On the General Solution for the Two-Dimensional Electrical Impedance Equation in Terms of Taylor Series in Formal Powers

M. P. Ramirez T., V. D. Sanchez N., O. Rodriguez T. and A. Gutierrez S.

***Abstract*— We study the general solution of the two-dimensional Electrical Impedance Equation in terms of Taylor series in formal powers, for the case when the electrical conductivity is a separable-variables function. We use the elements of Quaternionic Analysis in order to rewrite the Electrical Impedance Equation into a Vekua equation, for which it is possible to express the general solution using the elements of Bers Pseudoanalytic Function Theory. We finally discuss the contribution of thess results for approaching solutions of the Electrical Impedance Tomography problem.**

***Index Terms*— Electrical impedance, pseudoanalytic functions, quaternions, tomography.**

I. Introduction

The Electrical Impedance Equation

$$\operatorname{div}(\sigma \operatorname{grad} u) = 0, \tag{1}$$

also known as the Inhomogeneous Laplace equation [6], or Poisson equation [15], is the base for well understanding the inverse problem posed by A. P. Calderon in 1980 [3], often referred as Electrical Impedance Tomography. Here σ denotes the conductivity function and u represents the electric potential.

The two-dimensional case of Calderon problem is specially interesting for medical image reconstruction, and it kept the attention of many researchers from its very appearing. Beside the purely numerical approaches, a good alternative for solving the Electrical Impedance Tomography problem is to use a wide class of analytic solutions for (1), posing different conductivity functions, and comparing such solutions valued in the boundary points, with the collected data until the difference can be considered minimum. Nevertheless, the mathematical complexity for solving analytically (1) represented such a challenge, that many experts considered *impossible* to obtain its general solution in analytic form [5], even for the simplest cases of σ (not including the constant case, of course).

But in 2006, K. Astala and L. Päivärinta [1] proved the existence and uniqueness of the solution in the plain for the inverse problem of (1), trough the path of relating the two-dimensional Electrical Impedance Equation with the Vekua equation [20]. One year latter, V. V. Kravchenko and H. Oviedo [13], who had previously noticed the relation of the two-dimensional stationary Schrödinger equation with a special class of Vekua equation [11], used the elements of Bers Pseudoanalytic Functions Theory [2] in order to represent the general solution of (1) in terms of Taylor series in formal powers, and gave what can be considered its first explicit general solution, for a special class of σ.

Applying the elements of Quaternionic Analysis, we will analyze an alternative way for transforming the two-dimensional case of (1) into a Vekua equation [14][18], and we will express its general solution in terms of formal powers for the case when σ is a separable-variables function. We will finally discuss the contribution of these results within the theory of Electrical Impedance Tomography.

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II. Preliminaries

A. Elements of Quaternionic Analysis

We will denote the algebra of real quaternions (see e.g. [8] and [12]) by $\mathbb{H}(\mathbb{R})$. The elements q belonging to $\mathbb{H}(\mathbb{R})$ have the form $q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$, where $q_k,\ k = 0,1,2,3$ are real-valued functions depending upon the spacial variables x_1, x_2 and x_3; whereas e_k are the standard quaternionic units, satisfying the relations

$$\begin{aligned} e_1 e_2 &= e_3 = -e_2 e_1, \\ e_2 e_3 &= e_1 = -e_3 e_2, \\ e_3 e_1 &= e_2 = -e_1 e_3, \\ e_k^2 &= -1,\ k = 1,2,3. \end{aligned}$$

We will also use the notation

$$q = q_0 + \overrightarrow{q},$$

where $\overrightarrow{q} = \sum_{n=1}^{3} q_k e_k$ is usually known as the *vectorial part* of quaternion q, and q_0 is called the *scalar part*. Notice the subset of purely vectorial quaternions $q = \overrightarrow{q}$ can be identified with the set of three-dimensional vectors belonging to $\mathbb{R}^3$. This is, to every $\overrightarrow{E} = (E_1, E_2, E_3) \in \mathbb{R}^3$ corresponds one purely vectorial quaternion $\overrightarrow{E} = E_1 e_1 + E_2 e_2 + E_3 e_3$. It is easy to see this relation is one-to-one.

Due to this isomorphism, we can represent the multiplication of two quaternions q and p as

$$q \cdot p = q_0 p_0 + q_0 \overrightarrow{p} + p_0 \overrightarrow{q} - \langle \overrightarrow{q}, \overrightarrow{p} \rangle + [\overrightarrow{q} \times \overrightarrow{p}], \tag{2}$$

where $\langle \overrightarrow{q}, \overrightarrow{p} \rangle$ denotes the standard inner product

$$\langle \overrightarrow{q}, \overrightarrow{p} \rangle = \sum_{k=1}^{3} q_k p_k$$

and $[\overrightarrow{q} \times \overrightarrow{p}]$ is the vectorial product, written in the quaternionic sense. This is

$$\begin{aligned}&[\overrightarrow{q} \times \overrightarrow{p}] = \\ &= (q_2p_3 - q_3p_2)\, e_1 + \\ &+ (q_3p_1 - q_1p_3)\, e_2 + \\ &+ (q_1p_2 - q_2p_1)\, e_3.\end{aligned}$$

Because of this, we can notice that, in general,

$$q \cdot p \neq p \cdot q$$

so we will use the notation

$$M^p q = q \cdot p$$

to indicate the multiplication by the right-hand side of the quaternion q by the quaternion p.

The Moisil-Theodoresco differential operator D is defined as

$$D = e_1\partial_1 + e_2\partial_2 + e_3\partial_3,$$

where $\partial_k = \frac{\partial}{\partial x_k}$, and it acts on the set of at least once-derivable quaternionic-valued functions. Again, using the classic vectorial notation we can write

$$Dq = \operatorname{grad} q_0 - \operatorname{div} \overrightarrow{q} + \operatorname{rot} \overrightarrow{q}, \tag{3}$$

where

$$\operatorname{grad} q_0 = e_1\partial_1 q_0 + e_2\partial_2 q_0 + e_3\partial_3 q_0,$$

and

$$\begin{aligned}&\operatorname{rot} \overrightarrow{q} = \\ &= (\partial_2 p_3 - \partial_3 p_2)\, e_1 + \\ &+ (\partial_3 p_1 - \partial_1 p_3)\, e_2 + \\ &+ (\partial_1 p_2 - \partial_2 p_1)\, e_3.\end{aligned}$$

B. Elements of Pseudoanalytic Function Theory

Following [2], let F and G be a pair of complex-valued functions satisfying the inequality

$$\operatorname{Im}\left(\overline{F}G\right) > 0, \tag{4}$$

where $\overline{F}$ denotes the complex conjugation of F:

$$\overline{F} = \operatorname{Re} F - i \operatorname{Im} F,$$

and i is the standard complex unit $i^2 = -1$. Therefore any complex-valued function W can be expressed as the linear combination of F and G:

$$W = \phi F + \psi G,$$

where ϕ and ψ are purely real-valued functions. A pair of complex-valued functions satisfying (4) is called a *Bers generating pair*. The derivative in the sense of Bers, or (F,G)-*derivative* of a function W is defined as

$$\frac{d_{(F,G)}W}{dz} = (\partial_z \phi)\, F + (\partial_z \psi)\, G \tag{5}$$

where $\partial_z = \partial_1 - i\partial_2$, and it exists iff

$$(\partial_{\overline{z}} \phi)\, F + (\partial_{\overline{z}} \psi)\, G = 0 \tag{6}$$

where $\partial_{\overline{z}} = \partial_1 + i\partial_2$ (usually the operators ∂_z and $\partial_{\overline{z}}$ are introduced with the factor $\frac{1}{2}$, nevertheless it will result more convenient for us to work without it).

Let us introduce the following functions

$$\begin{aligned} A_{(F,G)} &= -\frac{\overline{F}\partial_z G - \overline{G}\partial_z F}{F\overline{G} - \overline{F}G}, \\ B_{(F,G)} &= \frac{F\partial_z G - G\partial_z F}{F\overline{G} - \overline{F}G}, \\ a_{(F,G)} &= -\frac{\overline{F}\partial_{\overline{z}} G - \overline{G}\partial_{\overline{z}} F}{F\overline{G} - \overline{F}G}, \\ b_{(F,G)} &= \frac{F\partial_{\overline{z}} G - G\partial_{\overline{z}} F}{F\overline{G} - \overline{F}G}. \end{aligned} \tag{7}$$

These functions are known as *the characteristic coefficients* of the generating pair (F,G). According to these notations, the equation (5) can be expressed as

$$\frac{d_{(F,G)}W}{dz} = \partial_z W - A_{(F,G)}W - B_{(F,G)}\overline{W}, \tag{8}$$

and the equation (6) will turn into

$$\partial_{\overline{z}} W - a_{(F,G)}W - b_{(F,G)}\overline{W} = 0. \tag{9}$$

This last equation is known as *the Vekua equation* [20], and the complex-valued functions W that fulfill (9) are named (F,G)-*pseudoanalytic*.

The following statements were originally posed in [2] by L. Bers.

Remark 1: The complex-valued functions of the generating pair (F,G) are (F,G)-pseudoanalytic, and according to (8) their (F,G)-derivatives satisfy

$$\frac{d_{(F,G)}F}{dz} = \frac{d_{(F,G)}G}{dz} = 0.$$

Definition 2: Let (F,G) and (F_1,G_1) be two generating pairs, and let their characteristic coefficients satisfy

$$a_{(F,G)} = a_{(F_1,G_1)} \quad \text{and} \quad B_{(F,G)} = -b_{(F_1,G_1)}. \tag{10}$$

Hence the generating pair (F_1,G_1) will be called *successor* pair of (F,G), as well (F,G) will be the *predecessor* pair of (F_1,G_1).

Theorem 3: Let W be a (F,G)-pseudoanalytic function, and let (F_1,G_1) be a successor pair of (F,G). Then the (F,G)-derivative of W

$$\frac{d_{(F,G)}W}{dz} = \partial_z W - A_{(F,G)}W - B_{(F,G)}\overline{W},$$

will be (F_1,G_1)-pseudoanalytic.

Definition 4: Let (F,G) be a generating pair. Its *adjoint* pair (F^*,G^*) will be defined by the formulas

$$F^* = -\frac{2\overline{F}}{F\overline{G} - \overline{F}G}, \qquad G^* = \frac{2\overline{G}}{F\overline{G} - \overline{F}G}.$$

The (F,G)-*integral* of a complex-valued function W is posed as

$$\int_{\Gamma} W d_{(F,G)} z = \\ = F(z_1)\operatorname{Re}\int_{\Gamma} G^{*} W dz + G(z_1)\operatorname{Re}\int_{\Gamma} F^{*} W dz,$$

where Γ is a rectifiable curve going from z_0 till z_1.

In particular, when $W = \phi F + \psi G$ is (F,G)-pseudoanalytic, then

$$\int_{z_0}^{z} \frac{d_{(F,G)}W}{dz} d_{(F,G)} z = \\ = W(z) - \phi(z_0) F(z) - \psi(z_0) G(z), \tag{11}$$

and since

$$\frac{d_{(F,G)}F}{dz} = \frac{d_{(F,G)}G}{dz} = 0,$$

the integral expression (11) represents the *antiderivative in the sense of Bers* of

$$\frac{d_{(F,G)}W}{dz}.$$

Also, a continuous complex-valued function w is said to be (F,G)-*integrable* iff

$$\operatorname{Re}\oint G^{*} w dz + i \operatorname{Re}\oint F^{*} w dz = 0.$$

Theorem 5: The (F,G)-derivative of a (F,G)-pseudoanalytic function W is (F,G)-integrable.

Theorem 6: Let (F,G) be a predecessor pair of (F_1,G_1). A complex-valued function $\mathcal{E}$ will be (F_1,G_1)-pseudoanalytic iff it is (F,G)-integrable.

Definition 7: Let $\{(F_m,G_m)\}$, $m = 0, \pm1, \pm2, \pm3, \ldots$ be a sequence of generating pairs. If every (F_{m+1},G_{m+1}) is a successor of (F_m,G_m) we say that $\{(F_m,G_m)\}$ is a *generating sequence*. If $(F_0,G_0) = (F,G)$ we say that (F,G) is *embedded* in $\{(F_m,G_m)\}$.

Let W be a (F,G)-pseudoanalytic function, and let $\{(F_m,G_m)\}$, $m = 0, \pm1, \pm2, \pm3, \ldots$ be a generating sequence in which (F,G) is embedded. We can express the higher derivatives in the sense of Bers of W as

$$W^{[0]} = W;$$
$$W^{[m+1]} = \frac{d_{(F_m,G_m)}W^{[m]}}{dz};\ m = 0,1,2,3..$$

where $\frac{d_{(F_m,G_m)}W^{[m]}}{dz}$ is defined by (8).

Definition 8: The formal power $Z_m^{(0)}(a, z_0; z)$ with center at z_0, coefficient a and exponent 0 is defined as

$$Z_m^{(0)}(a, z_0; z) = \lambda F_m(z) + \mu G_m(z)$$

where the coefficients λ and μ are real constants such that

$$\lambda F_m(z_0) + \mu G_m(z_0) = a.$$

The formal powers with exponents $n = 1, 2, 3, \ldots$ are defined by the formulas

$$Z_m^{(n)}(a, z_0; z) = n\int_{z_0}^{z} Z_{m+1}^{(n-1)}(a, z_0; \varsigma)\, d_{(F_m,G_m)}\varsigma.$$

It is possible to verify that formal powers posses the following properties:

1) $Z_m^{(n)}(a, z_0; z)$ is (F_m,G_m)-pseudoanalytic.
2) If a_1 and a_2 are real constants, then
$$Z_m^{(n)}(a_1 + ia_2, z_0; z) = \\ = a_1 Z_m^{(n)}(1, z_0; z) + a_2 Z_m^{(n)}(i, z_0; z). \tag{12}$$
3) The formal powers satisfy the differential relations
$$\frac{d_{(F_m,G_m)}Z_m^{(n)}(a, z_0; z)}{dz} = Z_{m+1}^{(n-1)}(a, z_0; z).$$
4) The formal powers satisfy the asymptotic formulas
$$\lim_{z\to z_0} Z_m^{(n)}(a, z_0; z) = a(z - z_0)^n.$$

Remark 9: As it has been proved in [2], any complex-valued function W, solution of (9), accepts the expansion

$$W = \sum_{n=0}^{\infty} Z^{(n)}(a_n, z_0; z), \tag{13}$$

where the missing subindex m indicates that all formal powers belong to the same generating pair. This is: *expression (13) is an analytic representation of the general solution of (9).*

The Taylor coefficients a_n are obtained according to the formulas

$$a_n = \frac{W^{[n]}(z_0)}{n!}.$$

III. Quaternionic reformulation of the electrical impedance equation, and its relation with the Vekua equation

Consider the electrical impedance equation (1)

$$\operatorname{div}(\sigma \operatorname{grad} u) = 0.$$

Indeed, the electric field vector $\vec{E}$ for the static case is defined as

$$\vec{E} = -\operatorname{grad} u, \tag{14}$$

so we can write

$$\operatorname{div}\left(\sigma \vec{E}\right) = 0. \tag{15}$$

But noticing

$$\operatorname{div}\left(\sigma \vec{E}\right) = \left\langle \operatorname{grad}\sigma, \vec{E}\right\rangle + \sigma \operatorname{div}\vec{E},$$

equation (15) yields

$$\operatorname{div}\vec{E} = -\left\langle \frac{\operatorname{grad}\sigma}{\sigma}, \vec{E}\right\rangle. \tag{16}$$

Beside, from (14) we immediately obtain

$$\operatorname{rot}\vec{E} = 0. \tag{17}$$

Following [14], [18], let us consider now $\vec{E}$ as a purely vectorial quaternionic-valued function

$$\vec{E} = E_1 e_1 + E_2 e_2 + E_3 e_3.$$

Substituting the equalities (16) and (17) in (3) we have

$$D\vec{E} = \left\langle \frac{\operatorname{grad}\sigma}{\sigma}, \vec{E}\right\rangle,$$

or using again (3)

$$D\overrightarrow{E} = \left\langle \frac{D\sigma}{\sigma}, \overrightarrow{E} \right\rangle. \tag{18}$$

Now, according to expression (2), we have that

$$\overrightarrow{q} \cdot \overrightarrow{p} + \overrightarrow{p} \cdot \overrightarrow{q} = -2 \left\langle \overrightarrow{q}, \overrightarrow{p} \right\rangle,$$

and since $\frac{D\sigma}{\sigma}$ is a purely vectorial quaternion, we can write

$$\left\langle \frac{D\sigma}{\sigma}, \overrightarrow{E} \right\rangle = -\frac{1}{2} \left(\frac{D\sigma}{\sigma} \overrightarrow{E} + \overrightarrow{E} \frac{D\sigma}{\sigma} \right).$$

Hence, it follows from (18)

$$D\overrightarrow{E} = -\frac{1}{2} \left(\frac{D\sigma}{\sigma} \overrightarrow{E} + \overrightarrow{E} \frac{D\sigma}{\sigma} \right). \tag{19}$$

But according to the Liapunov rule of derivation, when we apply the Moisil-Theodoresco operator to $\sqrt{\sigma}$ we obtain

$$D\sqrt{\sigma} = \frac{1}{2} \frac{1}{\sqrt{\sigma}} D\sigma,$$

thus we have

$$\frac{1}{2} \frac{D\sigma}{\sigma} = \frac{D\sqrt{\sigma}}{\sqrt{\sigma}}.$$

Taking into account this last expression, (19) turns into

$$D\overrightarrow{E} = -\left(\frac{D\sqrt{\sigma}}{\sqrt{\sigma}} \overrightarrow{E} + \overrightarrow{E} \frac{D\sqrt{\sigma}}{\sqrt{\sigma}} \right). \tag{20}$$

Let us consider now the Moisil-Theodoresco operator D applied to $\sqrt{\sigma}\overrightarrow{E}$. By the Leibiniz rule of derivation we obtain

$$D\left(\sqrt{\sigma}\overrightarrow{E}\right) = \left(D\sqrt{\sigma}\right) \overrightarrow{E} + \sqrt{\sigma} D\overrightarrow{E},$$

hence

$$D\overrightarrow{E} = \frac{1}{\sqrt{\sigma}} D\left(\sqrt{\sigma}\overrightarrow{E}\right) - \frac{D\sqrt{\sigma}}{\sqrt{\sigma}} \overrightarrow{E}.$$

Substituting this equality into the left side of (20) we obtain

$$\frac{1}{\sqrt{\sigma}} D\left(\sqrt{\sigma}\overrightarrow{E}\right) - \frac{D\sqrt{\sigma}}{\sqrt{\sigma}} \overrightarrow{E} =$$
$$= -\left(\frac{D\sqrt{\sigma}}{\sqrt{\sigma}} \overrightarrow{E} + \overrightarrow{E} \frac{D\sqrt{\sigma}}{\sqrt{\sigma}} \right),$$

and it follows

$$\frac{1}{\sqrt{\sigma}} D\left(\sqrt{\sigma}\overrightarrow{E}\right) = -\overrightarrow{E} \frac{D\sqrt{\sigma}}{\sqrt{\sigma}},$$

or

$$D\left(\sqrt{\sigma}\overrightarrow{E}\right) + \sqrt{\sigma}\overrightarrow{E} \frac{D\sqrt{\sigma}}{\sqrt{\sigma}} = 0. \tag{21}$$

Introducing the notations

$$\overrightarrow{\mathcal{E}} = \sqrt{\sigma}\overrightarrow{E}, \tag{22}$$

and

$$\overrightarrow{\sigma} = \frac{D\sqrt{\sigma}}{\sqrt{\sigma}}, \tag{23}$$

equality (21) turns into the equation

$$\left(D + M^{\overrightarrow{\sigma}}\right) \overrightarrow{\mathcal{E}} = 0, \tag{24}$$

which is a quaternionic reformulation of (1).

A. The two-dimensional case

Let us consider the special case when

$$\overrightarrow{\mathcal{E}} = \mathcal{E}_1 e_1 + \mathcal{E}_2 e_2 \tag{25}$$

and σ depends upon only two spatial variables $\sigma = \sigma(x_1, x_2)$. Thus, the expression (23) takes the form

$$\overrightarrow{\sigma} = \sigma_1 e_1 + \sigma_2 e_2,$$

where

$$\sigma_1 = \frac{\partial_1 \sqrt{\sigma}}{\sqrt{\sigma}}, \quad \sigma_2 = \frac{\partial_2 \sqrt{\sigma}}{\sqrt{\sigma}}. \tag{26}$$

Substituting (25) and (26) into (24) we have

$$D(\mathcal{E}_1 e_1 + \mathcal{E}_2 e_2) + (\mathcal{E}_1 e_1 + \mathcal{E}_2 e_2)(\sigma_1 e_1 + \sigma_2 e_2) = 0,$$

which is equivalent to the system

$$\begin{aligned} \partial_1 \mathcal{E}_1 + \partial_2 \mathcal{E}_2 &= -\mathcal{E}_1 \sigma_1 - \mathcal{E}_2 \sigma_2, \\ \partial_1 \mathcal{E}_2 - \partial_2 \mathcal{E}_1 &= \mathcal{E}_3 \sigma_1 - \mathcal{E}_1 \sigma_2, \\ \partial_3 \mathcal{E}_1 &= \partial_3 \mathcal{E}_2 = 0. \end{aligned}$$

Multiplying the second equation by $-i$ and adding to the first, it yields

$$\partial_{\overline{z}} (\mathcal{E}_1 - i\mathcal{E}_2) + (\sigma_1 - i\sigma_2)(\mathcal{E}_1 + i\mathcal{E}_2) = 0, \tag{27}$$

but according to (26) it is possible to see that

$$\sigma_1 - i\sigma_2 = \frac{\partial_z \sqrt{\sigma}}{\sqrt{\sigma}}.$$

Taking this into account and introducing the notation

$$\mathcal{E} = \mathcal{E}_1 - i\mathcal{E}_2, \tag{28}$$

the equation (27) becomes

$$\partial_{\overline{z}} \mathcal{E} + \frac{\partial_z \sqrt{\sigma}}{\sqrt{\sigma}} \overline{\mathcal{E}} = 0. \tag{29}$$

which is a special kind of Vekua equation [20].

We shall mention that in [4], the authors obtained a bicomplex Vekua equation similar to (29), starting from a quaternionic equation with the same structure that (24), but related to the Dirac equation with different classes of potentials.

In order to analyze the general solution of (29), it will be convenient to associate it with another Vekua equation of the form

$$\partial_{\overline{z}} W - \frac{\partial_{\overline{z}} \sqrt{\sigma}}{\sqrt{\sigma}} \overline{W} = 0, \tag{30}$$

as we shall expose in the following paragraphs [13].

Let

$$F = \sqrt{\sigma} \quad \text{and} \quad G = \frac{i}{\sqrt{\sigma}}. \tag{31}$$

It is easy to verify these functions satisfy (4), so they constitute a generating pair, and according to (7), their characteristic coefficients are

$$A_{(F,G)} = a_{(F,G)} = 0,$$
$$B_{(F,G)} = \frac{\partial_z \sqrt{\sigma}}{\sqrt{\sigma}}, \quad b_{(F,G)} = \frac{\partial_z \sqrt{\sigma}}{\sqrt{\sigma}}.$$

Let us notice that, in concordance with *Definition 2*, the characteristic coefficients corresponding to a successor pair (F_1, G_1) of the pair $\left(\sqrt{\sigma}, \frac{i}{\sqrt{\sigma}}\right)$ must verify the relations

$$a_{(F_1,G_1)} = 0, \quad b_{(F_1,G_1)} = -\frac{\partial_z \sqrt{\sigma}}{\sqrt{\sigma}}.$$

Beside, a (F_1, G_1)-pseudoanalytic function $\mathcal{E}$ must fulfill equation (29).

Remark 10: By *Theorem 3*, the $\left(\sqrt{\sigma}, \frac{i}{\sqrt{\sigma}}\right)$-derivative of any solution of (30) will be a solution of (29).

We have now established the relation between the Vekua equation (29) and the Vekua equation (30).

Moreover, since the general solution of (30) can be represented by means of (13), once we have a generating sequence where the pair $\left(\sqrt{\sigma}, \frac{i}{\sqrt{\sigma}}\right)$ is embedded, we will be able to express the general solution of (29) as the $\left(\sqrt{\sigma}, \frac{i}{\sqrt{\sigma}}\right)$-derivative of the general solution of (30).

It is important to mention that, in general, it is not clear how to build a generating sequence in which an arbitrary generating pair is embedded. Although, using new results in Applied Pseudoanalytic Functions Theory [9], we are able to write an explicit generating sequence for the case when the desired *embedded generating pair* belongs to a special class of functions that, without loss of generality, fulfills the requirements of the Electrical Impedance Tomography.

B. Explicit generating sequence for the case when σ is a separable-variables function

Since the early appearing of Bers Pseudoanalytic Function Theory [2], the development of methods for introducing explicit generating sequences, in which a specific generating pair is embedded, have represented a very interesting challenge. We shall remark that an explicit generating sequence is required if we desire to express the general solution of a Vekua equation in terms of Taylor series in formal powers.

When considering the Electrical Impedance Equation (1), a very important case is referred to a separable-variables conductivity σ function

$$\sigma(x_1, x_2) = U^2(x_1)V^2(x_2),$$

because it represents a very useful approach for the problem of Electrical Impedance Tomography (see e.g. [6]).

For this case, an explicit generating sequence was introduced by V. V. Kravchenko as follows.

Theorem 11: [10] Let (F, G) be a generating pair of the form

$$\begin{aligned} F &= \sqrt{\sigma} = U(x_1)V(x_2), \\ G &= \frac{i}{\sqrt{\sigma}} = \frac{i}{U(x_1)V(x_2)}. \end{aligned}$$

Then, it is embedded in the generating sequence $\{(F_m, G_m)\}$, $m = 0, \pm1, \pm2, \pm3, \ldots$ defined as

$$\begin{aligned} F_m &= 2^m U(x_1)V(x_2), \\ G_m &= i\frac{2^m}{U(x_1)V(x_2)}; \end{aligned}$$

when m is an even number, and

$$\begin{aligned} F_m &= 2^m \frac{V(x_2)}{U(x_1)}, \\ G_m &= i2^m \frac{U(x_1)}{V(x_2)}; \end{aligned}$$

when m is odd.

Remark 12: Given an explicit generating sequence where the generating pair

$$\begin{aligned} F &= \sqrt{\sigma} = U(x_1)V(x_2), \\ G &= \frac{i}{\sqrt{\sigma}} = \frac{i}{U(x_1)V(x_2)}, \end{aligned}$$

is embedded, we are in the possibility of building the Taylor series in formal powers in order to approach the general solution of the Vekua equation (30). According to *Theorem 3*, the $\left(\sqrt{\sigma}, \frac{i}{\sqrt{\sigma}}\right)$-derivative of such solution will be the general solution of the Vekua equation (29). Hence, the real and the imaginary components of the solution of (29) will constitute the general solution for the two-dimensional case of the quaternionic equation (24). Finally, using (22), it immediately follows we are able to write the general solution for the two-dimensional Electrical Impedance Equation (1).

Remark 13: [17] Introducing the notations

$$\begin{gathered} p = Ae^{-\int \sigma_1 dx_1 + \int \sigma_2 dx_2}, \\ \partial_{\bar{z}_1} = \partial_2 + i\partial_1, \\ V = \overline{\mathcal{E}}, \end{gathered}$$

where A is a real constant, σ_1 and σ_2 are defined in (26), and according to (28), $\overline{\mathcal{E}}$ has the form

$$\overline{\mathcal{E}} = \mathcal{E}_1 + i\mathcal{E}_2,$$

the equation (29) will turn into the equation

$$\partial_{\bar{z}_1} V + \frac{\partial_{\bar{z}_1} p}{p}\overline{V} = 0. \tag{32}$$

It is evident that p is a separable-variables function, and since (32) has identical structure to (30), its general solution can be approached with the same methods exposed before. This represents an alternative path for approaching the general solution of (1).

IV. Conclusions

Since the study of equation (1) is the base for the Electrical Impedance Tomography problem, the possibility of expressing the general solution of (1) by means of Taylor series in formal powers, opens a new path for improving the convergence speed of many numerical methods designed for medical image reconstruction.

We should notice that the mathematical methods studied before impose minimal restrictions to the conductivity function σ. Indeed, it is only necessary for σ to be a separable-variables function in the Cartesian plane, and to be at least once derivable. This is a very general case which includes most part of mathematical approaches for physical situations in Electrical Impedance Tomography (see e.g. [5], [6] and [15]).

We should also notice that the numerical methods that might be used when applying the techniques shown before, belong almost exclusively to the evaluation of the integral operators that are needed for approaching the formal powers. This task can be accomplished by quite standard numerical procedures, hence we can lead our further discussions to approach the constants for Taylor series at the moment of solving the problem of Electrical Impedance Tomography.

Notice also that the equivalence of the two-dimensional Electrical Impedance Equation with a Vekua equation is precisely the key that warrants the uniqueness of the solution of Calderon problem [1], hence from a proper point of view, the techniques exposed before may work as a powerful complement for the well developed electronic systems, designed for detecting with high accuracy the potentials around the domains of interest of tomography [7][16].

References

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