graphical representation of the proof by induction)

$$\begin{aligned}
Q^{(k,j)}_{m,n}(x) &= \varphi^{(k)}_{m\backslash k}(x)\varphi^{(j)}_{n\backslash j}(x) - \varphi^{(j)}_{m\backslash j}(x)\varphi^{(k)}_{n\backslash k}(x) \\
&= \left[\varphi^{(m-1)}_{m\backslash(m-1)}(x)\varphi^{(k)}_{(m-1)\backslash k}(x) - P^{(m-1)}_m(x)\varphi^{(k)}_{(m-2)\backslash k}(x)\right] \times \\
&\quad \left[\varphi^{(n-1)}_{n\backslash(n-1)}(x)\varphi^{(j)}_{(n-1)\backslash j}(x) - P^{(n-1)}_n(x)\varphi^{(j)}_{(n-2)\backslash j}(x)\right] - \\
&\quad \left[\varphi^{(m-1)}_{m\backslash(m-1)}(x)\varphi^{(j)}_{(m-1)\backslash j}(x) - P^{(m-1)}_m(x)\varphi^{(j)}_{(m-2)\backslash j}(x)\right] \times \\
&\quad \left[\varphi^{(n-1)}_{n\backslash(n-1)}(x)\varphi^{(k)}_{(n-1)\backslash k}(x) - P^{(n-1)}_n(x)\varphi^{(k)}_{(n-2)\backslash k}(x)\right] \\
&= \varphi^{(m-1)}_{m\backslash(m-1)}(x)\varphi^{(n-1)}_{n\backslash(n-1)}(x)Q^{(k,j)}_{m-1,n-1}(x) \\
&\quad + P^{(m-1)}_m(x)P^{(n-1)}_n(x)Q^{(k,j)}_{m-2,n-2}(x) \\
&\quad - \varphi^{(m-1)}_{m\backslash(m-1)}(x)P^{(n-1)}_n(x)Q^{(k,j)}_{m-1,n-2}(x) \\
&\quad - \varphi^{(n-1)}_{n\backslash(n-1)}(x)P^{(m-1)}_m(x)Q^{(k,j)}_{m-2,n-1}(x) \\
&= P^{(j)}_m(x)\left\{\varphi^{(m-1)}_{m\backslash(m-1)}(x)\varphi^{(n-1)}_{n\backslash(n-1)}(x)\varphi^{(m)}_{(n-1)\backslash m}(x)\right. \\
&\quad + P^{(n-1)}_n(x)\varphi^{(m-1)}_{(n-2)\backslash(m-1)}(x) \\
&\quad - \varphi^{(m-1)}_{m\backslash(m-1)}(x)P^{(n-1)}_n(x)\varphi^{(m)}_{(n-2)\backslash m}(x) \\
&\quad \left. -\varphi^{(n-1)}_{n\backslash(n-1)}(x)\varphi^{(m-1)}_{(n-1)\backslash(m-1)}(x)\right\}\varphi^{(k)}_{(j-1)\backslash k}(x) \\
&= P^{(j)}_m(x)\left\{\varphi^{(m-1)}_{m\backslash(m-1)}(x)\varphi^{(m)}_{n\backslash m}(x)\right. \\
&\quad \left. -\varphi^{(m-1)}_{n\backslash(m-1)}(x)\right\}\varphi^{(k)}_{(j-1)\backslash k}(x) \\
&= P^{(j)}_{m+1}(x)\varphi^{(m+1)}_{n\backslash(m+1)}(x)\varphi^{(k)}_{(j-1)\backslash k}(x).
\end{aligned}$$

□

4 Functions of the second kind

Suppose the ARFs $\varphi^{(k-1)}_{n\backslash(k-1)}$ of order $k-1 \geq 0$ form an orthonormal system with respect to a normalized HPDLF $M^{(k-1)}$, and let $\Phi_{n\backslash(k-1)}$ be given by

$$\Phi_{n\backslash(k-1)}(x,t) = (1 - t/\overline{\alpha}_{k-1})\varphi^{(k-1)}_{n\backslash(k-1)}(x). \quad (15)$$

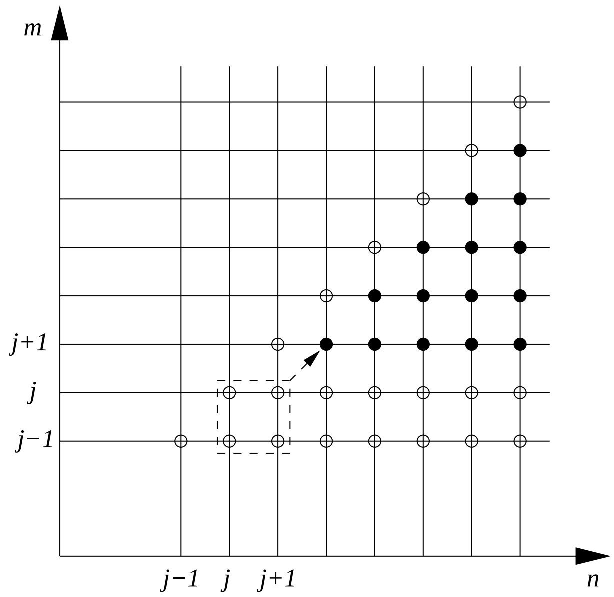

Figure 1: Graphical representation of the proof by induction of Theorem 3.3. The (n,m)-coordinates marked with an 'o' denote the initialization, while the induction step is represented by the rectangular and arrow. Consequently, the statement follows by induction for the (n,m)-coordinates marked with a black dot.

Then we define the rational functions of the second kind $\psi_{n\backslash k}$ by

$$\psi_{n\backslash k}(x) = \frac{(1-x/\alpha_k)}{\overline{E}_{k-1}C_k}\times \left[M_t^{(k-1)}\left\{\frac{\Phi_{n\backslash(k-1)}(t,x)-\Phi_{n\backslash(k-1)}(x,t)}{t-x}\right\} -\delta_{n,k-1}/\overline{\alpha}_{k-1}\right], \quad n\geq k-1, \quad (16)$$

where $\delta_{n,k-1}$ is the Kronecker Delta. Note that this definition is very similar to, but not exactly the same as the one given before in (5). We will then prove that the $\psi_{n\backslash k}$ satisfy the same three-term recurrence relation as $\varphi^{(k-1)}_{n\backslash(k-1)}$ with initial conditions $\psi_{(k-1)\backslash k}(x)\equiv 0$ and $\psi_{k\backslash k}(x)\equiv 1$, and hence, that $\psi_{n\backslash k}(x) = \varphi^{(k)}_{n\backslash k}(x)$. First, we need the following lemma.

Lemma 4.1. *Let $\psi_{n\backslash k}$, with $n\geq k-1\geq 0$, be defined as before in (16). Then it holds that $\psi_{(k-1)\backslash k}(x)\equiv 0$ and $\psi_{k\backslash k}(x)\equiv 1$, while $\psi_{n\backslash k}\in\mathcal{L}_{n\backslash k}$ for $n>k$.*

Proof. Define $q_{n-(k-2)}$ by

$$q_{n-(k-2)}(x) = (1-x/\overline{\alpha}_{k-1})\pi_{n\backslash(k-1)}(x).$$

For $n\geq k$ it then follows from (15) and (16) that

$$\begin{aligned}\overline{E}_{k-1}C_k\psi_{n\backslash k}(x) &= \frac{1}{\pi_{n\backslash k}(x)}M_t^{(k-1)}\left\{\frac{1}{t-x}\times \left[\varphi^{(k-1)}_{n\backslash(k-1)}(t)q_{n-(k-2)}(x)-(1-t/\overline{\alpha}_{k-1})p^{(k-1)}_{n-(k-1)}(x)\right]\right\}\\ &= \frac{\sum_{i=0}^{n-(k-1)}M_t^{(k-1)}\left\{a_i^{(k)}(t)\right\}x^i}{\pi_{n\backslash k}(x)}. \quad (17)\end{aligned}$$

Further, with

$$c_{n,k} = \lim_{x\to\infty}\frac{\pi_{n\backslash k}(x)}{x^{n-k}},$$

we have that

$$\begin{aligned}M_t^{(k-1)}\left\{a^{(k)}_{n-(k-1)}(t)\right\} &= \frac{c_{n,k-1}}{\overline{\alpha}_{k-1}}M_t^{(k-1)}\left\{\varphi^{(k-1)}_{n\backslash(k-1)}(t)\right\}\\ &= 0,\end{aligned}$$

so that $\psi_{n\backslash k}$ is of the form

$$\psi_{n\backslash k}(x) = \frac{p^{(k)}_{n-k}(x)}{\pi_{n\backslash k}(x)}\in\mathcal{L}_{n\backslash k}.$$

For $n=k$ we find that

$$\overline{E}_{k-1}C_k\psi_{k\backslash k}(x) = M_t^{(k-1)}\left\{\frac{\varphi^{(k-1)}_{k\backslash(k-1)}(t)q_2(x)-(1-t/\overline{\alpha}_{k-1})p_1^{(k-1)}(x)}{-x(1-t/x)}\right\}.$$

Note that

$$\lim_{x\to\overline{\alpha}_{k-1}} -\frac{q_2(x)}{x}M_t^{(k-1)}\left\{\frac{\varphi^{(k-1)}_{k\backslash(k-1)}(t)}{1-t/x}\right\} = 0,$$

so that

$$\begin{aligned}\psi_{k\backslash k}(x) &= \lim_{x\to\overline{\alpha}_{k-1}}M_t^{(k-1)}\left\{\frac{1-t/\overline{\alpha}_{k-1}}{1-t/x}\right\}\frac{p_1^{(k-1)}(x)}{\overline{E}_{k-1}C_kx}\\ &= \lim_{x\to\overline{\alpha}_{k-1}}\frac{\varphi^{(k-1)}_{k\backslash(k-1)}(x)}{\overline{E}_{k-1}C_kZ_k(x)} = \lim_{x\to\overline{\alpha}_{k-1}}\frac{E_k\left[1+F_k/Z_{k-1}(x)\right]}{\overline{E}_{k-1}C_k}\\ &= \frac{E_k\left[1+F_k/Z_{k-1}(\overline{\alpha}_{k-1})\right]}{\overline{E}_{k-1}C_k} = 1.\end{aligned}$$

Finally, in the special case in which $n=k-1$, we have that

$$\begin{aligned}&M_t^{(k-1)}\left\{\frac{\Phi_{(k-1)\backslash(k-1)}(t,x)-\Phi_{(k-1)\backslash(k-1)}(x,t)}{t-x}\right\}\\ &= M_t^{(k-1)}\left\{\frac{(1-x/\overline{\alpha}_{k-1})-(1-t/\overline{\alpha}_{k-1})}{t-x}\right\}\\ &= 1/\overline{\alpha}_{k-1}.\end{aligned}$$

$\square$

The following theorem now shows that these $\psi_{n\backslash k}$ satisfy the same three-term recurrence relation as the $\varphi^{(k-1)}_{n\backslash(k-1)}$.

Theorem 4.2. *Let $\psi_{n\backslash k}$ be defined as before in* (16). *The rational functions $\psi_{l\backslash k}$, with $l = n-2, n-1, n$ and $n \geq k+1$, then satisfy the three-term recurrence relation given by*

$$\psi_{n\backslash k}(x) = Z_n(x)\left\{E_n\left[1+\frac{F_n}{Z_{n-1}(x)}\right]\psi_{(n-1)\backslash k}(x) - \frac{C_n}{Z_{n-2*}(x)}\psi_{(n-2)\backslash k}(x)\right\}. \quad (18)$$

The initial conditions are $\psi_{(k-1)\backslash k}(x) \equiv 0$ and $\psi_{k\backslash k}(x) \equiv 1$.

Proof. First note that the ARFs $\varphi^{(k-1)}_{l\backslash(k-1)}$, with $l = n-2, n-1, n$, satisfy the three-term recurrence relation given by (18), and hence, so do the $\Phi_{l\backslash(k-1)}$. Consequently, we have that

$$\psi_{n\backslash k}(x) = Z_n(x)\left\{E_n\left[1+\frac{F_n}{Z_{n-1}(x)}\right]\psi_{(n-1)\backslash k}(x) - \frac{C_n}{Z_{n-2*}(x)}\psi_{(n-2)\backslash k}(x)\right\} + M_t^{(k-1)}\left\{\frac{f_n(x,t)}{t-x}\right\} - \delta_{n,k+1}\frac{(1-x/\alpha_k)}{\overline{\alpha}_{k-1}\overline{E}_{k-1}C_k}C_{k+1}\frac{Z_{k+1}(x)}{Z_{k-1*}(x)},$$

where $f_n(x,t) = \frac{(1-x/\alpha_k)}{\overline{E}_{k-1}C_k}g_n(x,t)$ and $g_n(x,t)$ is given by

$$\begin{aligned} g_n(x,t) &= E_n[Z_n(t) - Z_n(x)]\Phi_{(n-1)\backslash(k-1)}(t,x) \\ &+ E_nF_n\left[\frac{Z_n(t)}{Z_{n-1}(t)} - \frac{Z_n(x)}{Z_{n-1}(x)}\right]\Phi_{(n-1)\backslash(k-1)}(t,x) \\ &- C_n\left[\frac{Z_n(t)}{Z_{n-2*}(t)} - \frac{Z_n(x)}{Z_{n-2*}(x)}\right]\Phi_{(n-2)\backslash(k-1)}(t,x). \end{aligned}$$

Note that

$$\begin{aligned} Z_n(t) - Z_n(x) &= \frac{(t-x)}{(1-t/\alpha_n)(1-x/\alpha_n)} \\ \frac{Z_n(t)}{Z_{n-1}(t)} - \frac{Z_n(x)}{Z_{n-1}(x)} &= \frac{(t-x)/Z_{n-1}(\alpha_n)}{(1-t/\alpha_n)(1-x/\alpha_n)} \\ \frac{Z_n(t)}{Z_{n-2*}(t)} - \frac{Z_n(x)}{Z_{n-2*}(x)} &= \frac{(t-x)/Z_{n-2*}(\alpha_n)}{(1-t/\alpha_n)(1-x/\alpha_n)}, \end{aligned}$$

so that

$$\begin{aligned} \frac{f_n(x,t)}{t-x} &= \frac{(1-x/\alpha_k)}{\overline{E}_{k-1}C_k}\cdot\frac{Z_n(x)}{Z_{k-1*}(x)}(1-t/\alpha_n)^{-1}h_n(t) \\ &= \frac{(1-x/\alpha_k)}{\overline{E}_{k-1}C_k}\cdot\frac{Z_n(x)}{Z_{k-1*}(x)}\left(1+\frac{Z_n(t)}{\alpha_n}\right)h_n(t), \end{aligned}$$

where

$$h_n(t) = E_n\left[1+\frac{F_n}{Z_{n-1}(\alpha_n)}\right]\varphi^{(k-1)}_{(n-1)\backslash(k-1)}(t) - \frac{C_n}{Z_{n-2*}(\alpha_n)}\varphi^{(k-1)}_{(n-2)\backslash(k-1)}(t).$$

It clearly holds that

$$M_t^{(k-1)}\{h_n(t)\} = -\delta_{n,k+1}C_{k+1}/Z_{k-1*}(\alpha_{k+1}).$$

Further, note that

$$\frac{Z_n(t)}{Z_{n-2*}(\alpha_n)} = \frac{Z_n(t)}{Z_{n-2*}(t)} - 1$$

and

$$\frac{Z_n(t)}{Z_{n-1}(\alpha_n)} = \frac{Z_n(t)}{Z_{n-1}(t)} - 1.$$

Hence,

$$Z_n(t)h_n(t) = \varphi^{(k-1)}_{n\backslash(k-1)}(t) - E_nF_n\varphi^{(k-1)}_{(n-1)\backslash(k-1)}(t) + C_n\varphi^{(k-1)}_{(n-2)\backslash(k-1)}(t),$$

so that

$$\frac{M_t^{(k-1)}\{Z_n(t)h_n(t)\}}{\alpha_n} = \delta_{n,k+1}C_{k+1}/\alpha_{k+1}.$$

As a result,

$$M_t^{(k-1)}\left\{\frac{f_n(x,t)}{t-x}\right\} = \delta_{n,k+1}\frac{(1-x/\alpha_k)}{\overline{\alpha}_{k-1}\overline{E}_{k-1}C_k}C_{k+1}\frac{Z_{k+1}(x)}{Z_{k-1*}(x)},$$

which ends the proof. □

The next theorem directly follows from Lemma 4.1 and Theorem 4.2.

Theorem 4.3. *Let $\psi_{n\backslash k}$ be defined as before in* (16). *These $\psi_{n\backslash k}$ are the ARFs $\varphi^{(k)}_{n\backslash k}$ of order k with initial conditions $\varphi^{(k)}_{(k-1)\backslash k}(x) \equiv 0$ and $\varphi^{(k)}_{k\backslash k}(x) \equiv 1$.*

In the above lemma and theorems we have assumed that the ARFs $\varphi^{(k-1)}_{n\backslash(k-1)}$ form an orthonormal system with respect to a normalized HPDLF $M^{(k-1)}$. The assumption certainly holds for $k = 1$, and hence, the ARFs $\varphi^{(1)}_{n\backslash 1}$ are the rational functions of the second kind of the ORFs φ_n. The next question is then whether the ARFs $\varphi^{(1)}_{n\backslash 1}$ form an orthonormal system with respect to a normalized HPDLF $M^{(1)}$. Therefore, we need the following lemma.

Lemma 4.4. *Let the ARFs $\varphi^{(k)}_{n\backslash k}$ of order k be defined by* (16). *Then the leading coefficient $K^{(k)}_{n-k}$, i.e. the coefficient of $b_{n\backslash k}$ in the expansion of $\varphi^{(k)}_{n\backslash k}$ with respect to the basis $\{b_{k\backslash k}, \ldots, b_{n\backslash k}\}$, is given by*

$$K^{(k)}_{n-k} = \frac{K^{(k-1)}_{n-(k-1)}}{\overline{E}_{k-1}C_k}M_t^{(k-1)}\left\{\frac{1-t/\overline{\alpha}_{k-1}}{1-t/\alpha_n}\right\}, \quad n \geq k.$$

Proof. Note that the leading coefficient $K^{(k)}_{n-k}$ is given by (see also [3, Thm. 3.2])

$$K^{(k)}_{n-k} = \lim_{x\to\alpha_n} \frac{\varphi^{(k)}_{n\backslash k}(x)}{b_{n\backslash k}(x)} = \lim_{x\to\alpha_n} \frac{p^{(k)}_{n-k}(x)}{x^{n-k}}.$$

Further, let $q_{n-(k-2)}$ be defined as before in Lemma 4.1. Clearly, for $n \geq k$ it then holds that

$$\lim_{x\to\alpha_n} -\frac{q_{n-(k-2)}(x)}{x^{n-(k-1)}} M^{(k-1)}_t \left\{ \frac{\varphi^{(k-1)}_{k\backslash(k-1)}(t)}{1-t/x} \right\} = 0.$$

So, from (17) we deduce that

$$\begin{aligned} &\overline{E}_{k-1} C_k K^{(k)}_{n-k} \\ &\quad = \lim_{x\to\alpha_n} \frac{p^{(k-1)}_{n-(k-1)}(x)}{x^{n-(k-1)}} M^{(k-1)}_t \left\{ \frac{1-t/\overline{\alpha}_{k-1}}{1-t/x} \right\} \\ &\quad = K^{(k-1)}_{n-(k-1)} M^{(k-1)}_t \left\{ \frac{1-t/\overline{\alpha}_{k-1}}{1-t/\alpha_n} \right\}. \end{aligned}$$

This proves the statement. □

As a consequence, we now have the following theorem.

Theorem 4.5. *Let the ARFs $\varphi^{(k)}_{n\backslash k}$ of order k be defined by* (16) *and assume that $\alpha_{k-1} \in \overline{\mathbb{R}}_0$. Further, suppose that*

$$M^{(k-1)}_t \left\{ \frac{1-t/\alpha_{k-1}}{1-t/\alpha_n} \right\} \neq 0 \tag{19}$$

whenever $n > k$ and $\alpha_n \notin \{\alpha_{k-1}, \overline{\alpha}_k, \alpha_k\}$. Then it holds that the $\varphi^{(k)}_{n\backslash k}$ form an orthonormal system with respect to a normalized HPDLF $M^{(k)}$.

Proof. Note that $\varphi^{(k)}_{n\backslash k} \in \mathcal{L}_{n\backslash k} \setminus \mathcal{L}_{(n-1)\backslash k}$ iff $K^{(k)}_{n-k} \neq 0$. We now have that $K^{(k-1)}_{n-(k-1)} \neq 0$ for every $n > k$, due to the fact that the ARFs $\varphi^{(k-1)}_{n\backslash(k-1)} \in \mathcal{L}_{n\backslash(k-1)} \setminus \mathcal{L}_{(n-1\backslash(k-1)}$. Moreover, as $M^{(k-1)}$ is a normalized HPDLF and because $\varphi^{(k-1)}_{k\backslash(k-1)}$ is regular, we also have that

$$M^{(k-1)}_t \left\{ \frac{1-t/\alpha_{k-1}}{1-t/\alpha_n} \right\} \neq 0$$

whenever $\alpha_n \in \{\alpha_{k-1}, \overline{\alpha}_k, \alpha_k\}$. Thus, together with the assumption given by (19), it follows from Lemma 4.4 that $\varphi^{(k)}_{n\backslash k} \in \mathcal{L}_{n\backslash k} \setminus \mathcal{L}_{(n-1)\backslash k}$ for every $n > k$. Consequently, both assumptions in Theorem 2.2 are satisfied, which ends the proof. □

Finally, note that none of the ARFs form an orthonormal system whenever $(\mathcal{A}_\infty \cup \{\alpha_0\}) \subset (\mathbb{C} \setminus \mathbb{R})$. On the other hand, whenever the inner product is defined as a weighted infinite sum of as an integral over a subset of the real line with respect to a positive bounded Borel measure, and all the poles (including α_0) are real and outside the convex hull of the support of the measure, then the ARFs form an orthonormal system for every order $k \geq 1$.

5 ARFs and functions of the second kind

In the previous section, a generalization of (2) to the case of ARFs has been proved for the special case in which $k = j+1$. The aim of this section is to give a generalization for arbitrary k, with $j+1 \leq k \leq n+1$, and hence, to give a generalization of relation (3) and (4).

Suppose the ARFs $\varphi^{(j)}_{n\backslash j}$ of order $j \geq 0$ form an orthonormal system with respect to a normalized HPDLF $M^{(j)}$, and let $\chi^{(j)}_{n,x}$ be defined by

$$\chi^{(j)}_{n,x}(t) = (1 - x/\alpha_{j+1}) \frac{\Phi_{n\backslash j}(t,x) - \Phi_{n\backslash j}(x,t)}{t-x}. \tag{20}$$

Clearly, for fixed values of x we have that $\chi^{(j)}_{n,x} \in \mathcal{L}_{n\backslash j}$, and hence, there exist coefficients $a_{k,n}(x)$ so that

$$\chi^{(j)}_{n,x}(t) = \sum_{k=j}^{n} a_{k,n}(x) \varphi^{(j)}_{k\backslash j}(t),$$

with

$$a_{k,n}(x) = M^{(j)} \left\{ \chi^{(j)}_{n,x} \varphi^{(j)}_{(k\backslash j)*} \right\}.$$

For $k = j < n$, it already follows from the previous section that

$$a_{j,n}(x) = \overline{E}_j C_{j+1} \varphi^{(j+1)}_{n\backslash(j+1)}(x).$$

While for $k = n$ we have the following lemma.

Lemma 5.1. *For $k = n$, the coefficient $a_{k,n}(x)$ is given by*

$$a_{n,n}(x) = \begin{cases} (1 - x/\alpha_{n+1})/\overline{\alpha}_n, & n = j \\ \frac{Z_n(x)}{Z_{j*}(x)}(1 - x/\alpha_{j+1})/\alpha_n, & n > j \end{cases}. \tag{21}$$

In the special case in which $\alpha_j \in \overline{\mathbb{R}}_0$, we may rewrite (21) *as*

$$a_{n,n}(x) = \frac{Z_n(x)}{Z_j(x)}(1 - x/\alpha_{j+1})/\alpha_n, \quad n \geq j.$$

Proof. The expression for $a_{j,j}(x)$ is easily verified (see also the last step in the proof of Lemma 4.1). So, it remains to prove the case in which $n > j$. We then have that

$$\begin{aligned} a_{n,n}(x) &= \lim_{t\to\alpha_n} \frac{\chi^{(j)}_{n,x}(t)}{\varphi^{(j)}_{n\backslash j}(t)} \\ &= (1 - x/\alpha_{j+1}) \lim_{t\to\alpha_n} \left[\frac{1 - x/\overline{\alpha}_j}{t-x} - \frac{(1 - t/\overline{\alpha}_j)\varphi^{(j)}_{n\backslash j}(x)}{(t-x)\varphi^{(j)}_{n\backslash j}(t)} \right] \\ &= (1 - x/\alpha_{j+1}) \left[\frac{1 - x/\overline{\alpha}_j}{\alpha_n - x} \right] = \frac{Z_n(x)}{Z_{j*}(x)}(1 - x/\alpha_{j+1})/\alpha_n. \end{aligned}$$

□

In the remainder we will make the following assumptions:

(A1) $\alpha_j \in \overline{\mathbb{R}}_0$, $j \geq 0$;

(A2) $\mathcal{A}_{(n-2)\backslash j} \subset \overline{\mathbb{R}}_0$ whenever $n > j+2$.

Finally, note that ORFs are fixed up to a unimodular constant. Thus, without loss of generality we may as well assume that

(A3) $E_k \in \mathbb{R}_0$ whenever $\alpha_{k-1} \in \overline{\mathbb{R}}_0$.

We are now able to prove our second main result.

Theorem 5.2. *Let $\chi^{(j)}_{n,x}$, with $0 \le j \le n$, be defined by* (20). *Under the assumptions (A1)–(A3) it then holds that*

$$\chi^{(j)}_{n,x}(t) = E_{j+1}\sum_{k=j}^{n-1} P^{(j+1)}_{k+1}(x)\varphi^{(k+1)}_{n\backslash(k+1)}(x)\varphi^{(j)}_{k\backslash j}(t) + \frac{Z_n(x)}{Z_j(x)}\left(\frac{1-x/\alpha_{j+1}}{\alpha_n}\right)\varphi^{(j)}_{n\backslash j}(t), \quad (22)$$

where $P^{(j+1)}_{k+1}$ is defined as before in (13). *And hence,*

$$\varphi^{(k+1)}_{n\backslash(k+1)}(x) = \frac{1}{E_{j+1}P^{(j+1)}_{k+1}(x)} M^{(j)}\left\{\chi^{(j)}_{n,x}\varphi^{(j)}_{(k\backslash j)*}\right\}, \quad j \le k < n.$$

Proof. The equality in (22) clearly holds for $n \in \{j, j+1\}$. Thus, suppose the equality holds for $n-2$ and $n-1$, and let (see also the proof of Theorem 4.2)

$$h_n(t) = E_n\left[1 + \frac{F_n}{Z_{n-1}(\alpha_n)}\right]\varphi^{(j)}_{(n-1)\backslash j}(t) - \frac{C_n}{Z_{n-2}(\alpha_n)}\varphi^{(j)}_{(n-2)\backslash j}(t)$$

and

$$r_n(t) = Z_n(t)h_n(t) = \varphi^{(j)}_{n\backslash j}(t) - E_nF_n\varphi^{(j)}_{(n-1)\backslash j}(t) + C_n\varphi^{(j)}_{(n-2)\backslash j}(t).$$

By induction, we then find for $n \ge j+2$ that

$$\begin{aligned}\chi^{(j)}_{n,x}(t) &= \varphi^{(n-1)}_{n\backslash(n-1)}(x)\chi_{n-1,x}(t) - C_n\frac{Z_n(x)}{Z_{n-2}(x)}\chi_{n-2,x}(t) \\ &+ (1-x/\alpha_{j+1})\frac{Z_n(x)}{Z_j(x)}h_n(t) + \left(\frac{1-x/\alpha_{j+1}}{\alpha_n}\right)\frac{Z_n(x)}{Z_j(x)}r_n(t) \\ &= E_{j+1}\sum_{k=j}^{n-2} P^{(j+1)}_{k+1}(x)\varphi^{(k+1)}_{n\backslash(k+1)}(x)\varphi^{(j)}_{k\backslash j}(t) \\ &+ c_{n-2}(x)\varphi^{(j)}_{(n-2)\backslash j}(t) + c_{n-1}(x)\varphi^{(j)}_{(n-1)\backslash j}(t) \\ &+ \frac{Z_n(x)}{Z_j(x)}\left(\frac{1-x/\alpha_{j+1}}{\alpha_n}\right)\varphi^{(j)}_{n\backslash j}(t)\end{aligned}$$

where

$$c_{n-2}(x) = C_n\frac{Z_n(x)}{Z_j(x)}(1-x/\alpha_{j+1})\times \left(\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-2}} - \frac{1}{Z_{n-2}(\alpha_n)}\right) \equiv 0$$

and

$$\begin{aligned}c_{n-1}(x) &= E_n(1-x/\alpha_{j+1})\frac{Z_n(x)}{Z_j(x)}\times \\ &\left\{\left[1 + \frac{Z_{n-1}(x)}{\alpha_{n-1}}\right] + F_n\left[\frac{1}{\alpha_{n-1}} + \frac{1}{Z_{n-1}(\alpha_n)} - \frac{1}{\alpha_n}\right]\right\} \\ &= E_n\frac{Z_n(x)Z_{n-1}(x)}{Z_{j+1}(x)Z_j(x)} = E_{j+1}P^{(j+1)}_n(x)\varphi^{(n)}_{n\backslash n}(x).\end{aligned}$$

□

Finally, as a consequence of the previous theorem, we have the following corollary.

Corollary 5.3. *Let $\chi^{(j)}_{n,x}$, with $0 \le j \le n$, be defined by* (20). *Under the assumptions (A1)–(A3) it then holds that*

$$\begin{aligned}&(1-x/\alpha_j)(1-x/\alpha_{j+1})\frac{d}{dx}\left[\varphi^{(j)}_{n\backslash j}(x)\right] \\ &= E_{j+1}\sum_{k=j}^{n-1} P^{(j+1)}_{k+1}(x)\varphi^{(k+1)}_{n\backslash(k+1)}(x)\varphi^{(j)}_{k\backslash j}(x) \\ &+ \frac{1}{Z_j(\alpha_n)}\left[\frac{Z_n(x)}{Z_{j+1}(x)}\right]\varphi^{(j)}_{n\backslash j}(x),\end{aligned}$$

where $P^{(j+1)}_{k+1}$ is defined as before in (13).

Proof. From Theorem 5.2 it follows that

$$\chi^{(j)}_{n,x}(x) = E_{j+1}\sum_{k=j}^{n-1} P^{(j+1)}_{k+1}(x)\varphi^{(k+1)}_{n\backslash(k+1)}(x)\varphi^{(j)}_{k\backslash j}(x) + \frac{Z_n(x)}{Z_j(x)}\left(\frac{1-x/\alpha_{j+1}}{\alpha_n}\right)\varphi^{(j)}_{n\backslash j}(x).$$

On the other hand we have that

$$\begin{aligned}\chi^{(j)}_{n,x}(x) &= (1-x/\alpha_{j+1})(1-x/\alpha_j)^2 \times \lim_{t\to x}\left[\frac{\frac{\varphi^{(j)}_{n\backslash j}(t)}{1-t/\alpha_j} - \frac{\varphi^{(j)}_{n\backslash j}(x)}{1-x/\alpha_j}}{t-x}\right] \\ &= (1-x/\alpha_{j+1})(1-x/\alpha_j)^2\frac{d}{dx}\left[\frac{\varphi^{(j)}_{n\backslash j}(x)}{1-x/\alpha_j}\right] \\ &= (1-x/\alpha_{j+1})(1-x/\alpha_j)\frac{d}{dx}\left[\varphi^{(j)}_{n\backslash j}(x)\right] + \left(\frac{1-x/\alpha_{j+1}}{\alpha_j}\right)\varphi^{(j)}_{n\backslash j}(x).\end{aligned}$$

Consequently,

$$\begin{aligned}&(1-x/\alpha_j)(1-x/\alpha_{j+1})\frac{d}{dx}\left[\varphi^{(j)}_{n\backslash j}(x)\right] \\ &= E_{j+1}\sum_{k=j}^{n-1} P^{(j+1)}_{k+1}(x)\varphi^{(k+1)}_{n\backslash(k+1)}(x)\varphi^{(j)}_{k\backslash j}(x) \\ &+ (1-x/\alpha_{j+1})\left[\frac{Z_n(x)}{Z_j(x)}\cdot\frac{1}{\alpha_n} - \frac{1}{\alpha_j}\right]\varphi^{(j)}_{n\backslash j}(x).\end{aligned}$$

Finally, note that

$$\frac{Z_n(x)}{Z_j(x)} \cdot \frac{1}{\alpha_n} - \frac{1}{\alpha_j} = \frac{1}{Z_j(\alpha_n)(1 - x/\alpha_n)},$$

which ends the proof. □

6 Conclusion

In this paper, we have given a relation between associated rational functions (ARFs) of order j and $k \geq j+1$ in terms of rational functions of the second kind, assuming the ARFs of order j form an orthonormal system with respect to an Hermitian positive-definite inner product. Further, we have given a relation between ARFs of different order that holds in general; i.e. the relation holds independently of whether the ARFs involved form an orthonormal system with respect to an Hermitian positive-definite inner product. If all the poles are at infinity, we again obtain the polynomial case.

The results in this paper have been derived in the more general framework of the approximation of integrals on the interval $[-1, 1]$; more specific, to characterize rational quadrature formulas with positive weights and to derive asymptotic formulas for the weights (like has been done for the polynomial case in [6, 7]). At this moment of writing, however, this investigation is still in an early phase.

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