Facility Location Problems with Random Demands in a Competitive Environment

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Abstract— **This paper proposes a new location problem of competitive facilities, e.g. shops and stores, with uncertain demands in the plane. By representing the demands for facilities as random variables, the location problem is formulated to a stochastic programming problem, and for finding its solution, three deterministic programming problems: expectation maximizing problem, probability maximizing problem, and satisfying level maximizing problem are considered. After showing that one of their optimal solutions can be found by solving 0-1 programming problems, their solution method is proposed by improving the tabu search algorithm with strategic vibration. Efficiency of the solution method is shown by applying to numerical examples of the facility location problems.**

Keywords: facility location, competitiveness, stochastic programming, 0-1 programming, tabu search

1 Introduction

Competitive facility location problem (CFLP) is one of optimal location problems for commercial facilities, e.g. shops and stores, and the objective of a decision maker (DM) for the CFLP is mainly to obtain as many demands for her/his facilities as possible. Mathematical studies on CFLPs are originated by Hotelling [8]. He considered the CFLP under the conditions that (i) customers are uniformly distributed on a line segment, (ii) each of DMs can locate and move her/his own facility at any times, and (iii) all customers only use the nearest facility. CFLPs on the plane were studied by Okabe and Suzuki [13], etc. As an extension of Hotelling's CFLP, Wendell and McKelvey [20] assumed that there exist customers on a finite number of points, called demand points (DPs), and they considered the CFLP on a network whose nodes are DPs.

Based upon the CFLP proposed by Wendell and McKelvey, Hakimi [6] considered CFLPs under the conditions that the DM locates her/his facilities on a network that other competitive facilities were already located. Drezner [3] extended Hakimi's CFLPs to the CFLP on the plane that there are DPs and competitive facilities. In the above CFLPs, customers choose their using facilities by estimating only the distance from them to facilities. Huff [9] defined the attractive function of facility for customers by considering not only the distance but also quality of facility. Drezner's CFLPs with Huff's attractive function are studied by Uno and Katagiri [17], Fernández et al. [4], Bruno and Improta [2], and Zhang and Rushton [21]. As other extensions of Drezner's CFLPs, CFLPs with fuzziness are considered by Moreno Pérez et al [11], and CFLPs based on maximal covering are considered by Plastria and Vanhaverbeke [14].

In the above studies of CFLPs, the demands of customers for facilities are represented as deterministic values. Wagnera et al. [19] considered facility location models with random demands in a noncompetitive environment. For the details of noncompetitive location models with random demands, the reader can refer to the study of Berman and Krass [1].

In this paper, we proposes a new CFLP with random demands by extending Drezner's CFLP with Huff's attractive function. Then, the location problem can be formulated as a stochastic programming problem. For finding its optimal solutions, three deterministic programming problems: expectation maximizing problem, probability maximizing problem, and satisfying level maximizing problem are considered. These problems are nonconvex and nonlinear programming problems, and we need to find at least one optimal solution of them. However, for most CFLPs in the plane [3, 17], their optimal solutions cannot be found by directly applying general analytic solution methods with derivatives of the objective function, Kuhn-Tucker conditions, etc. Moreover, Uno and Katagiri [17] and Uno et al. [18] showed that optimal solutions of CFLPs in the plane cannot be found by directly applying heuristic solution methods for nonlinear programming problems, e.g. genetic algorithm for numerical optimization for constrained problem (GENOCOP) [10]. For solving the problems efficiently, we first show that one of their optimal solutions can be found by solving a 0-1 programming problems. Since the refor-

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mulated problems are NP-hard, we propose an efficient solution method for the problems. For discrete optimization problems, the tabu search algorithm, proposed by Glover [5], is one of the efficient solution algorithms; for the details of the tabu search algorithm, the reader can refer to the book of Reeves [15]. Hanafi and Freville [7] proposed an efficient tabu search approach for the 0-1 multidimensional knapsack problem, which is designed based on a strategic vibration. By utilizing characteristics of the CFLPs, we propose a solution method for the CFLPs improving their tabu search approach. Moreover, efficiency of the solution method is shown by applying to several examples of the CFLPs with random demands.

The remaining structure of this article is organized as follows. In Section 2, we formulate the CFLP with random demands as a stochastic programming problem, and for finding its solution, three deterministic programming problems are reformulated. Since the formulated problems cannot be solved directly, we show that one of their optimal solutions can be found by solving 0-1 programming problems in Section 3. In Section 4, we propose an efficient solution method based upon tabu search algorithms by utilizing characteristics of the CFLPs. We show the efficiency of the solution method by applying to numerical examples of the CFLPs with random demands in Section 5. Finally, in Section 6, concluding comments and future extensions are summarized.

2 Formulation of CFLP with random demands

In the proposed CFLPs, we assume that all customers only exist on DPs in plane R^2. For convenience sake, by aggregating all customers on the same DP, we regard one DP as one customer.

There are n DPs in R^2, and let $D = \{1, \dots, n\}$ be the set of indices of the DPs. Let m be the number of new facilities that the DM locates, and k be the number of competitive facilities which have been already located in R^2. The sets of indices of the new facilities and the competitive facilities are denoted by $F = \{1, \dots, m\}$ and $F_C = \{m+1, \dots, m+k\}$, respectively.

Let $\boldsymbol{u}_i \in R^2$ be the site of DP $i \in D$, and $\boldsymbol{x}_j \in R^2$ and $q_j > 0$ be the site and quality of facility $j \in F \cup F_C$, respectively. Then, attractive power of facility j for DP i is represented as the following function introduced by Huff [9]:

$$a_i(\boldsymbol{x}_j, q_j) \equiv \begin{cases} \dfrac{q_j}{||\boldsymbol{u}_i - \boldsymbol{x}_j||^2}, & \text{if } ||\boldsymbol{u}_i - \boldsymbol{x}_j|| > \varepsilon, \\ \dfrac{q_j}{\varepsilon^2}, & \text{if } ||\boldsymbol{u}_i - \boldsymbol{x}_j|| \le \varepsilon, \end{cases} \tag{1}$$

where $\varepsilon > 0$ is an upper limit of the distance that customers can move without any trouble. It is assumed that all customers only use one facility with the largest attractive power, and if the two or more attractive powers are the same, they use the facility in reverse numerical order of the indices of facilities; that is, in the order of competitive facilities and new facilities.

Let $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_m)$ be the location of the new facilities. Then we use the following 0-1 variable for representing whether DP i uses new facility $j \in F$:

$$\varphi_i^j(\boldsymbol{x}) = \begin{cases} 1, & \text{if DP } i \text{ uses the new facility } j, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Let $\bar{w}_i$ be the random variable meaning the buying power (BP) of DP i. New facility $j \in F$ can obtain the BP $\bar{w}_i$ if $\varphi_i^j(\boldsymbol{x}) = 1$. The objective of the DM is maximizing the sum of BP that all the new facilities obtain. Then, the CFLP with random demand is formulated as the following stochastic programming problem:

$$\left.\begin{array}{ll} \text{maximize} & f(\boldsymbol{x}) = \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{m} \bar{w}_i \varphi_i^j(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in R^{2m} \end{array}\right\} \tag{3}$$

For finding optimal solutions of (3), we consider the following three deterministic programming problems: (i) expectation maximizing problem

$$\left.\begin{array}{ll} \text{maximize} & E[f(\boldsymbol{x})] \\ \text{subject to} & \boldsymbol{x} \in R^{2n} \end{array}\right\} \tag{4}$$

(ii) probability maximizing problem

$$\left.\begin{array}{ll} \text{maximize} & Pr[f(\boldsymbol{x}) \ge f_0] \\ \text{subject to} & \boldsymbol{x} \in R^{2n} \end{array}\right\} \tag{5}$$

where f_0 means a given satisfying level of obtaining BP, and (iii) satisfying level maximizing problem

$$\left.\begin{array}{ll} \text{maximize} & f_0 \\ \text{subject to} & Pr[f(\boldsymbol{x}) \ge f_0] \ge \alpha \\ & \boldsymbol{x} \in R^{2n} \end{array}\right\} \tag{6}$$

where α is a given satisfying level of probability that the DM can obtain BP level f_0.

Problems (4), (5), and (6) are nonconvex nonlinear programming problems, and we need to find at least one optimal solution for each of the problems. In the next section, we show that the above three problems can be reformulated as 0-1 programming problems.

3 Reformulation to 0-1 programming problems

In the location model of the previous section, if the location of the new facilities are given, the values of (2) for all facilities and DPs are given, and then objective function values of (4), (5), and (6) can be computed. On the other hand, the outline of our solution method is as follows:

1. Give the set of DPs that the DM wants to obtain their BPs preferentially for each new facility, and
2. Find the location of all new facilities which can obtain BPs from the given all DPs, if any.

From (1), the set of DPs that new facility $j \in F$ cannot obtain their BPs wherever it is located can be represented as follows:

$$D_j^{\triangle} = \{i \in D \mid \sqrt{q_j / a_i^C} \leq \varepsilon\}, \tag{7}$$

where

$$a_i^C \equiv \min_{j \in F_C} \{a_i(\boldsymbol{x}_j, q_j)\}. \tag{8}$$

Then, the set of DPs that there is at least one location of new facility j which can obtain their BPs is denoted by $D_j = D \backslash D_j^{\triangle}$. For each new facility j, the DM gives the set of DPs $\bar{D}_j \subseteq D_j$ that she/he wants to obtain their BPs by locating it preferentially. Let

$$l_{ij} = \begin{cases} 1, & \text{if } i \in \bar{D}_j, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Then, $\bar{D}_j$ can be represented as 0-1 vector $\boldsymbol{l}_j = (l_{1j}, \ldots, l_{nj})$.

For new facility j and vector $\boldsymbol{l}_j$ given by the DM, we consider the following problem with an auxiliary variable $r_j \geq 0$:

$$\left.\begin{array}{ll} \text{minimize} & r_j^2 \\ \text{subject to} & ||\boldsymbol{x}_j - \boldsymbol{u}_i||^2 \leq \dfrac{q_j}{a_i^C} \cdot r_j, \\ & \qquad\qquad \forall i \in \{\bar{i} | l_{\bar{i}j} = 1\}, \\ & \boldsymbol{x}_j \in R^2, \ r_j \geq 0. \end{array}\right\} \tag{10}$$

Let $(\boldsymbol{x}_j^{\boldsymbol{l}_j}, r_j^{\boldsymbol{l}_j})$ be an optimal solution of (10). Then, the following theorem plays an important role to find an optimal location of the CFLP.

Theorem 1 *If $r_j^{\boldsymbol{l}_j} < 1$, the new facility j can obtain BPs from any DP i satisfying $l_{ij} = 1$ by locating it at $\boldsymbol{x}_j^{\boldsymbol{l}_j}$.*

PROOF: *For the constraint of (10) and $r_j^{\boldsymbol{l}_j} < 1$, $||\boldsymbol{x}_j^{\boldsymbol{l}_j} - \boldsymbol{u}_i||^2 < q_j / a_i^C$ is satisfied for any DP i satisfying $l_{ij} = 1$. Then, $a_i^C < q_j / ||\boldsymbol{x}_j^{\boldsymbol{l}_j} - \boldsymbol{u}_i||^2$ is satisfied. From (1), this relation means that the attractive power of new facility j is more than that of all competitive facilities if new facility j is located at $\boldsymbol{x}_l^{\boldsymbol{l}_j}$.* □

Note that because (10) is a convex programming problem, (10) can be solved by using the solution algorithms for convex programming problems, such as sequential quadratic programming (SQP) method; for the details of the SQP method, the reader can refer to the book of Nocedal and Wright [12]. From Theorem 1, the following corollary is derived for finding an optimal solution of the CFLP.

Corollary 2 *Let $L = {}^t(\boldsymbol{l}_1, \ldots, \boldsymbol{l}_m) \in \{0,1\}^{mn}$ and $\boldsymbol{x}^L = (\boldsymbol{x}_1^{\boldsymbol{l}_1}, \ldots, \boldsymbol{x}_m^{\boldsymbol{l}_m})$. Then, there exists L such that $\boldsymbol{x}^L$ is an optimal solution of (4), (5), and (6).*

PROOF: *Let $\boldsymbol{x}^*$ be an optimal solution of (4). We define the 0-1 matrix $\bar{L} = (\bar{\boldsymbol{l}}_1, \ldots, \bar{\boldsymbol{l}}_m) \in \{0,1\}^{mn}$, each of whose element for $i \in D$ and $j \in F$ is that $\bar{l}_{ij} = \varphi_i^j(\boldsymbol{x}^*)$. Then, from Theorem 1, $\boldsymbol{x}^{\bar{L}}$ is also an optimal solution of (4) because $\varphi_i^j(\boldsymbol{x}^{\bar{L}}) = \varphi_i^j(\boldsymbol{x}^*)$ for all i, j and $r_j^{\bar{\boldsymbol{l}}_j} < 1$ for all j. This is also shown for the cases of (5) and (6).* □

Let $r^L = \max\{r_1^{\boldsymbol{l}_1}, \ldots, r_m^{\boldsymbol{l}_m}\}$. From Corollary 2, finding an optimal solution of (4), (5), and (6) can be formulated as the following 0-1 programming problems respectively:

$$\left.\begin{array}{rl} \text{maximize} & E[f(\boldsymbol{x}^L)] \\ \text{subject to} & r^L < 1, \\ & L \in \{0,1\}^{mn} \end{array}\right\} \tag{11}$$

$$\left.\begin{array}{rl} \text{maximize} & Pr[f(\boldsymbol{x}^L) \geq f_0] \\ \text{subject to} & r^L < 1, \\ & L \in \{0,1\}^{mn} \end{array}\right\} \tag{12}$$

$$\left.\begin{array}{rl} \text{maximize} & f_0 \\ \text{subject to} & r^L < 1, \\ & Pr[f(\boldsymbol{x}^L) \geq f_0] \geq \alpha, \\ & L \in \{0,1\}^{mn} \end{array}\right\} \tag{13}$$

Because the number of solving (10) for finding an optimal 0-1 matrix is 2^{mn}, the above three problems are NP-hard. In the next section, we propose an efficient solution method for the problems.

4 Tabu search algorithm with strategic vibration

Tabu search is one of the local search methods. In our solution method, we define a move from a current solution, denoted by L^{now}, as an increase or a decrease of its one element. The neighborhood of a current solution of (11), (12), or (13) is represented as a set of all solutions that can be transferred by only one move from the current solution. In the tabu search including our solution method, the next searching solution from L^{now}, denoted by L^{next}, is basically chosen to the best solution for given criteria, e.g. objective function value, in the neighborhood of L^{now}. However, if we use such a search without modification, a circulation of certain chosen moves occurs after a local optimal solution is found, and then it can only search in a narrow part of the feasible set. For preventing such a circulation, if a move is chosen in the search, the tabu constraint for its opposite move is activated for given terms, called the tabu term and denoted by T_1. Then the activated moves are forbidden to choose in T_1 terms, called tabu, even if the moves make the objective function value best in all solutions in a neighborhood. Such tabu moves are memorized in the tabu list for the search.

Although the tabu search method has advantage for searching in local areas intensively, there are generally many local optimal solutions of (11), (12), or (13). For searching an optimal solution efficiently, we propose the solution method by utilizing characteristics of CFLP.

First, we introduce an important theorem showing a similarity between the 0-1 programming problems in the previous section and multidimensional knapsack problems. Let $\boldsymbol{l}_j^{k+} := \boldsymbol{l}_j + \boldsymbol{e}^k$, where $\boldsymbol{e}^k$ is the k-th unit vector, and $L_j^{k+} := {}^t(\boldsymbol{l}_1, \ldots, \boldsymbol{l}_j^{k+}, \ldots, \boldsymbol{l}_m)$.

Theorem 3 *Let $L \in \{0,1\}^{mn}$ be the matrix that $\bar{l}_{ij} = \varphi_i^j(\boldsymbol{x}^L)$ for all i, j and $\boldsymbol{x}^L$ be an optimal solution of (11), (12), or (13). Then, if there exists DP $k \in D$ satisfying $l_{kj} = 0$ for any new facility j, $r^{\boldsymbol{l}_j^{k+}} \geq 1$ for any j.*

PROOF: *We assume that there exists k, j such that $r^{\boldsymbol{l}_j^{k+}} < 1$. Then, from Corollary 2, the DM can obtain BP of DP k by locating facility j at $\boldsymbol{x}^{\boldsymbol{l}_j^{k+}}$, adding to the BPs that she/he can obtain by locating facility j at $\boldsymbol{x}^{\boldsymbol{l}_j}$. This contradicts the fact that $\boldsymbol{x}^L$ is an optimal solution.* □

From Theorem 3, an optimal solution of the three 0-1 programming problems exists on the neighborhood of their common constraint $r^L < 1$. This is similar to the multidimensional knapsack problem whose optimal solution exists on the neighborhood of its constraints.

Moreover, the multidimensional knapsack problem has the characteristic that if any element of solution is changed from zero to one, the objective function value is improved. Similarly, for the three 0-1 programming problems, if L and L_j^{k+} hold that $r^L < 1$ and $r^{L_j^{k+}} < 1$, $\boldsymbol{x}^{L_j^{k+}}$ is mostly superior to $\boldsymbol{x}^L$. This is because (10) for L_j^{k+} includes the constraint for obtaining BP of DP k, adding to the constraints of (10) for L.

From these two characteristics, the solution methods for multidimensional knapsack problems are also efficient for (11), (12), and (13). For multidimensional knapsack problems, Hanafi and Freville [7] proposed an efficient solution method based upon the tabu search algorithm with strategic vibration. We apply their solution method to the problems with some modifications for the CFLP. Then, our proposing solution method is described as follows:

Tabu search algorithm with strategic vibration

Step 0: Generate the initial searching solution L^{now}, and initialize the tabu list and other variables. If $r^{L^{\text{now}}} < 1$, then go to Step 4.

Step 1: Move L^{now} to L^{next} by decreasing an element of L^{now} with the purpose of decreasing $r^{L^{\text{next}}}$ as much as possible. This step is repeated until it is satisfied $r^{L^{\text{next}}} < 1$.

Step 2: Move L^{now} to L^{next} with the purpose of improving the objective function value of (11), (12), or (13). This step is repeated at given certain terms, denoted by T_2.

Step 3: Move L^{now} to L^{next} by decreasing an element of L^{now} with the purpose of decreasing $r^{L^{\text{next}}}$ as much as possible. This step is repeated until $r^{L^{\text{next}}}$ is less than a certain vector, denoted by r^{low}.

Step 4: Move L^{now} to L^{next} by increasing an element of L^{now} with the purpose of improving the objective function value of (11), (12), or (13). This step is repeated until it is not satisfied $r^{L^{\text{next}}} < 1$.

Step 5: Do the same operations as Step 2.

Step 6: Move L^{now} to L^{next} by increasing an element of L^{now} with the purpose of improving the objective function value of (11), (12), or (13). This step is repeated until $r^{L^{\text{next}}}$ is more than a certain vector, denoted by r^{upp}.

Step 7: If given terminal conditions are satisfied, then this algorithm is terminated. The obtaining approximate solution is the best solution about the objective function value of (11), (12), or (13) in all searched solutions. Otherwise, return to Step 1.

5 Numerical experiments

In this section, we show the efficiency of the solution algorithm in the previous section by applying to three examples of the CFLPs. In these examples, the numbers of DPs are $n = 30, 40, 50$. The sites of DPs $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ are given in $[0, 100] \times [0, 100]$ randomly. Their random BPs $\bar{w}_1, \ldots, \bar{w}_n$ are represented as random variables each of which has three scenarios whose probabilities are 0.5, 0.3, and 0.2, and BP of each DP for each scenario is given in $[5, 12]$ randomly. We give fifteen competitive facilities, that is $k = 15$, and for competitive facility $j \in F_C$, its site $\boldsymbol{x}_j$ and quality q_j are randomly given in $[0, 100] \times [0, 100]$ and $\{1, \ldots, 5\}$, respectively. In this plane, the DM locates one facility, that is $m = 1$, whose quality is that $q_1 = 3$. For (12) and (13), we give $f_0 = 30 + n$ and $\alpha = 0.8$.

Next, we give parameters about our solution method. We set the tabu term $T_1 = n/2 - 10$. At Step 2, we set that $T_2 = 10$. At Steps 3 and 6, we set that $r^{\text{low}} = 0.3$ and $r^{\text{upp}} = 3$. The terminal condition at Step 7 is satisfied if the tabu search algorithm is iterated at more than 10 times.

For showing the efficiency of our solution method, we compare its computational results to that of the genetic algorithm; for the details of the genetic algorithms, the readers can refer to the study of Sakawa et al. [16]. We set generation gap $G = 0.9$, population size $N_{GA} = 150$, and terminal generation $T_{GA} = 2000$. Probabilities of crossover, mutation, and inversion are $p_C = 0.9$, $p_M = 0.01$, and $p_I = 0.03$, respectively.

We apply the tabu search algorithm and the genetic algorithm to three examples of the CFLPs, where each of these algorithms is implemented 20 times for each example by using DELL Optiplex GX620 (CPU: Pentium(R) 4 2.33GHz, RAM: 512MB). The computational results of solving the CFLPs are shown in Tables 1-6. From Tables 1-6, the tabu search algorithm can obtain better solutions for (11), (12), and (13) than those of the genetic algorithm with shorter computational times. This means that our solution method is efficient for the CFLPs with random demands.

Table 1: Computational results by the tabu search algorithm with strategic vibration for (11)

n	30	40	50
Best	64.87	67.79	86.77
Mean	64.87	67.79	86.77
Worst	64.87	67.79	86.77
CPU times (sec)	9.83	20.73	48.28

Table 2: Computational results by the genetic algorithm for (11)

n	30	40	50
Best	64.87	67.79	86.77
Mean	64.87	66.30	84.78
Worst	64.87	64.54	70.07
CPU times (sec)	47.86	52.93	69.14

6 Conclusions and future researches

In this paper, we have proposed a new CFLP on the plane with random demands. We have formulated the CFLP as a stochastic programming problem, and for finding an optimal solution of the problem, the three deterministic programming problems: expectation maximizing problem, probability maximizing problem, and satisfying level maximizing problem are considered. Because these problems cannot be solved directly, we have shown that one of their optimal solutions can be found by solving 0-1 programming problems. Since the 0-1 programming problems are NP-hard, we have proposed an efficient solution method based upon the tabu search algorithm with strategic vibration by utilizing characteristics of the CFLPs. Efficiency of the solution method is shown by applying to several examples of the CFLPs.

These three reformulated deterministic programming problems have the characteristic that the more the DPs whose BPs are obtained by the new facilities are increased, the more their objective function values are improved. We can improve efficiency of our solution method by utilizing the characteristic. However, if the CFLP with random demands is reformulated to deterministic programming problems for considering risk, e.g. variance and VaR minimizing problems, these problems do not necessarily have the characteristic and then cannot be found their optimal solutions by applying our solution method. Therefore, to propose an efficient solution method for such problems is a future study.

Table 3: Computational results by the tabu search algorithm with strategic vibration for (12)

n	30	40	50
Best	0.7	0.5	0.7
Mean	0.7	0.5	0.7
Worst	0.7	0.5	0.7
CPU times (sec)	10.41	22.73	53.70

Table 4: Computational results by the genetic algorithm for (12)

n	30	40	50
Best	0.7	0.5	0.7
Mean	0.7	0.5	0.64
Worst	0.7	0.5	0.5
CPU times (sec)	47.86	59.20	77.79

Table 5: Computational results by the tabu search algorithm with strategic vibration for (13)

n	30	40	50
Best	45.86	48.20	58.96
Mean	45.86	48.20	58.96
Worst	45.86	48.20	58.96
CPU times (sec)	11.47	29.20	48.81

Table 6: Computational results by the genetic algorithm for (13)

n	30	40	50
Best	45.86	48.20	58.96
Mean	45.66	47.23	56.27
Worst	44.85	44.85	50.10
CPU times (sec)	54.11	61.70	80.28

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