Laplace Transform Of The Time Of Ruin For A Perturbed Risk Process Driven By A Subordinator

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Abstract—**In this paper, we construct, by Bochner subordination, a new model (an extension of the Sparre-Andersen model with investments that is perturbed by diffusion). For this risk process, we derive a general integro-differential equation for the Laplace transform of the time of ruin with positive surplus initial via the elementary properties of the classical conditional expectation. The special cases, for different inter-arrival time distributions, are given in some details. We also deduce a comparison for Laplace transforms of the time of ruin, for different inter-arrival time distributions.**

Keywords: Ruin theory; Bochner Subordination; Laplace transform; Integro-differential equation.

1 Introduction

The classical risk model perturbed by a diffusion was introduced by Gerber (1970) and has been further studied by many authors during the last few years; such as Dufresne and Gerber (1991), Furrer and Schmidli (1994), Gerber and Landry (1998), Wang and Wu (2000), Tsai and Willmot (2002a), Li and Garrido (2005) and the references therein.

The purpose of the paper are twice. At first time, we consider a generalization of the perturbed risk model of Li and Garrido (2005). We substitute the Brownian motion by a subordinated process in the sense of Bochner by unspecified subordinator. This substitution yields a new model given by equation (3.6) (see Section 3 for more details). At second time, for the above risk process, we derive an integro-differential equation for the Laplace transform of the time of ruin with positive surplus initial via the elementary properties of the classical conditional expectation.

The organization of this paper is as follows. The next Section starts with a brief description of the classical risk model, and some attention is payed to the quantities: the Laplace transform of the time of ruin, the elementary properties of the classical conditional expectation and the infinitesimal generator. In Section 3, we construct a new model, Sparre-Andersen model with investments perturbed by diffusion under subordination, and we end this section by a particular case of subordination which shows the coincidence of the results in the Sparre-Andersen model with investments perturbed by diffusion (2.1) and in our model (3.6). In Section 4, for the new model, we start by deriving an integro-differential equation for the Laplace transform of the time of ruin with positive surplus initial under very general conditions regarding the claim sizes, the claim arrivals and the returns from investments, via the elementary properties of the classical conditional expectation, and we end this section by giving several examples of the integro-differential equation satisfied by the Laplace transform of the time of ruin, for different inter-arrival time distributions. Finally, in Section 5, we present a comparison of Laplace transforms of the time of ruin, for different inter-arrival time distributions.

2 Model Description and Notations

We will assume that all processes and random variables are defined on a filtered complete probability space $(\Omega, \mathcal{F}, \mathbf{P}, (\mathcal{F}_t)_{t\geq 0})$. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and all the stochastic processes to be defined in this paper are adapted.

Consider a time-continuous Sparre-Andersen surplus process perturbed by a diffusion

$$U(t) = u + ct - \sum_{k=1}^{N(t)} \xi_k + \lambda B(t), \quad t \geq 0, \tag{2.1}$$

where $u \geq 0$ is the initial capital and $c > 0$ is the incoming premium rate. The claim sizes $(\xi_k)_{k\in\mathbf{N}}$ are positive i.i.d. random variables with common probability distribution function F_ξ and density function f_ξ, representing the k-th claim amount, with finite mean $\mu = \mathbf{E}[\xi_1]$, and variance $\sigma^2 = Var(\xi_1) < \infty$.

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The ordinary renewal process $\{N(t), t \geq 0\}$ denotes the number of claims up to time t, with $N(t) = \sup\{n \geq 1 : T_n \leq t\}$, $t \geq 0$, with, by convention, $\sup\emptyset = 0$, where the $(T_n)_{n\in\mathbf{N}}$ denotes the claim times, with $T_0 = 0 < T_1 < T_2 < ...$
The inter-arrival times: $\tau_1 = T_1$, $\tau_k = T_k - T_{k-1}$, $k = 2, 3, ...$ are i.i.d. distributed with finite mean.
Finally, $\{B(t) : t \geq 0\}$ is a standard Wiener process that is independent of the compound ordinary renewal process $S(t) = \sum_{k=1}^{N(t)} \xi_k$ and the dispersion parameter $\lambda > 0$. Further assume that the sequences ξ_k and τ_k are independent of each other.

Next, we denote by f_τ the density for the time in between claims $(\tau_k)_{k\in\mathbf{N}}$ that is satisfies an ordinary differential equation of order $n \geq 1$ and with constant coefficients, formally denoted by

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = 0, \tag{2.2}$$

with $\mathcal{L}^\star$ denoting the formal adjoint of the linear operator $\mathcal{L}$. In general, the linear operator $\mathcal{L}$ is defined by

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = \sum_{j=0}^{n} \alpha_j \frac{d^j}{dt^j} f_\tau(t), \tag{2.3}$$

with the adjoint $\mathcal{L}^\star$,

$$\mathcal{L}^\star(\frac{d}{dt})f_\tau(t) = \sum_{j=0}^{n} (-1)^j \alpha_j \frac{d^j}{dt^j} f_\tau(t). \tag{2.4}$$

If the investment price is modeled by geometric Brownian motion with drift a and volatility σ^2, then the equation of the surplus model is expressible as:

$$\begin{aligned} U(t) &= u + ct + a\int_0^t U(s)\,ds + \sigma\int_0^t U(s)\,dB(s) \\ &\quad - \sum_{k=1}^{N(t)} \xi_k + \lambda B(t), \quad t \geq 0, \end{aligned} \tag{2.5}$$

where $B(.)$ is a standard Brownian motion.

Remark 1:
If $N(t)$ is Poisson distributed, the process (2.1) is a compound Poisson process and refers to the classical Cramér-Lundberg model perturbed by a diffusion (see e.g. Furrer and Schmidli 1994: [7]). But if $N(t)$ is renewal process, then the process (2.1) is called the Sparre-Andersen model perturbed by a diffusion (see e.g. Li and Garrido 2005: [9]) and consequently (2.5) is referred to as the Sparre-Andersen model perturbed by a diffusion with investments.

Now define

$$T_u = \inf\{t \geq 0\,;\, U(t) < 0 \mid U(0) = u\} \quad (\infty,\ \text{otherwise}),$$

to be the time of ruin of (2.1) and

$$\begin{aligned} \Psi(u) &= \mathbf{P}(T_u < \infty \mid U(0) = u) \\ &= \mathbf{P}[\inf\{t \geq 0\,;\, U(t) < 0 \mid U(0) = u\}], \end{aligned} \tag{2.6}$$

to be the ultimate ruin probability with an initial surplus u.
Next, for $\delta \geq 0$ define

$$\Phi_\delta(u) = \mathbf{E}(e^{-\delta T_u} 1_{\{T_u<\infty\}} \mid U(0) = u), \tag{2.7}$$

where $1_{\{.\}}$ is the usual indicator function, to be the Laplace transform of the time of ruin with an initial surplus u.
The loading of security is defined by $r = c - \mu\mathbf{E}[N(t)]$. If $r > 0$, then the activity is known as profitable. Indeed, the Law of Large Numbers ensures that, in this case, the process $U(t) \longrightarrow +\infty$ almost surely $(a.s.)$ as $t \longrightarrow +\infty$, and consequently $\Psi(u) \neq 1$. If $r < 0$, then $U(t) \longrightarrow -\infty$ $a.s.$ as $t \longrightarrow +\infty$. Generally, we will make the assumption that the activity is profitable.

Recall that the transition operator K_t of a Markov process $U(t)$ is given by: $K_t f(u) = \mathbf{E}(f(U(t)) \mid U(0) = u)$, and the infinitesimal generator of $\{K_t, t > 0\}$, is the linear operator A defined by: $Af(x) = \lim_{t\longrightarrow 0} \frac{K_t f(x) - f(x)}{t}$ for all real-valued, bounded, Borel measurable function f defined on a metric space S. The domain of A is denoted by D_A. Using Itô's formula we find that the infinitesimal generator of (2.5)

$$A = (\frac{\sigma^2}{2}u^2 + \frac{\lambda^2}{2})\frac{d^2}{du^2} + (au + c)\frac{d}{du}. \tag{2.8}$$

Paulsen and Gjessing (1997) (see [12]) introduce a relationship between the infinitesimal generator of the risk process and the two quantities of ruin (probability of ruin, Laplace transform of the time of ruin). Actually, for example, they show that a function $q_\delta(u)$ that satisfies the equation $Aq_\delta(u) = \delta q_\delta(u)$ with some boundary conditions, is the Laplace transform of the time of ruin. The following theorem is an adapted form of their theorem to the Cramér-Lundberg case with no investments. Our Theorem 4.2 (see Section 4) is based on the theorem given below.

Theorem 1: (See [12])
Assume that on the event $\{T_u = \infty\}$, $U_t \longrightarrow \infty\, a.s.$ as $t \longrightarrow \infty$. Then with the above notation we have the following.
1)Assume that $g(u)$ is a bounded and twice continuously differentiable function on $u \geq 0$ with a bounded first derivative there, where we at $u = 0$ mean the right-hand derivative. If $g(u)$ solves

$$Ag(u) = 0 \ \text{ on } \ u > 0,$$

together with the boundary conditions:

a) $g(u) = 1$ on $u < 0$,

b) $g(0) = 1$ if $\lambda^2 > 0$,

c) $\lim_{u \longrightarrow \infty} g(u) = 0$,

then

$$g(u) = \mathbf{P}(T_u < \infty \mid U(0) = u).$$

2) Assume that q_δ, $\delta \geq 0$ is a bounded and twice continuously differentiable function on $u \geq 0$ with a bounded first derivative there, where we at $u = 0$ mean the right-hand derivative. If $q_\delta(u)$ solves

$$Aq_\delta(u) = \delta q_\delta(u) \text{ on } u > 0,$$

together with the boundary conditions:

a) $q_\delta(u) = 1$ on $u < 0$,

b) $q_\delta(0) = 1$ if $\lambda^2 > 0$,

c) $\lim_{u \longrightarrow \infty} q_\delta(u) = 0$,

then

$$q_\delta(u) = \mathbf{E}(e^{-\delta T_u} 1_{\{T_u < \infty\}} \mid U(0) = u).$$

3 Model Construction

3.1 Subordination of Brownian motion

For the following classical notions, we refer the reader to [10] and [13].
Let $(E, \mathcal{E})$ be a measurable space and let m be a σ-finite positive measure on $(E, \mathcal{E})$. Let $\mathbf{B} = (B_t)_{t \geq 0}$ be a Brownian motion on $\mathbf{R}$. The associated semigroup $P = (P_t)_{t \geq 0}$ is defined by $P_t = g_t * f$ for $t > 0; f \in L^2(\lambda)$ and $g_t(x) = \frac{1}{\sqrt{2\Pi t}} \exp\{\frac{-x^2}{2t}\}$ is the function of Gauss on $\mathbf{R}$.
The associated $L^2(m)$- generator (or generator) M is defined by $Mf(x) = \lim_{t \longrightarrow 0} \frac{P_t f(x) - f(x)}{t} = \frac{1}{2}\Delta$, where Δ is the Laplacian operator, on its domain $D(M)$ which is the set of all functions $f \in L^2(m)$ for which this limit exists in $L^2(m)$.
Let $Z = (Z_t)_{t \geq 0}$ a unspecified Bochner subordinator, i.e. $Z = (Z_t)_{t \geq 0}$ is a convolution semigroup of probability measures on $\mathbf{R}$ such that, for each $t > 0$, we have $Z_t \neq \varepsilon_0$ and Z_t is supported by $[0, \infty[$.
The associated *Bernstein function* h is defined by its Laplace transform $LZ_t(r) = \exp(-th(r))$ for all $r, t > 0$. It is known that h admits the representation

$$h(r) = kr + \int_0^\infty (1 - \exp(-rs))\, \nu(ds), \qquad r > 0, \quad (3.1)$$

where $k \geq 0$ and ν is a measure on $]0, \infty[$ verifying $\int_0^\infty (1 \wedge s)\, \nu(ds) < \infty$. Moreover, $k \geq 0$ and ν are uniquely determined, they are called parameters of the Bernstein function of Z.
Then $Y = (Y_t)_{t \geq 0}$, where $Y_t = B_{Z_t}$ is called the subordinated process of the Brownian motion B_t by means of the subordinator Z_t. The associated semigroup $P^Z = (P_t^Z)_{t \geq 0}$ be the (Bochner) subordinated semigroup of P by means Z is given by:

$$P_t^Z = \int_0^\infty P_s\, Z_t(ds) \text{ for every } t > 0.$$

The associated generator is denoted by M_Z on its domain $D(M_Z)$. Moreover, it is known that $D(M) \subset D(M_Z)$ and

$$M_Z u = kMu + \int_0^\infty (P_t u - u)\, \nu(dt), \quad u \in D(M), \quad (3.2)$$

where k and ν are given in (3.1).

3.2 Examples of subordinators

Case 1: Dirac subordinator
Let $\varepsilon = (\varepsilon_t)_{t \geq 0}$ the dirac subordinator. Then in the case, the subordinated process Y of the Brownian motion B by means of the dirac subordinator ε coincide to the original Brownian motion i.e. ($Y_t = B_{\varepsilon_t} = B_t$ for every $t > 0$). Consequently $P^\varepsilon = P$ and

$$M_\varepsilon = M = \frac{1}{2}\Delta, \quad (3.3)$$

where Δ is the Laplacian operator.

Case 2: One-sided stable subordinator
Let η^α "fractional powers" be the *one-sided stable* subordinator of order $\alpha \in]0, 1[$, i.e. the unique convolution semigroup $\eta^\alpha := \eta^\alpha_{t>0}$ on $[0, \infty[$ such that for each $t > 0$, the Laplace transform $L(\eta_t^\alpha)(x) = \exp(-tx^\alpha)$, $x > 0$. It is well known that the associated Bernstein function $h(r) = r^\alpha$ admits the representation

$$h(r) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty (1 - \exp(-rs)) \frac{ds}{s^{\alpha+1}}. \quad (3.4)$$

In the case, the subordinated process Y of the Brownian motion B by means of η^α is expressible as: $Y_t = B_t^{\eta^\alpha}$, for each $t \geq 0$. The associated semigroup $P^{\eta^\alpha} = (P_t^{\eta^\alpha})_{t>0}$ be the (Bochner) subordinated semigroup of P by means η^α (*i.e.* $P_t^{\eta^\alpha} = \int_0^\infty P_s\, \eta_t^\alpha(ds)$ for every $t > 0$).
Hence, (see [13] for more details) the infinitesimal generator M_{η^α} of Y_t on its domain $D(M_{\eta^\alpha})$ is expressible as:

$$M_{\eta^\alpha} = -2^{-\alpha} c' (-\overline{\Delta})^\alpha, \quad c' > 0, \quad (3.5)$$

where $\overline{\Delta}$ is the closure of the Laplacian operator.

3.3 The model

Now, we construct, by a Bochner subordination, a new model in the following way: we take the model (2.1) and we substitute the Brownian motion: B_t by the subordinated process of B_t in the sense of Bochner by means of

Z_t that is denoted by Y_t.
In this way the model (2.1), (see Section 2), is expressible as:

$$U(t) = u + ct - \sum_{k=1}^{N(t)} \xi_k + \lambda Y(t), \quad t \geq 0, \tag{3.6}$$

where $u \geq 0$ is the initial capital and $c > 0$ is the incoming premium rate. The claim sizes $(\xi_k)_{k\in\mathbf{N}}$ are positive i.i.d. random variables with common probability distribution function F_ξ and density function f_ξ, representing the k-th claim amount, with finite mean $\mu = \mathbf{E}[\xi_1]$, and variance $\sigma^2 = Var(\xi_1) < \infty$.
The ordinary renewal process $\{N(t), t \geq 0\}$ denotes the number of claims up to time t, with $N(t) = \sup\{n \geq 1 : T_n \leq t\}$, $t \geq 0$, with, by convention, $\sup \emptyset = 0$, where the $(T_n)_{n\in\mathbf{N}}$ denotes the claim times, with $T_0 = 0 < T_1 < T_2 < ...$
The inter-arrival times: $\tau_1 = T_1$, $\tau_k = T_k - T_{k-1}$, $k = 2, 3, ...$ are i.i.d. distributed with finite mean.
Finally, $\{Y(t) = B_{Z_t} : t \geq 0\}$ is a subordinated process of the standard Wiener process $\{B(t) : t \geq 0\}$ in the sense of Bochner by means of Z that is independent of the compound ordinary renewal process $S(t) = \sum_{k=1}^{N(t)} \xi_k$ and the dispersion parameter $\lambda > 0$. Further assume that the sequences ξ_k and τ_k are independent of each other.

In this paper, a particular case is considered, namely the investment price is modeled by geometric Brownian motion with drift a and volatility σ^2, then the equation of the surplus model is given by:

$$\begin{aligned} U(t) &= u + ct + a\int_0^t U(s)\,ds + \sigma\int_0^t U(s)\,dB(s) \\ &\quad - \sum_{k=1}^{N(t)} \xi_k + \lambda Y(t), \quad t \geq 0, \end{aligned} \tag{3.7}$$

where $B(.)$ is a standard Brownian motion and $Y(.)$ is subordinated process of $B(.)$ in the sense of Bochner by means of subordinator $Z(.)$.
Hence, by the equation (3.2) and It$\hat{o}$'s formula, the infinitesimal generator of (3.7) is expressible as:

$$A_Z = (\frac{\sigma^2}{2}u^2 + \frac{k\lambda^2}{2})\frac{d^2}{du^2} + (au+c)\frac{d}{du} + \lambda\int_0^\infty (P_t - I)\,\nu(dt), \tag{3.8}$$

where I denoted the identity operator.

Since the ruin may occur only at the claim times, T_k, the surplus process (3.6) may be discredited. The discrete version

$$U_k = U(T_k) = u + cT_k - \sum_{k=1}^{N(T_k)} \xi_k + \lambda Y(T_k), \tag{3.9}$$

is a discrete time Markov process (Markov chain). The process U_k may be written immediately after the payment of the $k-$th claim ξ_k.

$$U_k = V_{\tau_k}^{U_{k-1}} - \xi_k, \tag{3.10}$$

where $V_{\tau_k}^{U_{k-1}}$ represents the worth of a portfolio that results from investing the capital U_{k-1} (immediately after the payment of the $k-1$ claim) and the premiums collected over the time τ_k, into a risky asset. Recall that for the discrete time Markov process $\{(U_k)_{k\in\mathbf{N}} \mid U(0) = u\}$, on the set of all real-valued, bounded, Borel-measurable functions φ, define the transition operator $P\varphi : \mathbf{R} \longrightarrow \mathbf{R}$, $Ph(u) := \mathbf{E}(h(U_1) \mid U(0) = u) =$

$$\int_0^\infty \int_0^\infty f_\tau(t)\mathbf{E}(h(Z_t^u - x) \mid U(0) = u) f_X(x)\,dx\,dt\,, \tag{3.11}$$

the generator of the time discrete Markov process is given by: $A_U\varphi(u) = (P - I)\varphi(u)$, where I denoted the identity operator, and D_{A_U} denoted the domain of the operator A_U.

Remark 2:
For the president trivial case of subordination (Dirac subordinator), if the investment price is modeled by geometric Brownian motion with drift a and volatility σ^2, then the equation of the surplus model is given by:

$$\begin{aligned} U(t) &= u + ct + a\int_0^t U(s)\,ds + \sigma\int_0^t U(s)\,dB(s) \\ &\quad - \sum_{k=1}^{N(t)} \xi_k + \lambda Y(t), \quad t \geq 0, \end{aligned} \tag{3.12}$$

where $B(.)$ is a standard Brownian motion and $Y(.)$ is a subordinated process of $B(.)$ in the sense of Bochner by means of subordinator ε.
Hence, by the equation (3.3) and the It$\hat{o}$'s formula, the infinitesimal generator of (3.12) is expressible as:

$$A_\varepsilon = (\frac{\sigma^2}{2}u^2 + \frac{\lambda^2}{2})\frac{d^2}{du^2} + (au+c)\frac{d}{du}. \tag{3.13}$$

We remark that the equation (2.5) and the equation (3.12) possess the same infinitesimal generator. This explains a coincidence between the Sparre-Andersen model perturbed by diffusion given by the equation (2.1) and the subordinated Sparre-Andersen model perturbed by diffusion in the sense of Bochner by means of ε given by the equation (3.6).

4 The integro-differential equation of the Laplace transform of the time of ruin

The classical approach in deriving equations satisfied by the Laplace transform of the time of ruin is conditioning on the time of the first claim and its size, followed by differentiation [Dickson and Hipp (1998), Jun Cai (2004) in the classical risk model without perturbation] and [Dufresne and Gerber (1991), Furrer and Schmidli (1994), Gerber and Landry (1998), Wang and Wu (2000), Li and Garrido (2005) and the references therein in the classical risk model with perturbation]. In contrast, the uniform approach of this paper (see [6] for the classical risk model without perturbation) consists in deriving a general equation for the classical conditional expectation that relates to the Laplace transform of the time of ruin via our Theorem 4.1 and Theorem 4.2 given below.

Theorem 4.1 Let $q_\delta \in D_{A_Z}, \delta \geq 0$ with $Pq_\delta(u,0) = q_\delta(u,0)$. If f_τ satisfies the ordinary differential equation with constant coefficients

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = 0$$

and

1) $f_\tau^{(k)}(0) = 0$, for $k = 0, ..., n-2$,

2) $\lim_{x \longrightarrow \infty} f_\tau^{(k)}(x) = 0$, for $k = 0, ..., n-1$,

then

$$\mathcal{L}^\star(A_Z - \delta I)Pq_\delta(u,0) = f_\tau^{(n-1)}(0)\,\mathbf{E}[q_\delta(u,\xi_1)]. \quad (4.1)$$

Proof.
For $\delta \neq 0$, we take the same technical of the proof of the theorem 3 in thesis of Corina D. Constantinescu (see page 31-33).

Theorem 4.2 *Assume that on the event* $\{T_u = \infty\}$, $U_t \longrightarrow \infty$ *as* $t \longrightarrow \infty$. *Assume that* Φ_δ *is* P *invariant. Then the following axiom are equivalent:*

1) *Any bounded function* $q_\delta \in D_{A_U}$ *such that* $q_\delta(u) = e^{-\delta T_u} g(u)$; $\delta \geq 0$ *satisfies*

$$A_U q_\delta(u) = (P - I)q_\delta(u) = 0,$$

together with the boundary conditions for the function g:

   a) $g(u) = 1$ *on* $u < 0$,

   b) $g(0) = 1$ *if* $\lambda^2 > 0$,

   c) $\lim_{u \longrightarrow \infty} g(u) = 0$,

2) $q_\delta(u)$ *is the Laplace transform of the time of ruin, in other words*

$$q_\delta(u) = \Phi_\delta(u).$$

Proof.
First part "1) $\Longrightarrow$ 2)".
Let

$$\begin{aligned} \mathbf{E}_u[q_\delta(U_k)] &:= \mathbf{E}_u[q_\delta(U(T_k))] \\ &= \mathbf{E}(q_\delta(U(T_k)) \mid U(0) = u). \end{aligned}$$

It is known that for any $n \geq 1$

$$\begin{aligned} M_n &= q_\delta(U_n) - \sum_{k=0}^{n-1} A_U q_\delta(U_k) \\ &= q_\delta(U_n) - \sum_{k=0}^{n-1} (P - I)q_\delta(U_k) \end{aligned}$$

is a martingale. Indeed

$$\begin{aligned} \mathbf{E}(M_{n+1} \mid \mathcal{F}(U_0, ..., U_n)) &= \mathbf{E}(q_\delta(U_{n+1}) \mid U_0, ..., U_n) \\ &\quad - \sum_{k=0}^{n} (P - I)q_\delta(U_k) \\ &= Pq_\delta(U_n) - Pq_\delta(U_n) \\ &\quad + q_\delta(U_n) \\ &\quad - \sum_{k=0}^{n-1} (P - I)q_\delta(U_k) \\ &= M_n. \end{aligned}$$

The assumption: $A_U q_\delta(u) = 0$ implies that $q_\delta(U_k)$ is a martingale, i.e. for any k,

$$q_\delta(u) = \mathbf{E}_u[q_\delta(U(T_k))] = \mathbf{E}_u[e^{-\delta T_k} g(U(T_k))].$$

The time of ruin T_u is a stopping time, thus $q_\delta(u) = \mathbf{E}_u[q_\delta(U(T_u))] = \mathbf{E}_u[e^{-\delta T_u} g(U(T_u))]$ and moreover

$$\begin{aligned} q_\delta(u) &= \mathbf{E}_u[q_\delta(U(T_u \wedge T_k))] \\ &= \mathbf{E}_u[e^{-\delta(T_u \wedge T_k)} g(U(T_u \wedge T_k))] \\ &= \mathbf{E}_u[e^{-\delta(T_u \wedge T_k)} g(U(T_u \wedge T_k))1_{\{T_u < T_k\}}] \\ &\quad + \mathbf{E}_u[e^{-\delta(T_u \wedge T_k)} g(U(T_u \wedge T_k))1_{\{Tu > T_k\}}] \\ &= \mathbf{E}_u[g(U(T_u))]\,\mathbf{E}_u[e^{-\delta T_u} 1_{\{T_u < T_k\}}] \\ &\quad + \mathbf{E}_u[g(U(T_k))]\,\mathbf{E}_u[e^{-\delta T_k} 1_{\{T_u > T_k\}}]. \end{aligned}$$

The result thus follows by letting $T_k \longrightarrow \infty$ and using the boundary conditions for the function g,

$$\begin{aligned} q_\delta(u) &= 1 \times \mathbf{E}_u[e^{-\delta T_u} 1_{\{T_u < \infty\}}] + 0 \times 0 \\ &= \mathbf{E}_u[e^{-\delta T_u} 1_{\{T_u < \infty\}}] \\ &= \Phi_\delta(u). \end{aligned}$$

When $\lambda^2 > 0$ the process starting from 0 will immediately assume a negative value, hence the extra boundary condition $q_\delta(0) = g(0) = 1$ in the case. This proves part (i).
Second part "2) $\Longrightarrow$ 1)".
Since the process U_k is a renewal process and since ruin cannot occur in the interval $(0, T_1)$, where T_1 represents

the time of the first claim, then the Laplace transform of the time of ruin satisfies the renewal equation,

$$q_\delta(u) = \mathbf{E}(q_\delta(U_1) \mid U(0) = u) = Pq_\delta(u).$$

It is proved in the previous Theorem that Pq_δ satisfies the equation for any $q_\delta \in D_{A_Z}$. Since, $Pq_\delta = q_\delta$ it follows that q_δ satisfies the equation. Since q_δ is the Laplace transform of the time of ruin, it also satisfies the boundary conditions.

Combining Theorem 4.1 with Theorem 4.2 above, we get that the Laplace transform of the time of ruin satisfies the following integro-differential equation:

$$\mathcal{L}^\star(A_Z - \delta I)\Phi_\delta(u) = f_\tau^{(n-1)}(0)\int_0^\infty \Phi_\delta(u-x)f_\xi(x)\,dx, \tag{4.2}$$

together with the boundary conditions:

a) $\lim_{u\longrightarrow\infty}\Phi_\delta(u) = 0$,

b) $\Phi_\delta(0) = 1$ if $\lambda^2 > 0$,

c) (BC),

where (BC) stands for boundary conditions and n represents the degree of the ordinary differential equation satisfied by the density of the inter-arrival times. The boundary conditions (BC) may be derived from "compatibility" conditions assuming that the integro-differential equation and its derivatives hold at zero. For instance, if the investment is a geometric Brownian motion then the equation has order "$2n$".

4.1 Applications

Many well-known equations are a particular form of the equation (4.2). For instance, in the subordinated Sparre-Andersen model with investments perturbed by diffusion given by (3.7), the equations and their boundary conditions can be derived for different inter-arrival times.

Example 1: Erlang$(2,\beta)$
Consider that the surplus model (3.7) has inter-arrival times τ_k that are Erlang$(2,\beta)$, distributed with the density function

$$f_\tau(t) = \beta^2 t\exp\{-\beta t\}, \quad t \geq 0.$$

Then for an Erlang$(2,\beta)$ distribution, $\mathcal{L}(\frac{d}{dt}) = (\frac{d}{dt}+\beta)^2$, and $\mathcal{L}^\star(\frac{d}{dt}) = (-\frac{d}{dt}+\beta)^2$. Hence, the equation (4.2) is specifically:

$$(-A_Z+\delta I+\beta I)^2\Phi_\delta(u) = f_\tau^{(1)}(0)\int_0^\infty \Phi_\delta(u-x)f_\xi(x)\,dx, \tag{4.3}$$

with the boundary conditions:

a) $\lim_{u\longrightarrow\infty}\Phi_\delta(u) = 0$,

b) $\Phi_\delta(0) = 1$ if $\lambda^2 > 0$,

c) $A_ZA_Z\Phi_\delta(0) - 2(\beta+\delta)A_Z\Phi_\delta(0) + (\beta+\delta)^2\Phi_\delta(0) = \beta^2$,

d) the first two derivatives of the equation (4.3) evaluated at zero.

The equation (4.3) is equivalent to

$$A_ZA_Z\Phi_\delta(u) - 2(\beta+\delta)A_Z\Phi_\delta(u) + (\beta+\delta)^2\Phi_\delta(u)$$
$$= \beta^2\int_0^u \Phi_\delta(u-x)f_\xi(x)\,dx + \beta^2\int_u^\infty f_\xi(x)\,dx, \tag{4.4}$$

where

$$A_Z = (\frac{\sigma^2}{2}u^2+\frac{k\lambda^2}{2})\frac{d^2}{du^2}+(au+c)\frac{d}{du}+\lambda\int_0^\infty (P_t-I)\,\nu(dt)$$

and

$$\begin{aligned} A_ZA_Z &= (\frac{\sigma^2}{2}u^2+\frac{k\lambda^2}{2})A_Z'' + (au+c)A_Z' \\ &+\lambda\int_0^\infty (P_tA_Z - A_Z)\,\nu(dt)\end{aligned}$$

with (by taking account of derivation theorem under integral sign)

$$\begin{aligned} A_Z'\Phi_\delta(u) &= (\frac{\sigma^2}{2}u^2+\frac{k\lambda^2}{2})\Phi_\delta^{(3)}(u) \\ &+(\sigma^2u+au+c)\Phi_\delta^{(2)}(u) + a\Phi_\delta^{(1)}(u) \\ &+\lambda\int_0^\infty (P_t\Phi_\delta^{(1)}(u) - \Phi_\delta^{(1)}(u))\,\nu(dt)\end{aligned}$$

and

$$\begin{aligned} A_Z''\Phi_\delta(u) &= (\frac{\sigma^2}{2}u^2+\frac{k\lambda^2}{2})\Phi_\delta^{(4)}(u) \\ &+(2\sigma^2u+au+c)\Phi_\delta^{(3)}(u) \\ &+(2a+\sigma^2)\Phi_\delta^{(2)}(u) \\ &+\lambda\int_0^\infty (P_t\Phi_\delta^{(2)}(u) - \Phi_\delta^{(2)}(u))\,\nu(dt).\end{aligned}$$

With the boundary conditions obtained from the fact that the equation holds at zero and so do the first two derivatives of the equation.

Example 2: Erlang(n,β)
Consider that the surplus model (3.7) has inter-arrival times τ_k that are Erlang(n,β), distributed with the density function

$$f_\tau(t) = \frac{\beta^n}{\Gamma(n)}t^{n-1}\exp\{-\beta t\}, \quad t \geq 0 \ \textit{and}\ n \in \mathbf{N}.$$

Then for an Erlang(n,β) distribution, $\mathcal{L}(\frac{d}{dt}) = (\frac{d}{dt}+\beta)^n$, and $\mathcal{L}^\star(\frac{d}{dt}) = (-\frac{d}{dt}+\beta)^n$. Hence, the equation (4.2) is specifically:

$$\sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!} A_Z^k(\delta+\beta)^{n-k}\Phi_\delta(u) =$$
$$\beta^n \int_0^u \Phi_\delta(u-x) f_\xi(x)\,dx + \beta^n \int_u^\infty f_\xi(x)\,dx, \qquad (4.5)$$

with the boundary conditions:

a) $\lim_{u\longrightarrow\infty}\Phi_\delta(u) = 0$,

b) $\Phi_\delta(0) = 1$ if $\lambda^2 > 0$,

c) $\sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!} A_Z^k(\delta+\beta)^{n-k}\Phi_\delta(0) = \beta^n$,

d) the first "$2n-2$" derivatives of the equation (4.5) evaluated at zero.

Example 3: Sum of two exponentials
Consider that the surplus model (3.7) has inter-arrival times τ_k that are sum of two exponentials, with the density function

$$f_\tau(t) = \frac{\beta_1\beta_2}{\beta_2-\beta_1}[\exp\{-\beta_1 t\} \ - \exp\{-\beta_2 t\}],$$

$t \geq 0$ *and* $\beta_1 \neq \beta_2$.
In the case of inter-arrival times distributed as a mixture of exponentials, the density f_τ satisfies an ordinary differential equation of order 2, the linear operator $\mathcal{L}$ is

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = (\frac{d}{dt}+\beta_1)(\frac{d}{dt}+\beta_2)f_\tau(t) = 0,$$

and the adjoint linear operator $\mathcal{L}^\star$ is expressible as:

$$\mathcal{L}^\star(\frac{d}{dt})f_\tau(t) = (-\frac{d}{dt}+\beta_1)(-\frac{d}{dt}+\beta_2)f_\tau(t).$$

Assume that

$$H(u) := \beta_1\beta_2 \int_0^u \Phi_\delta(u-x) f_\xi(x)\,dx + \beta_1\beta_2 \int_u^\infty f_\xi(x)\,dx.$$

Thus the equation (4.2) is specifically:

$$\begin{aligned} H(u) \ &= \ A_Z A_Z \Phi_\delta(u) - (\beta_1+\beta_2+2\delta)A_Z\Phi_\delta(u) \\ &\quad +[\beta_1\beta_2 + (\beta_1+\beta_2)\delta + \delta^2]\Phi_\delta(u), \quad (4.6) \end{aligned}$$

with the boundary conditions:

a) $\lim_{u\longrightarrow\infty}\Phi_\delta(u) = 0$,

b) $\Phi_\delta(0) = 1$ if $\lambda^2 > 0$,

c) $A_Z A_Z \Phi_\delta(0) - (\beta_1+\beta_2+2\delta)A_Z\Phi_\delta(0) + [\beta_1\beta_2 + (\beta_1+\beta_2)\delta + \delta^2]\Phi_\delta(0) = \beta_1\beta_2$,

d) the first two derivatives of the equation (4.6) evaluated at zero.

Thus, the explicit equation for the Laplace transform of the time of ruin in case on investments in a geometric Brownian motion with inter-arrival times distributed as a sum of two exponentials is a forth order integro-differential equation:

$$\begin{aligned} H(u) \ = \ & [\frac{\sigma^2}{2}u^2 + \frac{k\lambda^2}{2}]^2\Phi_\delta^{(4)}(u) \\ & + (\sigma^2u^2 + k\lambda^2)(au + \sigma^2 u + c)\Phi_\delta^{(3)}(u) \\ & + [(\sigma^2u^2 + k\lambda^2) \times \\ & \qquad (a + \frac{\sigma^2}{2} - (\frac{\beta_1+\beta_2}{2})u - \delta u)]\Phi_\delta^{(2)}(u) \\ & + (au+c)(\sigma^2 u + au + c)\Phi_\delta^{(2)}(u) \\ & + (au+c)(a - \beta_1 - \beta_2 - 2\delta)\Phi_\delta^{(1)}(u) \\ & + \lambda \int_0^\infty (P_t A_Z \Phi_\delta(u) - A_Z\Phi_\delta(u))\,\nu(dt) \\ & + \lambda(\frac{\sigma^2}{2}u^2 + \frac{k\lambda^2}{2}) \times \\ & \qquad \int_0^\infty (P_t\Phi_\delta^{(2)}(u) - \Phi_\delta^{(2)}(u))\,\nu(dt) \\ & + \lambda(au+c) \int_0^\infty (P_t\Phi_\delta^{(1)}(u) - \Phi_\delta^{(1)}(u))\,\nu(dt) \\ & - \lambda(\beta_1+\beta_2+2\delta) \times \\ & \qquad \int_0^\infty (P_t\Phi_\delta(u) - \Phi_\delta(u))\,\nu(dt) \\ & + [\beta_1\beta_2 + (\beta_1+\beta_2)\delta + \delta^2]\Phi_\delta(u), \end{aligned}$$

with the boundary conditions obtained from the fact that the equation holds at zero and so do the first two derivatives of the equation.

5 Ordering of the Laplace transform of the time of ruin

Since the Laplace transform of the time of ruin are solutions of the newly introduced integro-differential equations, a new comparison of the Laplace transforms of the time of ruin when only the inter-arrival times distributions are different is possible.

Let

$$\begin{aligned} U^{(1)}(t) \ &= \ u + ct + a\int_0^t U^{(1)}(s)\,ds + \sigma\int_0^t U^{(1)}(s)dB(s) \\ &\quad - \sum_{k=1}^{N^{(1)}(t)} \xi_k + \lambda Y(t), \quad t \geq 0, \qquad (5.1) \end{aligned}$$

be a subordinated Cramér-Lundberg risk model with investments perturbed by diffusion in the sense of Bochner in risky asset with a price which follows a geometric Brownian motion. The inter-arrival times $\{\tau_k^{(1)}\}_{k\in\mathbf{N}}$ are independent, $\exp(\beta)$ distributed random variables. The claims arrival process $N^{(1)}(t)$ is a Poisson process. The

Laplace transform of the time of ruin for this process will be denoted by

$$\Phi^1_\delta(u) = \mathbf{E}(e^{-\delta T^{(1)}_u} 1_{\{T^{(1)}_u<\infty\}} \mid U^{(1)}(0) = u),$$

where

$T^{(1)}_u = \inf\{t \geq 0;\ U^{(1)}(t) < 0 \mid U^{(1)}(0) = u\}$ $(\infty,$ otherwise), is the time of ruin of (5.1).

Let

$$\begin{aligned} U^{(2)}(t) &= u + ct + a\int_0^t U^{(2)}(s)\,ds + \sigma\int_0^t U^{(2)}(s)dB(s) \\ &\quad - \sum_{k=1}^{N^{(2)}(t)} \xi_k + \lambda Y(t), \quad t \geq 0, \end{aligned} \tag{5.2}$$

be a subordinated Sparre-Andersen risk model with investments perturbed by diffusion in the sense of Bochner in risky asset as in (5.1), but with inter-arrival times $\{\tau^{(2)}_k\}_{k\in\mathbf{N}}$ are independent, Erlang$(2,\beta)$ distributed random variables. The claims arrival process $N^{(2)}(t)$ is a renewal process. The corresponding Laplace transform of the time of ruin is

$$\Phi^2_\delta(u) = \mathbf{E}(e^{-\delta T^{(2)}_u} 1_{\{T^{(2)}_u<\infty\}} \mid U^{(2)}(0) = u),$$

where

$T^{(2)}_u = \inf\{t \geq 0;\ U^{(2)}(t) < 0 \mid U^{(2)}(0) = u\}$ $(\infty,$ otherwise), is the time of ruin of (5.2).
The comparison of the surplus processes under these different inter-arrival times distributions may be achieved by a coupling of both processes derived from the common underlying Brownian motion. To be precise, one uses

$$\begin{aligned} X(t) &= X(0)\exp\{(a-\frac{\sigma^2}{2})t + \sigma B_t\} \\ &\quad +(c-\lambda\sigma)\int_0^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\qquad +\sigma(B_t - B_u)\}]\,du \\ &\quad +\lambda\int_0^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\qquad +\sigma(B_t - B_u)\}]\,dY(u), \end{aligned} \tag{5.3}$$

the explicit representation in terms of the Brownian motion of the solution of the stochastic differential equation governing the investment process

$$dX_t = (aX_t + c)\,dt + \sigma X_t\,dB_t + \lambda\,dY_t, \tag{5.4}$$

this can be verified using Itô lemma.

Lemma 5.1 *If $X(t)$ satisfies the equation (5.3), then for any $0 \leq s \leq t$,*

$$\begin{aligned} X(t) &= X(s)\exp\{(a-\frac{\sigma^2}{2})(t-s) + \sigma(B_t - B_s)\} \\ &\quad +(c-\lambda\sigma)\int_s^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\qquad +\sigma(B_t - B_u)\}]\,du \\ &\quad +\lambda\int_s^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\qquad +\sigma(B_t - B_u)\}]\,dY(u). \end{aligned}$$

Proof.
As $X(t)$ satisfies the equation (5.3) one has

$$\begin{aligned} X(t) &= X(0)\exp\{(a-\frac{\sigma^2}{2})t + \sigma B_t\} \\ &\quad +(c-\lambda\sigma)\int_0^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\qquad +\sigma(B_t - B_u)\}]\,du \\ &\quad +\lambda\int_0^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\qquad +\sigma(B_t - B_u)\}]\,dY(u). \end{aligned}$$

Then for any $0 \leq s \leq t$, one has

$$\begin{aligned} X(t) &= X(0)[\exp\{(a-\frac{\sigma^2}{2})(t-s+s) \\ &\qquad +\sigma(B_t - B_s + B_s)\}] \\ &\quad +(c-\lambda\sigma)\int_0^s [\exp\{(a-\frac{\sigma^2}{2})[(t-s)+(s-u)] \\ &\qquad +\sigma[(B_t - B_s)+(B_s - B_u)]\}]\,du \\ &\quad +(c-\lambda\sigma)\int_s^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\qquad +\sigma(B_t - B_u)\}]\,du \\ &\quad +\lambda\int_0^s [\exp\{(a-\frac{\sigma^2}{2})[(t-s)+(s-u)] \\ &\qquad +\sigma[(B_t - B_s)+(B_s - B_u)]\}]\,dY(u) \\ &\quad +\lambda\int_s^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\qquad +\sigma(B_t - B_u)\}]\,dY(u) \\ &= X(s)\exp\{(a-\frac{\sigma^2}{2})(t-s) + \sigma(B_t - B_s)\} \\ &\quad +(c-\lambda\sigma)\int_s^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\qquad +\sigma(B_t - B_u)\}]\,du \\ &\quad +\lambda\int_s^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\qquad +\sigma(B_t - B_u)\}]\,dY(u). \end{aligned}$$

Proposition 5.2 *For the processes $U^{(1)}$ and $U^{(2)}$ defined above, the Laplace transform of the time of ruin have the following order:*

$$\Phi^1_\delta(u) \geq \Phi^2_\delta(u). \tag{5.5}$$

Proof.

In order to compare the two Laplace transforms of the time of ruin: $\Phi_\delta^1(u)$ and $\Phi_\delta^2(u)$, one compares the two surplus processes $U^{(1)}$ and $U^{(2)}$ along each sample path of the Brownian motion. Both start with the same initial surplus u and have the same underling Brownian motion. Let $T_1^{(1)}$ denote the time of the first claim in the surplus process $U^{(1)}$.

Then for any $0 \le t \le T_1^{(1)}$, according to the equation (5.3) one has

$$\begin{aligned} U^{(2)}(t) &= u\exp\{(a-\frac{\sigma^2}{2})t+\sigma B_t\} \\ &\quad +(c-\lambda\sigma)\int_0^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\quad\quad +\sigma(B_t-B_u)\}]\,du \\ &\quad +\lambda\int_0^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\quad\quad +\sigma(B_t-B_u)\}]\,dY(u) \\ &= U^{(1)}(t). \end{aligned}$$

For $t = T_1^{(1)}$, one has

$$\begin{aligned} U^{(2)}(T_1^{(1)}) &= u\exp\{(a-\frac{\sigma^2}{2})T_1^{(1)}+\sigma B_{T_1^{(1)}}\} \\ &+ (c-\lambda\sigma)\int_0^{T_1^{(1)}} [\exp\{(a-\frac{\sigma^2}{2})(T_1^{(1)}-u) \\ &\quad +\sigma(B_{T_1^{(1)}}-B_u)\}]\,du \\ &+ \lambda\int_0^{T_1^{(1)}} [\exp\{(a-\frac{\sigma^2}{2})(T_1^{(1)}-u) \\ &\quad +\sigma(B_{T_1^{(1)}}-B_u)\}]\,dY(u) \\ &\ge u\exp\{(a-\frac{\sigma^2}{2})T_1^{(1)}+\sigma B_{T_1^{(1)}}\} \\ &+ (c-\lambda\sigma)\int_0^{T_1^{(1)}} [\exp\{(a-\frac{\sigma^2}{2})(T_1^{(1)}-u) \\ &\quad +\sigma(B_{T_1^{(1)}}-B_u)\}]\,du \\ &+ \lambda\int_0^{T_1^{(1)}} [\exp\{(a-\frac{\sigma^2}{2})(T_1^{(1)}-u) \\ &\quad +\sigma(B_{T_1^{(1)}}-B_u)\}]\,dY(u) \\ &- \xi_1^{(1)} \\ &= U^{(1)}(T_1^{(1)}). \end{aligned}$$

For $T_1^{(1)} \le t \le T_1^{(2)}$, according to lemma 5.1

$$\begin{aligned} U^{(2)}(t) &= U^{(2)}(T_1^{(1)})[\exp\{(a-\frac{\sigma^2}{2})(t-T_1^{(1)}) \\ &\quad +\sigma(B_t-B_{T_1^{(1)}})\}] \\ &\quad +(c-\lambda\sigma)\int_{T_1^{(1)}}^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\quad\quad +\sigma(B_t-B_u)\}]\,du \\ &\quad +\lambda\int_{T_1^{(1)}}^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\quad\quad +\sigma(B_t-B_u)\}]\,dY(u) \\ &\ge U^{(1)}(T_1^{(1)})[\exp\{(a-\frac{\sigma^2}{2})(t-T_1^{(1)}) \\ &\quad +\sigma(B_t-B_{T_1^{(1)}}\}) \\ &\quad +(c-\lambda\sigma)\int_{T_1^{(1)}}^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\quad\quad +\sigma(B_t-B_u)\}]\,du \\ &\quad +\lambda\int_{T_1^{(1)}}^t [\exp\{(a-\frac{\sigma^2}{2})(t-u) \\ &\quad\quad +\sigma(B_t-B_u)\}]\,dY(u) \\ &= U^{(1)}(t). \end{aligned}$$

It follows by induction that $U^{(1)}(t) \le U^{(2)}(t)$ for any t. Therefore, $\Phi_\delta^1(u) \ge \Phi_\delta^2(u)$ for any u.

Proposition 5.3 *For $m < n$, in the case of Erlang(m, β), Erlang(n, β) risk processes*

$$\Phi_\delta^m(u) \ge \Phi_\delta^n(u). \tag{5.6}$$

Proof.

Analogously to the previous proof, it can be shown that

$$\Phi_\delta^m(u) \ge \Phi_\delta^n(u).$$

Inductively, this means

$$\Phi_\delta^1(u) \ge \Phi_\delta^2(u) \ge \Phi_\delta^3(u) \ge ... \ge \Phi_\delta^n(u), \quad n \in \mathbf{N}.$$

Thus, for any $m < n$, the result follows.

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