Note on an $\langle s, S \rangle$ Inventory System with Decay

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Abstract

We analyze a single-product continuous review $\langle s, S \rangle$ inventory system in which the stock-on-hand decays over time, and the demand stream occurs in a Poisson process. Our model allows for the ordering cost to depend on whether an order is initiated by decay to the order point level s, or by a demand that causes the stock-on-hand to jump below level s. We obtain the steady-state probability distribution of the stock- on-hand, develop a cost function, and determine related quantities.

Keywords: $\langle s, S \rangle$ inventory with decay, probability distribution of stock-on-hand, order-size dependent costs, WhittakerM function, level crossing method

1. Introduction

Inventory models with decay have attracted attention recently (e.g., [4]). In the present paper we consider an $\langle s, S \rangle$ inventory system with continuous stock decay. Our model allows the replenishment cost (order cost) to depend on the order size. For exposition, we assume that demands are satisfied immediately, and replenishment orders are received immediately. The order-up-to-S policy implies that an order initiated by decay to level s is smaller than an order initiated by a demand that causes a deficit below level s. Each order caused by decay will be of size $S - s$. Each order initiated by a demand will be of size $S-s+\gamma$ where γ is the deficit below level s. We obtain the steady-state probability density function (pdf) of the stock-on-hand, develop a cost function, and compute related quantities. For exposition, we do not incorporate lead time, backlogging, lost sales, etc., into the present model. However, such generalizations can be analyzed using the method presented (see, e.g., [1], [2]).

2. Model Description

Assume demands occur at a Poisson rate λ. Denote the demand sizes by $D_i, i = 1, 2, ...,$ which are iid (independent and identically distributed) with common cdf (cumulative distribution function) $B(x)$. Let $\bar{B}(x) = 1 - B(x), x \geq 0$. Denote the stock-on-hand at time t by $I(t), t \geq 0$. The decay rate depends on the current inventory level. Thus $\frac{dI(t)}{dt} = -r(I(t)) < 0$, $I(t) \in (s, S]$. If the stock decays to level s, or jumps downward to, or below level s due to a demand, then an order is placed with the supplier and is received immediately, replenishing the stock up to level S. A sample path of $\{I(t)\}$ is depicted in Fig. 1. The leading point of the sample path is called the system point (SP). The SP is convenient for describing the motion of the sample path as it evolves over time.

The limiting distribution of $I(t)$ as $t \to \infty$ exists, since $\{I(t)\}$ is a bounded Markov process. Let $F(x)$, $x \leq S$, denote the steady-state cdf; $f(x) = \frac{dF(x)}{dx}$, $s < x < S$, is the pdf, wherever the derivative exists. We assume the decay rate is $r(x) = kx > 0, x \in (s, S]$. This choice of $r(x)$ results in a negative exponential decay pattern for the sample path, and generalizes the $\langle s, S \rangle$ model with constant decay rate (e.g., [1], [3]). We illustrate the solution technique by taking $\bar{B}(x) = e^{-\mu x}, x > 0, \mu > 0$.

3. Integral Equation for PDF of Inventory

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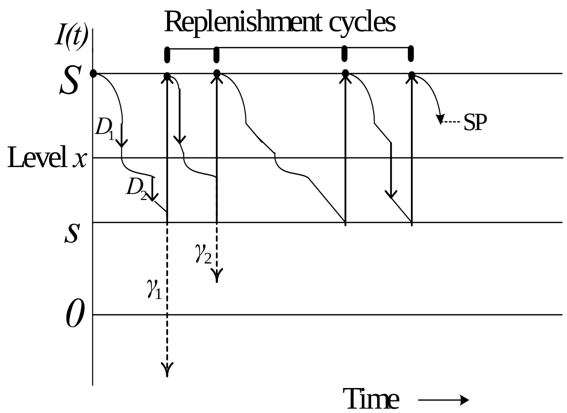

Figure 1: Sample path of stock-on-hand in $\langle s, S \rangle$ inventory model with continuous decay of stock.

A sample path of $\{I(t)\}$ is similar to Fig. 1, with negative exponential decay curves between jumps. Fix level $x \in (s, S)$. Let $\mathcal{D}_t^c(x)$ = number of continuous downcrossings of x during the time interval $(0, t)$. Let $\mathcal{D}_t^j(x)$ = number of jump downcrossings of x during $(0, t)$. The SP *decays* into level $x \in (s, S)$ at rate

$$\lim_{t\to\infty} \frac{E\left(\mathcal{D}_t^c(x)\right)}{t} = r(x)f(x) = kxf(x),$$

and into level s at rate

$$\lim_{t\to\infty} \frac{E\left(\mathcal{D}_t^c(s)\right)}{t} = r(s)f(s) = ksf(s)$$

(see, e.g., [3]). The SP *jumps* over level x, to a level below $x \in [s, S)$ due to demands at rate

$$\lim_{t\to\infty} \frac{E\left(\mathcal{D}_t^j(x)\right)}{t} = \lambda \int_{y=x}^{S} \bar{B}(y-x)f(y)dy = \lambda \int_{y=x}^{S} e^{-\mu(y-x)}f(y)dy$$

(jumps start at $y > x$ and span a distance greater than $y - x$). The *total* SP *downcrossing* rate of level $x \in (s, S)$ is

$$\lim_{t\to\infty} \frac{E\left(D_t^c(x)\right)}{t} + \lim_{t\to\infty} \frac{E\left(\mathcal{D}_t^j(x)\right)}{t} = r(x)f(x) + \lambda \int_{y=x}^{S} \bar{B}(y-x)f(y)dy = kxf(x) + \lambda \int_{y=x}^{S} e^{-\mu(y-x)}f(y)dy. \quad (1)$$

The total "downcrossing" rate of the reorder point s is the constant

$$\lim_{t\to\infty} \frac{E(\mathcal{D}_t(s))}{t} = r(s)f(s) + \lambda \int_{y=s}^{S} \overline{B}(y-s)f(y)dy = r(S)f(S);$$
$$\text{thus } ksf(s) + \lambda \int_{y=s}^{S} e^{-\mu(y-s)}f(y)dy = kSf(S). \quad (2)$$

The SP downcrossing rate of level s is equal to the SP egress rate out of level S, due to the one-to-one correspondence between orders up to S and downcrossings of the order point level s. Rate balance into and out of state $\{S\}$ yields the second and third equalities in (2).

An important feature of the sample path structure is that the total SP *upcrossing* rate of *every* level $x \in (s, S]$ is equal to the total *downcrossing* rate of level s = the total ordering rate. Applying sample-path rate balance across level x by equating (1) and (2) yields an integral equation for $f(x)$, namely

$$kxf(x) + \lambda \int_{y=x}^{S} e^{-\mu(y-x)}f(y)dy = ksf(s) + \lambda \int_{y=s}^{S} e^{-\mu(y-s)}f(y)dy = kSf(S), x \in (s, S]. \quad (3)$$

The normalizing condition is

$$\int_{x=s}^{S} f(x)dx = 1. \quad (4)$$

4. The PDF of Inventory

In (3) differentiation with respect to x and some algebra yields a first order differential equation with non-constant coefficients,

$$kxf'(x) + (k - \mu kx - \lambda) f(x) + \mu kSf(S) = 0. \quad (5)$$

As an illustration of the analysis, we solve (5) for arbitrary parameter values $k = 3.9$, $\mu = 1.5$, $\lambda = 1.2$, $s = 0.25$, $S = 4$. The solution is (with the aid of

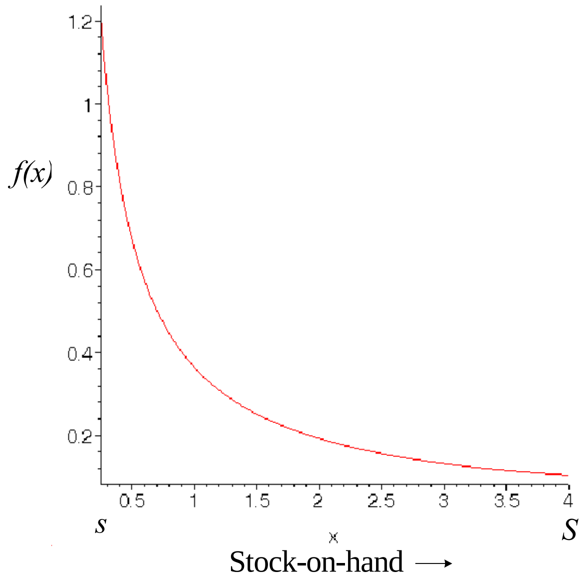

Figure 2: Steady-state pdf of stock-on-hand: $k = 3.9$, $\mu = 1.5$, $\lambda = 1.2$, $s = 0.25$, $S = 4$

Maple software), for $s < x < S$,

$$\begin{aligned} f(x) &= .224467 \cdot 10^{-19}(-.206059 \cdot 10^{20} \cdot x^{(9/26)} \\ &\cdot WhittakerM(.346154, .846154, 1.5x) \\ &-.401260 \cdot 10^{20} \cdot e^{(-.75x)} \cdot x^{(9/13)} \\ &+.274942 \cdot 10^{20} \cdot e^{(.75x)}) \cdot e^{(.75x)}/x^{(9/13)}, \end{aligned} \tag{6}$$

where the function WhittakerM(β, ν, z) solves the differential equation

$$y'' + \left(-\frac{1}{4} + \frac{\beta}{z} + \frac{\frac{1}{4} - \nu^2}{z^2}\right) y = 0.$$

A plot of $f(x)$ is given in Fig. 2. It can be shown that in (6) $f(x)$ is convex since $f''(x) > 0, x \in (s, S)$.

Figure 2 shows that the probability accumulates near level s. A factor contributing to this effect is that the decay rate kx is small for values of x near s. Thus the inventory spends a relatively large proportion of time at levels near s. Assigning different parameter values will give different shapes to the pdf. The ordering policy orders up to S and tends to maintain the inventory level near S. This interplay between opposing tendencies differs from the case when the decay rate is constant for all $x \in (s, S)$ (see, e.g., [1], [3]).

5. Model Characteristics

We present several model characteristics based on the foregoing analysis.

5.1. Expected Duration of an Order Cycle

An **order cycle** (replenishment cycle) is the time between two successive instants when an order is *received*. Let $d_S^{(i)}$ = the duration of the i^{th} order cycle. Then $d_S^{(i)}$ is the time between successive SP hits of level S (see Fig. 1). The sequence $\left\{d_S^{(i)}\right\}$ is a renewal process due to the Poisson demand stream. Let $d_S^{(i)} \underset{dist}{\equiv} d_S$. The total ordering rate is equal to the rate at which the SP downcrosses level s and consequently hits level S from below. It is equal to the egress rate out of level S, namely $kSf(S)$, the right side of (3). By the elementary renewal theorem (e.g., [5]),

$$\begin{aligned} E(d_S) &= \frac{1}{kSf(S)} = \frac{1}{1.621253} \\ &= .6168069 \text{ time units}, \end{aligned} \tag{7}$$

where the *total* order *rate* is $k\mathrm{S}f(S) = 1.621253$ orders per unit time. With the given parameter values, the order rate turns out to be faster than the demand rate. A plausible explanation is the relatively fast decline in inventory due to decay when the stock-on-hand is in the range $(1, 4)$ (discussed further on Subsection 5.2).

5.2. Two Types of Orders

A **type-c** order occurs when the SP *decays continuously into level s*. A **type-j** order occurs when the SP *jumps downward and ends at or below s due to a demand*. An order initiating a cycle of duration d_S is either type-c or type-j. Let $P_c = P$(the order initiating a cycle is type-c). Let $P_j = P$(the order initiating a cycle is type-j). Then $P_c + P_j = 1$.

We now determine P_c and P_j. The long-run ordering rate due to decay into level s is $\lim_{t\to\infty} \frac{E(\mathcal{D}_t^c(s))}{t} = ksf(s)$. Since there is exactly one order in a cycle,

$$\begin{aligned} &E(\text{number of type-c orders in a cycle}) \\ &= 1 \cdot P_c + 0 \cdot P_j = P_c. \end{aligned}$$

By the theory of regenerative processes,

$$\frac{E(\text{number of type-c orders in a cycle})}{E(\text{duration of a cycle })} = \frac{P_c}{E(d_S)}$$
$$= \lim_{t\to\infty} \frac{E(\mathcal{D}_t^c(s))}{t} = ksf(s).$$

Solving for P_c gives

$$P_c = ksf(s)\cdot E(d_S) = \frac{ksf(s)}{kSf(S)} = \frac{sf(s)}{Sf(S)}. \qquad (8)$$

In (8) the numerator of $\frac{ksf(s)}{kSf(S)}$ is the rate of type-c orders; the denominator is the overall rate of orders. The ratio is the proportion of orders that are type-c.

Similarly,

$$\frac{E(\text{number of type-j orders in a cycle})}{E(\text{duration of a cycle})}$$
$$= \frac{1.P_j + 0\cdot P_c}{E(d_S)} = \lim_{t\to\infty}\frac{E(\mathcal{D}_t^j(s))}{t}.$$

Thus

$$P_j = \lim_{t\to\infty}\frac{E(\mathcal{D}_t^j(s))}{t}\cdot E(d_S) = \frac{\lambda\int_{y=s}^{S} e^{-\mu(y-s)}f(y)dy}{kSf(S)}. \qquad (9)$$

In (9) the numerator is the rate of type-j orders; the denominator is the overall rate of orders. The ratio is the proportion of orders that are type-j. In our example we obtain P_c =.729743 and P_j =.270257. These values bolster the explanation of why the order rate is less than the demand rate. That is, approximately 73% of orders are initiated by decay, and only 27% by demands.

5.3. Expected Order Size

Denote the order size by R (Fig. 1). If an order is caused by decay into level s, then $R = S - s$. If an order is caused by a downward jump ending below level s, then $R = S - s + \gamma$. Due to exponentially distributed demand sizes with mean $\frac{1}{\mu}$, the expected order size is (P_j is given in (9))

$$E(R) = (S-s)P_c + \left(S - s + \frac{1}{\mu}\right)P_j$$
$$= S - s + P_j\cdot\frac{1}{\mu} = 3.930171 \text{ units.} \qquad (10)$$

5.4. Cost Rate

Since there is no backlogging or lead-time costs in the model considered here, the cost function includes only the setup cost of placing orders, and the holding cost of inventory. Let $\mathcal{C}$ be the total average *cost rate*, $\mathcal{C}_{Oc}$ the setup cost per order when initiated by continuous product decay to level s, and C_{Oj} the setup cost per order when initiated by a demand. Let $\mathcal{C}_H$ be the holding cost per unit per unit time. Using $f(x)$ in (6), we obtain

$$\begin{aligned} C &= \mathcal{C}_{Oc}ksf(s) + \mathcal{C}_{Oj}\lambda\int_{x=s}^{S} e^{-(x-s)\mu}f(x)dx \\ &\quad +\mathcal{C}_H\int_{x=s}^{S} xf(x)dx, \\ &= 1.183099\mathcal{C}_{Oc} + .438154\mathcal{C}_{Oj} + 3.930171\mathcal{C}_H \end{aligned} \qquad (11)$$

6. Future Work

Potential future work includes: a sensitivity analysis for the effect of varying the model parameter values, on the analytical properties of the pdf $f(x)$; a comparison of the characteristics of $f(x)$ when the decay rate is $r(x) = kx$ versus $r(x) = m$ (a constant); determination of the expected number of orders in an order cycle.

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