Lower Bounds of Ramsey Numbers $R(k,l)$

Decha Samana and Vites Longani

***Abstract*—For positive integers k and l, the Ramsey number $R(k,l)$ is the least positive integer n such that for every graph G of order n, either G contains K_k as a subgraph or $\overline{G}$ contains K_l as a subgraph. In this paper it is shown that Ramsey numbers**

$$R(k,l) \geq 2kl - 3k - 3l + 6 \quad \text{when } 3 \leq k \leq l,$$

and

$$R(k,l) \geq 2kl - 3k + 2l - 12 \quad \text{when } 5 \leq k \leq l.$$

***Index Terms*—Ramsey numbers, lower bounds, graph.**

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I. Introduction

For positive integers k and l, Ramsey number $R(k,l)$ is the least positive integer n such that for every graph G of order n, either G contains K_k as a subgraph or $\overline{G}$ contains K_l as a subgraph. Some known $R(k,l)$ are shown in the table [1]:

Table 1: Some known $R(k,l)$

k \ l	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4		18	25				

For upper bounds, Erdös and Szekeres [2] have shown that

$$R(k,l) \leq \binom{k+l-2}{k-l}, \text{ for } k \geq 1, l \geq 1.$$

Some know results of $R(k,l)$, in recurrence forms, are described in Lemma 1 and Lemma 2.

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Lemma 1: [3] For $k,l \geq 3$,

$$R(k,l) \geq R(k,l-1) + 2k - 3.$$

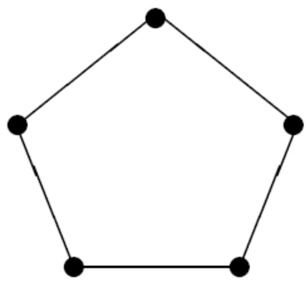
Lemma 2: [4] For $l \geq 5$, $k \geq 2$,

$$R(2k-1,l) \geq 4R(k,l-1) - 3.$$

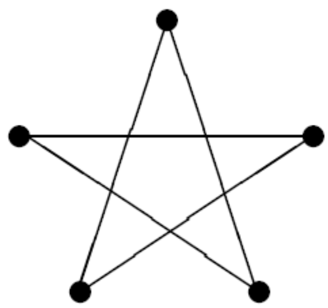
II. Lower Bound of $R(k,l)$

First, we define cycle-power C_n^d for the proof of Lemma 3 from which the main results could be derived.

The cycle-power C_n^d is constructed by placing n vertices on a circle and making each vertex adjacent to d nearest vertices in each direction on the circle. See Figure 1 and 2, for the examples of C_5^1 and C_8^2.

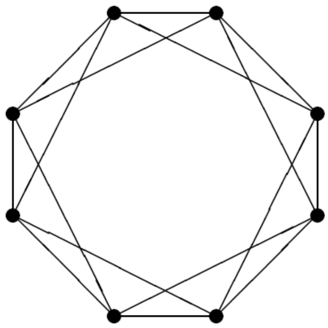


C_5^1

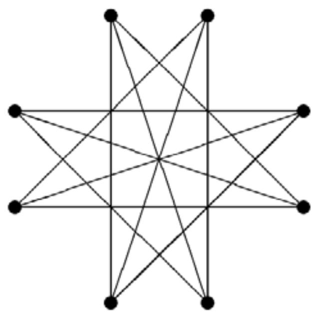


$\overline{C}_5^1$

Figure 1: Cycle-power C_5^1 and $\overline{C}_5^1$



C_8^2



$\bar{C}_8^2$

Figure 2: Cycle-power C_8^2 and $\bar{C}_8^2$

Lemma 3: For $k \geq 3$,

$$R(3,k) \geq 3(k-1) .$$

Proof: Let $\{1,2,3,\ldots,3k-4\}$ be the points of the cycle C_{3k-4}. We say that the line $\{i,j\}$ has line distance l_{ij} if the distance of the two points i and j of C_{3k-4} is equal to l_{ij}. For example, the line $\{1,4\}$ in Figure 3 has line distance 3. From the definition of cycle-power C_{3k-4}^{k-2}, the point 1 is adjacent to the $k-2$ nearest vertices in each direction on the circle. From the definitions, the $2(k-2)$ lines of C_{3k-4}^{k-2} that are adjacent to 1 have distances 1, 2, 3, …, or $k-2$.

Also, there are $(3k-4)-2(k-2)-1=k-1$ consecutive points of C_{3k-4}^{k-2} that are not adjacent to the point 1. We note that the lines that join each pair of these consecutive $k-1$ points have line distance 1, 2, 3, …, or $k-2$, and so these lines are lines of C_{3k-4}^{k-2}. We shall use this note in the second part of the proof. For example when $k=5$, see Figure 3 for the lines of C_{3k-4}^{k-2} and $\bar{C}_{3k-4}^{k-2}$ that are adjacent to 1.

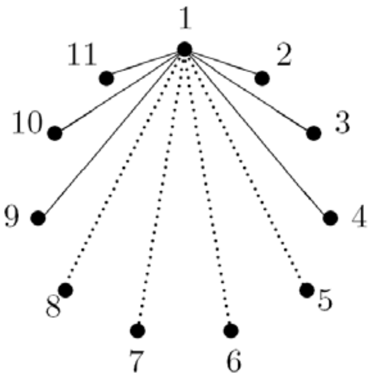


Figure 3: Cycle-power C_{3k-4}^{k-2} , $k=5$

First, we want to show that there is no K_k in C_{3k-4}^{k-2}, and then we shall show that there is no K_3 in $\bar{C}_{3k-4}^{k-2}$. Suppose there is K_k in C_{3k-4}^{k-2}. Due to the symmetry of C_{3k-4}^{k-2}, it is without lost of generality if we say that the point 1 is a point of a K_k. Therefore, the k points of this K_k are the point 1 and some $k-1$ points among the $2(k-2)$ points that are adjacent to 1. The k points of K_k are on the circle C_{3k-4} and so some two of these points has line distance greater than $k-2$. This is a contradiction, since the line distances of lines in C_{3k-4}^{k-2} are 1, 2, 3,…, or $k-2$. Therefore, there is no K_k in C_{3k-4}^{k-2}.

Next, we show that there is no K_3 in $\bar{C}_{3k-4}^{k-2}$. Suppose there is K_3 in $\bar{C}_{3k-4}^{k-2}$. Again, it is without lost of generality if we say that 1 is a point of one of K_3 in $\bar{C}_{3k-4}^{k-2}$. Since, in $\bar{C}_{3k-4}^{k-2}$, 1 is adjacent to the $k-1$ consecutive points, so the point 1 with some two points from these $k-1$ points form a K_3. This is a contradiction, since we have noted that the lines formed by these $k-1$ points are lines of C_{3k-4}^{k-2} only. So, there is no K_3 in $\bar{C}_{3k-4}^{k-2}$.

Hence, we have shown that there are no K_k in C_{3k-4}^{k-2}, and no K_3 in $\bar{C}_{3k-4}^{k-2}$.

Therefore $\quad R(k,3) > 3k-4$

or $\quad R(3,k) \geq 3(k-1)$

■

Some lower bounds of $R(3,l)$, using Lemma 3, are shown in the following table:

Table 2: Some lower bounds of $R(3,l)$ using Lemma 3

l / k	3	4	5	6	7	8	9	10	11
3	6	9	12	15	18	21	24	27	30

Using Lemma 1 and Lemma 3, we can derive Theorem 1.

Theorem 1: For $3 \le k \le l$,

$$R(k,l) \ge 2kl - 3k - 3l + 6$$

Proof: From Lemma 1 and Lemma 3, and using $R(k,l) = R(l,k)$, we have

$$\begin{aligned} R(k,l) &\ge R(l,k-1) + 2l - 3 \\ &\ge R(l,k-2) + +2l - 3 + 2l - 3 \\ &\vdots \\ &\ge R(l,k-i) + i(2l-3), \qquad i = 1,2,...,k-3 \\ &\vdots \\ &= R(l,3) + (k-3)(2l-3) \\ &\ge 3(l-1) + (k-3)(2l-3), \qquad 3 \le k \le l \\ &= 2kl - 3k - 3l + 6\,. \end{aligned}$$

Therefore

$$R(k,l) \ge 2kl - 3k - 3l + 6 \quad \text{for} \quad 3 \le k \le l\,.$$

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Some lower bounds of $R(k,l)$, using Theorem 1, are shown in the following table:

Table 3: Some lower bounds of $R(k,l)$ using Theorem 1

l / k	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	12	15	18	21	24	27	30	33	36	39	42
4		14	19	24	29	34	39	44	49	54	59	64	69
5			26	33	40	47	54	61	68	75	82	89	96

Also, using Lemma 1, Lemma 2, and Lemma 3, we can derive Theorem 2

Theorem 2: For $5 \le k \le l$

$$R(k,l) \ge 2kl - 3k + 2l - 12$$

Proof: From Lemma 1, Lemma 2 and Lemma 3, we have

$$\begin{aligned} R(k,l) &\ge R(l,k-1) + 2l - 3 \\ &\ge R(l,k-2) + +2l - 3 + 2l - 3 \\ &\vdots \\ &\ge R(l,k-i) + i(2l-3), \qquad i = 1,2,...,k-5 \\ &\vdots \\ &\ge R(l,5) + (k-5)(2l-3) \\ &\ge 4R(3,l-1) - 3 + (k-5)(2l-3), \quad 5 \le k \le l \\ &= 2kl - 3k + 2l - 12\,. \end{aligned}$$

Therefore

$$R(k,l) \ge 2kl - 3k + 2l - 12, \quad \text{for} \quad 5 \le k \le l\,.$$

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Some lower bounds of $R(k,l)$, using Theorem 2, are shown
in the following table:

Table 4: Some lower bounds of $R(k,l)$ using Theorem 2

l / k	5	6	7	8	9	10	11	12	13	14	15
5	33	45	57	69	81	93	105	117	129	141	153
6		54	68	82	96	110	124	138	152	166	180
7			79	95	111	127	143	159	175	191	207
8				108	126	144	162	180	198	216	234
9					144	164	184	204	224	244	264
10						178	200	222	244	266	288

We note that Theorem 2 can generally give better results than those from Theorem 1 when $5 \le k \le l$. However, Theorem 1 could not provide results when $k < 5$ or $l < 5$.

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