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Characterization of lp-Summing Sublinear Operators

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Abstract—**In this paper, we introduce the concept of l_p-summing sublinear operators in the non commutative case and characterize this class of operators by giving the extension of the Pietsch domination theorem. Some properties are shown.**

Keywords: Banach lattice, completely bounded operator, operator space, l_p-summing operator, sublinear operator

1 Introduction

In the recent theory of operator spaces (or non-commutative Banach spaces) developed by [2] [3] [5] [6] [7] [8] [9] [11] [12] [13]; bounded operator is replaced by completely bounded operator, isomorphism by complete isomorphism and Banach space by operator space.

Precisely, we view in this new category every Banach space as a subspace of $B(H)$ for some Hilbert space H ($B(H)$ is the Banach space of all bounded linear operators on H) which is non-commutative, instead of viewing them as a subspace of $C(K)$ (the space of all continuous functions on a compact K) which is commutative.

The abstract characterization given in [13] signed the beginning of this theory. In [10] Pisier constructed the operator Hilbert space OH (i.e. the unique space verifying $\overline{OH^{\star}} = OH$ completely isometrically as in the case of Banach spaces because there are Hilbert spaces in this category which are non completely isometrically) and generalized in [11] (also Junge [8]) the notion of $p-$summing operators to the non-commutative case.

Ms. T. Belaib & L. Mezrag, generalized in [4] the notion of $p-$summing operators to the $p-$summing sublinear operators.

In this paper, we generalize this concept (of $p-$summing sublinear operators) in the non commutative case. We characterize this type of operators by given the extension of the Pietsch domination theorem. In the proof we using a deferent and necessary proposition concerning this notion of operators.

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2 Preliminaries

In this section, we recall some basic definitions and properties concerning the notion of sublinear operators and the theory of operator spaces (we consider that the reader is familiarized with this category).

If H is a Hilbert space, we let $B(H)$ denote the space of all bounded operators on H and for every n in $\mathbb{N}$ we let M_n denote the space of all $n \times n-$matrices of complex numbers, i.e., $M_n = B(l_2^n)$.

If X is a subspace of some $B(H)$ and $n \in \mathbb{N}$, then $M_n(X)$ denotes the space of all $n \times n-$matrices with $X-$valued entries which we in the natural manner consider as a subspace of $B(l_2^n(X))$.

Definition 2.1. An operator space X is a norm closed subspace of some $B(H)$ equipped with the distinguished matrix norm inherited by the spaces $M_n(X)$, $n \in \mathbb{N}$.

An operator space which is a Banach lattice (resp. complete Banach lattice) is called a quantum Banach lattice (resp. quantum complete Banach lattice).

Let H be a Hilbert space. We denote by $S_p(H)$ $(1 \leq p < \infty)$ the Banach space of all compact operators $u : H \longrightarrow H$ such that $Tr(|u|^p) < \infty$, equipped with the norm

$$\|u\|_{S_p(H)} = (Tr(|u|^p))^{\frac{1}{p}}.$$

If $H = l_2$ (resp. l_2^n), we denote simply $S_p(l_2)$ by S_p (resp. $S_p(l_2^n)$ by S_p^n).

We denote also by $S_\infty(H)$ (resp. S_∞) the Banach space of all compact operators equipped with the induced norm by $B(H)$ (resp. $B(l_2)$) ($S_\infty^n = B(l_2^n)$). Recall that if $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ $(1 \leq p, q, r < \infty)$, then $u \in B_{S_p(H)}$ iff there are $u_1 \in B_{S_q(H)}$, $u_2 \in B_{S_r(H)}$ such that $u = u_1 u_2$. Where $B_{S_p(H)}$ is the closed unit ball of $S_p(H)$. We also denote by $S_p^+(H) = \{a \in S_p(H) : a \geq 0\}$. Let H_1, H_2 be Hilbert spaces. Let $X \subset B(H_1)$ and $Y \subset B(H_2)$ be operator spaces. A linear operator $u : X \longrightarrow Y$ is called completely bounded (in short $c.b.$) if the operators

$$\begin{array}{rcccl} u_n & : & M_n(X) & \longrightarrow & M_n(Y) \\ & & (x_{ij})_{1 \leq i,j \leq n} & \longmapsto & (u(x_{ij}))_{1 \leq i,j \leq n} \end{array}$$

are uniformly bounded when $n \longrightarrow \infty$, i.e., $\sup\{\|u_n\|, n \geq 1\} < \infty$. In this case we put, $\|u\|_{cb} =$

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$\sup\{\|u_n\|, n \geq 1\}$ and we denote by $cb(X,Y)$ the Banach space of all *c.b.* maps from X into Y which is also an operator space $(M_n(cb(X,Y)) = cb(X, M_n(Y)))$ (see [5] and [6]). We denote also by $X \otimes_{\min} Y$ the subspace of $B(H \otimes_2 \mathcal{K})$ with induced norm.
We continue our preliminaries by mentioning briefly some properties concerning completely bounded operators.
Consider $Y \subset A$ (a commutative C^*-algebra) $\subset B(H)$ and let X be an arbitrary operator space. Then,

$$B(X,Y) = cb(X,Y)$$

and

$$\|u\| = \|u\|_{cb}. \tag{1}$$

Because $M_n \otimes_{\min} Y \equiv M_n \otimes_{\epsilon} Y$ isometrically ($M_n \otimes_{\epsilon} Y$ is the injective tensor product of M_n by Y in the commutative case).
Let OH be the Hilbert operator space introduced by Pisier in [12, Proposition 1.5, p. 18]. We recall that OH is homogeneous, namely, every bounded linear operator $u : OH \longrightarrow OH$ is automatically *c.b.* and

$$\|u\| = \|u\|_{cb}. \tag{2}$$

Before continuing our notation we announce the following properties. It will be needed in the sequel.

Let $X \subset B(H)$ be an operator space. For all n in $\mathbb{N}$ and $1 \leq p \leq \infty$, we have

$$\begin{aligned} \|v\|_{cb} &= \sup_{a,b \in B^+_{S_{2p}(H)}} \left(\sum_1^n \|a x_i b\|^p_{S_p(H)}\right)^{\frac{1}{p}} \\ &= \left\| \sum_1^n e_j \otimes x_j \right\|_{l_p^n \otimes_{\min} X} \end{aligned} \tag{3}$$

if p is finite and

$$\begin{aligned} \|v\|_{cb} &= \left\| \sum_1^n e_j \otimes x_j \right\|_{l_\infty^n \otimes_{\min} X} \\ &= \left\| \sum_1^n e_j \otimes x_j \right\|_{l_\infty^n \otimes_{\epsilon} X} = \|v\| \end{aligned} \tag{4}$$

if $p = \infty$. Where $v : l^n_{p^*} \longrightarrow X$ such that $v(e_i) = x_i$ (p^* is the conjugate of p i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$).
Let now X be an operator space. As usual we denote by $l_p(X)$ (resp. $l_p^n(X)$) for $1 \leq p < \infty$ the space of sequences $(x_1, ..., x_n, ...)$ (resp. finite sequences $(x_1, ..., x_n)$) in X equipped with the norm $(\sum_{n=1}^{\infty} \|x_n\|^p)^{\frac{1}{p}} < \infty$ (resp. $(\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}}$) which becomes an operator space. For more informations on this the reader can consult [12].
For the convenience of the reader, we recall the definition of sublinear operators and some propreties. For more details see [1] ,[10].

Definition 2.2. *An operator T from a Banach space X into a Banach lattice Y is said to be sublinear if for all x, y in X and λ in $\mathbb{R}_+$, we have*

$$\begin{array}{lll} (i) & T(\lambda x) & = \lambda T(x) \text{ (i.e., positively homogeneous)}, \\ (ii) & T(x+y) & \leq T(x) + T(y) \text{ (i.e., subadditive)}. \end{array}$$

Note that the sum of two sublinear operators is a sublinear operator and the multiplication of sublinear operator by a positive number is also a sublinear operator.
Let us denote by

$$SL(X,Y) = \{\text{sublinear operators } T : X \longrightarrow Y\}$$

and

$$L(X,Y) = \{\text{linear operators } U : X \longrightarrow Y\}.$$

We equip the set $SL(X,Y)$ with the natural order induced by Y

$$T_1 \leq T_2 \Longleftrightarrow T_1(x) \leq T_2(x), \quad \forall x \in X \tag{5}$$

and

$$\nabla T = \{u \in L(X,Y) : u \leq T \text{ (i.e. } \forall x \in X,\ u(x) \leq T(x))\}.$$

The set ∇T is not empty by Proposition 2.4 below. As a consequence

$$u \leq T \Longleftrightarrow -T(-x) \leq u(x) \leq T(x), \quad \forall x \in X \tag{6}$$

and

$$\lambda T(x) \leq T(\lambda x). \tag{7}$$

Now, we will give the following well-known fact and we leave the details to the reader.
Let T be a sublinear operator from a Banach space X into a (quasi-) Banach lattice Y.

$$T \text{is continuous} \Leftrightarrow \exists C > 0 : \forall x \in X, \|T(x)\| \leq C\|x\|.$$

In this case we also say that T is bounded and we put

$$\|T\| = \sup\{\|T(x)\| : \|x\|_{B_X} = 1\}.$$

We denote by

$$SB(X,Y) = \{\text{bounded sublinear operators } T : X \longrightarrow Y\}$$

and by

$$B(X,Y) = \{\text{bounded linear operators } u : X \longrightarrow Y\}.$$

We will need the following remark.

Remark 2.3. *Let X be an arbitrary Banach space. Let Y, Z be (quasi-) Banach lattices.*

(i) Consider T in $SL(X,Y)$ and u in $L(Y,Z)$. Assume that u is positive. Then, $u \circ T \in SL(X,Z)$.

(ii) Consider u in $L(X,Y)$ and T in $SL(Y,Z)$. Then, $T \circ u \in SL(X,Z)$.

The following proposition, will be useful in the sequel for the proof of Corollary 2.5.

Proposition 2.4. *Let X be a Banach space and let Y be a complete (quasi-) Banach lattice.*

Let $T \in SL(X,Y)$. Then, for all x in X there is $u_x \in \nabla T$ such that, $T(x) = u_x(x)$, (i.e. the supremum is attained, $T(x) = \sup\{u(x) : u \in \nabla T\}$).

As an immediate consequence of Proposition 2.4 we have.

Corollary 2.5. *Under the same conditions of the above proposition, we have*

(i) $\forall x \in X, \|T(x)\| \leq \sup_{u\in\nabla T} \|u(x)\| \leq \|T(x)\| + \|T(-x)\|$.

(ii) $\|T\| \leq \sup_{u\in\nabla T} \|u\| \leq 2\|T\|$.

Proposition 2.6. *Let X and Z be Banach spaces and let Y be a Banach lattice. Let $T \in SB(X,Y)$, $v \in B(X,Z)$ and a positive constant C with v injective and*

$$\|T(x)\| \leq C\|v(x)\|$$

for all $x \in X$, Then, there is a bounded sublinear operator $\tilde{T} \in SB(\overline{v(X)}, Y)$ such that $T = \tilde{T}ov$ and $\left\|\tilde{T}\right\| \leq C$.

Proof. Take $\tilde{T}(z) = T(v^{-1}(z))$ extenced to $\overline{v(X)}$.

3 Main result

We introduce the concept of l_p-summing sublinear operators in the non commutative case and characterize this class of operators by giving the extension of the "Pietsch domination theorem". Some properties are shown.

We define the class of l_p-summing sublinear operators as follows.

Definition 3.1. *Let H be a Hilbert space and let $X \subset B(H)$ be an operator space. Let $T : X \longrightarrow Y$ be a sublinear operator from X into a quantum Banach lattice Y.*

We will say that T is l_p-summing $(1 \leq p < \infty)$ if there is a positive constant C such that for all finite sequences $\{x_i\}_{1\leq i\leq n}$ in X, we have

$$(\sum_{i=1}^{n} \|T(x_i)\|^p)^{\frac{1}{p}} \leq C \sup_{a,b\in B^+_{S_{2p}}} (\sum_{i=1}^{n} \|ax_ib\|^p_{S_p(H)})^{\frac{1}{p}}.$$

We denote by $\pi_{l_p}(T)$ the smallest constant C for which this holds and $\pi_{l_p}(X,Y)$ the set of all sublinear l_p-summing operators.

We can show that

$$\begin{aligned} \sup_{a,b\in B^+_{S_{2p}(H)}} (\sum_{n=1}^{\infty} \|ax_ib\|^p_{S_p(H)})^{\frac{1}{p}} &= \|\{x_i\}\|_{l_p^n \otimes_{\min} X} \\ &= \left\|\sum_{i=1}^{n} e_i \otimes x_i\right\|_{cb(l_q^n, X)} \\ &= \|u\|_{cb} \qquad (8) \end{aligned}$$

where q is the conjugate of p and $\{e_i\}_{1\leq i\leq n}$ the canonical basis of l_q^n. By (8) the definition 3.1 is equivalent to: For all n in $\mathbb{N}$, $\{x_i\}_{1\leq i\leq n}$ in X and u in $cb(l_q^n, X)$ such that $u(e_i) = x_i$, we have

$$(\sum_{i=1}^{n} \|Tu(e_i)\|^p)^{\frac{1}{p}} \leq C\|u\|_{cb}. \qquad (9)$$

As a consequence of (8) and (9) we have the following proposition.

Proposition 3.2. *Let E, X be Banach spaces and Y, F be quantum Banach lattices. Let $T \in SB(X,Y)$, $R \in B(Y,F)$ and $S \in B^+(E,X)$ (i.e., $S(x) \geq 0$, $\forall x \geq 0$).*
(i) *If T is l_p-summing, then $R \circ T$ is l_p-summing and $\pi_{l_p}(RT) \leq \|R\|\pi_{l_p}(T)$.*
(ii) *If T is l_p-summing, then $T \circ S$ is l_p-summing and $\pi_{l_p}(T \circ S) \leq \pi_{l_p}(T)\|S\|$.*

Proposition 3.3. *Let E, X be operator spaces and let Y, F be quantum Banach lattices. Let $E \xrightarrow{v} X \xrightarrow{T} Y \xrightarrow{w} F$ such that v is completely bounded linear operator, $T \in SB(X,Y)$ is l_p-summing and w is a positive bounded linear operator. Then,*

$$\pi_{l_p}(wTv) \leq \|w\|\pi_{l_p}(T)\|v\|_{cb}.$$

Proof. Let the linear operator $u : l_q^n \longrightarrow E$. The operator $wTvu$ is a sublinear by the Remark 2.3.
We have

$$\left(\sum_{i=1}^{n} \|wTvu(e_i)\|^p\right)^{\frac{1}{p}} \leq \|w\| \sum_{i=1}^{n} (\|Tvu(e_i)\|^p)^{\frac{1}{p}}$$

and

$$\begin{aligned} \sum_{i=1}^{n} (\|Tvu(e_i)\|^p)^{\frac{1}{p}} &\leq \pi_{l_p}(T) \sum_{i=1}^{n} (\|vu(e_i)\|^p)^{\frac{1}{p}} \\ \text{by (9)} \quad &= \pi_{l_p}(T)\|vu\|_{cb} \\ &= \pi_{l_p}(T)\|v\|_{cb}\|u\|_{cb}. \end{aligned}$$

We deduce that

$$\left(\sum_{i=1}^{n} \|wTvu(e_i)\|^p\right)^{\frac{1}{p}} \leq \|w\|\pi_{l_p}(T)\|v\|_{cb}\|u\|_{cb}.$$

As a consequence

$$wTv \in \pi_{l_p}(X,Y)$$

and

$$\pi_{l_p}(wTv) \leq \|w\| \, \pi_{l_p}(T) \, \|v\|_{cb} .$$

This completes the proof.

Proposition 3.4. *Let H be a Hilbert space and let $X \subset B(H)$ be an operator space. Let $T : X \longrightarrow Y$ be a sublinear operator from X into a quantum Banach lattice Y.*

*If T is $p-$summing $(1 \leq p < \infty)$ then, T is l_p-*summing or *$\pi_p(X, Y) \subset \pi_{l_p}(X, Y)$ and*

$$\pi_{l_p}(T) \leq \pi_p(T) . \tag{10}$$

Proof. Let $T \in \pi_p(X, Y)$ and $\pi_p(T) \leq C$.

For all n in $\mathbb{N}$ and all $v : l_{p^*}^n \longrightarrow X$, such that $v(e_i) = x_i$ we have

$$\left(\sum_1^n \|Tv(e_i)\|^p \right)^{\frac{1}{p}} \leq C \|v\| .$$

We have by (3) that $\|v\| \leq \|v\|_{cb}$. Hence that for all $n \in \mathbb{N}$ and $\{x_1, ..., x_n\}$ in X we have v is in $cb(l_q^n, X)$ by (9)

$$\left(\sum_{i=1}^n \|Tv(e_i)\|^p\right)^{\frac{1}{p}} \leq C \|v\|_{cb} .$$

Hence $T \in \pi_{l_p}(X, Y)$ and

$$\pi_{l_p}(T) \leq \pi_p(T) .$$

This completes the proof.

Remark 3.5. *Let $X \subset A$ (a commutative C^* algebras) $\subset B(H)$ be an operator space and let Y be an operator space. Then, by (1) and (9) we have*

$$\pi_{l_p}(X, Y) = \pi_p(X, Y) .$$

The main result of this paper is the following extension of the Pietsch domination theorem for sublinear operators.

Theorem 3.6. *Let $X \subset B(H)$ be an operator space and let Y be a Banach space. Let $T : X \longrightarrow Y$ be a sublinear operator.*

Let $1 \leq p < \infty$. *The following properties of a positive constant C are equivalent.*

(i) The operator T is l_p-summing and $\pi_{l_p}(T) \leq C$.

(ii) There is a set I and families a_α, b_α in $B_{S_{2p}}^+$ and an ultrafilter $\mathcal{U}$ on I such that

$$\forall x \in X, \quad \|T(x)\| \leq C \lim_{\mathcal{U}} \|a_\alpha x b_\alpha\|_{S_p(H)} . \tag{11}$$

(iii) T factors of the form $T = \tilde{T}(M/E_\infty)i$ and $\left\|\tilde{T}\right\| \leq C$

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ i \downarrow & & \uparrow \tilde{T} \\ E_\infty & \xrightarrow{M/E_\infty} & E_p \\ \cap & & \cap \\ \hat{B}(H) & \xrightarrow{M} & \hat{S}_p(H) \end{array}$$

where
$i(x) = \{x_\alpha\}_{\alpha \in I}$ with $x_\alpha = x$, $\forall \alpha \in I$, $E_\infty = i(X)$ which is a closed subspace of $\hat{B}(H)$ and $\|i\|_{cb} = 1$.
$\hat{B}(H) = (B_\alpha(H))/\mathcal{U}$ with $B_\alpha(H) = B(H)$, $\forall \alpha \in I$.
M is the operator associated to $\{M_\alpha\}_{\alpha \in I}$ where $M_\alpha : B(H) \longrightarrow S_p(H)$ defined by $M_\alpha(x) = a_\alpha x b_\alpha$ and $\pi_{l_p}(M) \leq 1$.
$\hat{S}_p(H) = (S_\alpha(H))/\mathcal{U}$ with $S_\alpha(H) = S_p(H)$, $\forall \alpha \in I$ and $E_p = M(E_\infty)$ which is a closed subspace of $\hat{S}_p(H)$.
$\hat{B}(H)$ and $\hat{S}_p(H)$ are operator spaces.
Proof. (i)$\Rightarrow$(ii). Let

$$S = \left\{(a, b) \in B_{S_{2p}(H)} \times B_{S_{2p}(H)};\ a, b \geq 0\right\}$$

and

$$F = \Big\{\text{of all real functions on Sof the form } f_{\{x_i\}_{1 \leq i \leq n}}(a, b) = C^p \sum_{i=1}^n \|a x_i b\|^p - \sum_{i=1}^n \|T(x_i)\|^p . \Big\}$$

F is a convex cone. Indeed, let $\lambda \geq 0$

$$\begin{cases} \lambda f_{\{x_i\}_{1 \leq i \leq n}} & = f_{\left\{\lambda^{\frac{1}{p}} x_i\right\}_{1 \leq i \leq n}} \\ f_{\{x_i\}_{1 \leq i \leq n}} + f_{\{y_j\}_{1 \leq j \leq m}} & = f_{\{z_k\}_{1 \leq k \leq n+m}} . \end{cases}$$

Where $\{z_k\}_{1 \leq k \leq n+m} = \{x_1, ..., x_n, y_1, ..., y_m\}$. By hypothesis we have

$$\begin{aligned} \sup_{(a,b) \in S} f_{\{x_i\}_{1 \leq i \leq n}}(a, b) &= C^p \sup_{(a,b) \in S} \sum_{i=1}^n \|a x_i b\|^p \\ &\quad - \sum_{i=1}^n \|T(x_i)\|^p \geq 0 \end{aligned}$$

and this for all f in the convex cone F which is in $l_\infty(S)$, hence there are a set I, an ultrafilter $\mathcal{U}$ on I and a family $\{\lambda_\alpha\}_{\alpha \in I}$ of finitely supported probability on S such that

$$\lambda_\alpha \underset{\sigma(l_\infty^*(S),\ l_\infty(S))}{\longrightarrow} \lambda \ \left\{\lambda \text{ probability on } \hat{S} \ / \ \lambda(f) \geq 0\right\}$$

where $\hat{S}$ is the Stone Cech compactification of S and

$$\forall f \in F, \int_{\hat{S}} f(a, b) \, d\lambda(a, b) = \lim_{\mathcal{U}} \int_{\hat{S}} f(a, b) \, d\lambda_\alpha(a, b) \geq 0.$$

Particularly if

$$f(a, b) = C^p \|axb\|^p - \|T(x)\|^p$$

we have

$$\begin{aligned}\lim_{\mathcal{U}} \int_{\hat{S}} f(a,b)\, d\lambda_\alpha(a,b) &= \\ C^p \lim_{\mathcal{U}} \int_{\hat{S}} (\|axb\|^p_{S_p(H)} - \|T(x)\|^p) d\lambda_\alpha(s) &\geq 0\end{aligned}$$

$$(\lambda_\alpha = \sum_{j=1}^{n_\alpha} \lambda_{\alpha_j} \delta\left(a_{\alpha_j}, b_{\alpha_j}\right) \text{ with } \sum_{\mathrm{j}=1}^{\mathrm{n}_\alpha} \lambda_{\alpha_\mathrm{j}} = 1 \,\text{and}\, \lambda_{\alpha_\mathrm{j}} \geq 0).$$

Whence by Lemma 1.4 [12]

$$\begin{gathered}\|T(x)\|^p \leq \\ C^p \lim_{\mathcal{U}} \sum_{j=1}^{n_\alpha} \lambda_{\alpha_j} \left\|a_{\alpha_j} x b_{\alpha_j}\right\|^p_{S_p(H)} \; (a_{\alpha_j}, b_{\alpha_j} \geq 0) \leq \\ C^p \lim_{\mathcal{U}} \left\| (\sum_{j=1}^{n_\alpha} \lambda_{\alpha_j} a^{2p}_{\alpha_j})^{\frac{1}{2p}} .x. (\sum_{j=1}^{n_\alpha} \lambda_{\alpha_j} b^{2p}_{\alpha_j})^{\frac{1}{2p}} \right\|^p_{S_p(H)} \leq \\ C^p \lim_{\mathcal{U}} \|a_\alpha x b_\alpha\|^p_{S_p(H)}.\end{gathered}$$

(ii)⇒(iii).

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ i\downarrow & & \uparrow \tilde{T} \\ E_\infty & \xrightarrow{M/E_\infty} & E_p \\ \cap & & \cap \\ \hat{B}(H) & \xrightarrow{M} & \hat{S}_p(H) \end{array}$$

$$\|T(x)\| \leq C \lim_{\mathcal{U}} \|a_\alpha x b_\alpha\|_{S_p(H)} = C \|M/E_\infty \circ i(x)\|_{E_p},$$

by Proposition 2.6 hence there is $\tilde{T} : E_p \to Y$ such that $\left\|\tilde{T}\right\| \leq C$, $T(x) = \tilde{u} \circ (M/E_\infty) \circ i(x)$ and $\pi_{l_p}(M/E_\infty) \leq 1$.
(iii)⇒(i).

Is obvious and this completes the proof.

Lemma 3.7. *Let $X \subset B(H)$ be an operator space. Let $a, b \in B^+_{S_{2p}}$ and $1 \leq p \leq q < \infty$. Then,*

$$\forall x \in X, \quad \|axb\|_{S_p(H)} \leq \left\|a^{\frac{p}{q}} x b^{\frac{p}{q}}\right\|_{S_q(H)}.$$

Proof. Let x in X and consider a, b in $B^+_{S_{2p}}$.

We have

$$\begin{aligned}\|axb\|_{S_p(H)} &= \left\|a^{1-\frac{p}{q}} a^{\frac{p}{q}} x b^{\frac{p}{q}} b^{1-\frac{p}{q}}\right\|_{S_p(H)} \\ &\leq \left\|a^{1-\frac{p}{q}}\right\|_{S_{\frac{2pq}{q-p}}} \left\|a^{\frac{p}{q}} x b^{\frac{p}{q}} b^{1-\frac{p}{q}}\right\|_{S_{\frac{2pq}{q-p}}} \\ &\leq \left\|a^{1-\frac{p}{q}}\right\|_{S_{\frac{2pq}{q-p}}} \left\|a^{\frac{p}{q}} x b^{\frac{p}{q}}\right\|_{S_q(H)} \left\|b^{1-\frac{p}{q}}\right\|^{1-\frac{p}{q}}_{S_{\frac{2pq}{q-p}}} \\ &\leq \left\|a^{\frac{p}{q}} x b^{\frac{p}{q}}\right\|_{S_q(H)}\end{aligned}$$

(because $\left\|a^{1-\frac{p}{q}}\right\|_{S_{\frac{2pq}{q-p}}} = \|a\|^{\frac{q-p}{q}}_{S_{2p}} \leq 1$)

$$\begin{array}{ccc} X & \xrightarrow{M(a,b)} & S_p(H) \\ M(a^{\frac{p}{q}}, b^{\frac{p}{q}}) \searrow & & \nearrow M(a^{1-\frac{p}{q}}, b^{1-\frac{p}{q}}) \\ & S_q(H) & \end{array}$$

This give the commutative diagram.

Proposition 3.8. *Consider $1 \leq p_1, p_2 < \infty$ such that $p_1 \leq p_2$.*

If $T \in \pi_{l_{p_2}}(X,Y)$ then $T \in \pi_{l_{p_1}}(X,Y)$ and $\pi_{l_{p_1}}(T) \leq \pi_{l_{p_2}}(T)$.

Proof. It is immediate by the inequality (11) and lemma 3.7.

As an immediate consequence of Proposition 2.4, Corollary 2.5 and Theorem 3.6 we have the following corollary.

Corollary 3.9. *Let $T \in \pi_{l_p}(X,Y)$ then for any u in ∇T we have $u \in \pi_{l_p}(X,Y)$.*

Question: We do not know if the converse of the Corollary 3.9 is true?

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