On the Exit Laws for Semidynamical Systems and Bochner Subordination

Hassen Mejri * and Ezzedine Mliki †

Abstract—Let \( \Phi : [0, \infty] \times E \to E \) be a semidynamical system and \( \beta = (\beta_t)_{t \geq 0} \) be a Bochner subordinator. It is proved in this paper that, every \( \beta \)-Liapunov function \( l \) for \( \Phi \) is of the form \( l(x) = \int_0^\infty f(t, x) \, dt \) where \( f : [0, \infty] \times E \to [0, \infty] \) be a solution of the following functional equation

\[
\int_0^\infty f(t, \Phi(r, x)) \beta_r(dr) = f(s + t, x), \quad s, t > 0, x \in E.
\]

We deduce an explicit formula for \( \alpha \)-Liapunov functions defined by the fractional power subordinator of order \( \alpha \in [0, 1] \).

Keywords: semidynamical system, Bochner subordinator, exit law.

1 Introduction

Let \( \Phi : [0, \infty] \times E \to E \) be a measurable semidynamical system on a measurable space \( E \) and let \( F \) be the space of measurable finite functions defined on \( E \). Let \( \beta = (\beta_t)_{t \geq 0} \) be a Bochner subordinator, i.e., a convolution semigroup of probability measures on \([0, +\infty[\). We may define

\[
Q_t u(x) := \int_0^\infty u(\Phi(s, x)) \beta_s(ds), \quad u \in F, \quad t \geq 0, x \in E.
\]

A \( \beta \)-exit law associated to \( \Phi \) is a family \( f = (f_t)_{t \geq 0} \) of positive measurable function satisfying the functional equation (using the notation \( f_t := f(t, \cdot) \))

\[
Q_s f_t = f_{s+t}, \quad s, t > 0.
\]

The integral representation in terms of exit law is originally given by Dynkin [4] and its studied by several authors [6, 7, 8, 9, 10] and [12, 13, 14, 15]. In this paper, we investigate first the representation by \( \beta \)-exit laws. In this case, if the function \( \int_0^{\infty} f_t \, dt \) is finite then it belongs to the cone of \( Q \)-Liapunov functions defined by

\[
L^\beta := \{ u \in F : u \geq 0, Q_t u \leq u, \lim_{t \to 0} Q_t u = u \}
\]

Conversely, there are elementary examples for which elements from \( L^\beta \) do not admit an integral representation by a \( \beta \)-exit law (cf. [14], Example 2.7.1). In fact, as it is observed in many papers related to this problem (cf. [6, 7, 8, 9, 10, 12, 13, 14, 15]), some finiteness assumptions are needed, in order to represent elements of \( L^\beta \) in terms of \( \beta \)-exit laws. Along this paper, elements from \( L^\beta \) which is bounded on each trajectory of \( \Phi \) will be called \( \beta \)-Liapunov functions.

For our context, it is proved in [14] that, for each \( \eta^\alpha \)-Liapunov function \( l \) such that \( \lim_{t \to \infty} Q_t^\alpha u = 0 \), there exists a unique \( \eta^\alpha \)-exit law \( f^\alpha = (f^\alpha_t)_{t \geq 0} \) such that

\[
l(x) = \int_0^\infty f^\alpha_t(x) \, dt, \quad x \in E \quad (1)
\]

The aim of the present paper is to show that a similar, and in fact more general that (1). In what follows we shall denote by \( K \) the set of all Bochner subordinator \( \beta \) such that \( t \to \beta_t \) is continuously differentiable from \([0, \infty[ \) to the Banach algebra of complex borel measures on \( S \) such that \( \| \beta_t^\alpha \|_S < \infty \) for each \( t > 0 \). We prove the following integral representation result:

Let \( \beta \) be in \( K \). For each \( \beta \)-Liapunov function \( l \) there exists a unique (up to equivalence) \( \beta \)-exit law \( f = (f_t)_{t \geq 0} \) for \( \Phi \) such that

\[
l(x) = \int_0^\infty f_t(x) \, dt, \quad x \in E.
\]

Moreover, \( f = (f_t)_{t \geq 0} \) is explicitly given by

\[
f_t(x) = -\int_0^\infty \Phi(s, x) \frac{\partial}{\partial t} \beta_t(ds), \quad t > 0, x \in E.
\]

As application, we consider the fractional power subordinator \( \eta^\alpha := (\eta^\alpha_t) \) of order \( \alpha \in [0, 1] \). It is defined by its Laplace transform \( \mathcal{L}(\eta^\alpha_t)(r) = \exp(-tr^\alpha) \). In this case, under some regular assumption we prove that each \( \eta^\alpha \)-Liapunov function \( l \) admits the integral representation.

\[
l(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \varphi_t(x) t^{\alpha-1} \, dt, \quad x \in E
\]

where

\[
\varphi_t(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \left( l(\Phi(t, x)) - l(\Phi(s + t, x)) \right) ds
\]

Moreover, formulas like (2) and (3) will be also deduced for the \( \Gamma \)-subordinator and for the Poisson subordinator.

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Let \( \Phi \) be a SDS on \( \mathbb{E} \), we define a semigroup

\[
B(t) := \Phi(t,x) = \int_0^t \Phi(s,x - \xi(s)) ds,
\]

where \( \xi(s) \) is a stochastic process. We consider \( \Phi \) associated to the SDS \( \Phi \). Let \( \sigma \) be a semimartingale, and let \( \tau \) be the first hitting time of \( \mathbb{E} \). We denote by \( \Phi(\tau, t) := \Phi(\tau, \cdot) \) is the solution of the stochastic differential equation

\[
d\xi(s) = \xi(s) \, dt + \sigma(s) \, dW(s),
\]

where \( \sigma(s) \) is a \( \mathcal{F}_s \)-adapted process and \( W(s) \) is a standard Brownian motion. We denote by \( \sigma(s) \) the drift and \( \sigma(s) \) the volatility.

We refer to the references [8, 9, 10, 12] for more details.

2 Preliminary

A semidynamical system (SDS) on \( \mathbb{E} \) is a measurable mapping \( \Phi : \mathbb{E} \rightarrow \mathbb{E} \) which satisfies

\[
\Phi(0, x) = x, \quad \Phi(s + t, x) = \Phi(s, \Phi(t, x)), \quad s, t \geq 0, \quad x \in \mathbb{E}.
\]

We consider \( R \) endowed with its Borel field, we denote by \( \lambda \) the Lebesgue measure on \( [0, \infty] \). Let \( \mathcal{F}_t \) be the set of all finite sets defined on \( R \) and \( \mathcal{F}_t + \) be the subset of positive elements of \( \mathcal{F}_t \). Note that any linear operator defined on the space \( \mathcal{B} \) may be extended to any positive measurable function in the usual way. The space \( [0, \infty] \times \mathbb{E} \) is always endowed with product \( \sigma \)-algebra \( \mathcal{E} \otimes \mathcal{A} \).

We denote by \( \nu \) a measure on \( [0, \infty] \times \mathbb{E} \) which satisfies \( \nu([0, \infty] \times \mathbb{E}) = 1 \) and supported on \( \mathbb{E} \). We denote by \( \nu(\cdot, \mathbb{E}) \) the total variation measure at point \( \mathbb{E} \).

\[
\int_0^\infty \nu([0, \infty) \times \mathbb{E}) ds = 1
\]

We shall give some sufficient condition for the Bernstein function. The associated Bernstein function is defined by \( k := \int_0^\infty \beta_s \, ds \). Following (cf. [2], Proposition 14.1) \( k \) is a Bochner measure. The associated Bernstein function \( k \) is defined by the Laplace transform \( \mathcal{L}(\beta)(r) = \exp(-tr(r)) \) for all \( r, t > 0 \). It is known that \( k \) admits the representation (cf. [2], Theorem 9.8)

\[
k(r) = br + \int_0^\infty (1 - \exp(-rs)) \nu(ds), \quad r > 0
\]

where \( b \geq 0 \) and \( \nu \) is a measure on \( [0, \infty] \). \( \mathcal{L} \) is a convolution semigroup.

The most important example of Bochner subordinator in the class \( \kappa \) is the one-sided or fractional power stable subordinator of index \( \beta \in (0, 1] \).

Examples 2.2 Let \( \beta \) be a Bochner subordinator and let \( k \) be the associated Bernstein function given by (4). We shall give some sufficient condition for the Bernstein function in order to get a subordinator in \( \kappa \). We exhibit such examples of subordinator be in \( \kappa \) which contains a number of important functions, including fractional powers, the logarithm, the inverse hyperbolic cosine. We refer to [3] and [16].

1. If \( \sup_{t \in S} |F^\beta(t, u)| = O(t^{-1}) \), \( t \downarrow 0 \) where

\[
F^\beta(t, u) := \int_0^\infty \int_0^\infty u(r) \frac{\partial}{\partial r} (\beta_t(r-s) - \beta_t(r)) \nu(ds) dr
\]

and \( S \) is the unit sphere of the complex space of exponential polynomials with respect to sup-norm on \( R_+ \). Then \( \beta \in \kappa \). For example:

\[
i. \quad \alpha \in [0, 1], \quad c \geq 0 \quad \text{and} \quad k(r) = (c + r)^\alpha - c^\alpha.
\]

Then \( \beta \in \kappa \).

\[
ii. \quad 0 < \alpha < \gamma < 1 \quad \text{and} \quad k(r) = r^\alpha - (\exp(-r^\gamma) - 1).
\]

Then \( \beta \in \kappa \).

2. Let \( r \mapsto \beta_t([r-r, r]) \) is monotone decreasing function on \( [s, \infty) \) for all \( s \geq 0 \) for each sufficiently small \( t > 0 \). If

\[
\int_0^\infty \beta_t([0, s]) \nu(ds) = O(t^{-1}) \quad \text{as} \quad t \downarrow 0,
\]

then \( \beta \in \kappa \). For examples:

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i) Let \( b > 0 \) and \( k(r) = \log(b + r) - \log b \), then \( \beta \in \mathcal{K} \).

ii) Let \( b, s \geq 0 \) and \( k(r) = \acosh(b + r) - \acosh b \), then \( \beta \in \mathcal{K} \).

3. \( \varepsilon \ast \beta \) is not in \( \mathcal{K} \).

4. If \( \beta^1 \) and \( \beta^2 \) are in \( \mathcal{K} \) then so is \( \beta^1 \ast \beta^2 \).

Let \( \Phi \) be a SSD and \( \beta \) be a Bochner subordinator. Define \( Q = (Q_t)_{t \geq 0} \) by

\[
Q_t u(x) := \int_0^\infty u(\Phi(t, x)) \beta_t(dr)
\]

for all \( u \in \mathcal{B}, t \geq 0 \) and \( x \in E \). Then \( Q \) is a semigroup of linear operator on \( \mathcal{B} \). This is clear by using the translation equation of \( \Phi \) and semigroup property of \( \beta \). The potential kernel associated to \( Q \) is defined by \( V^\beta := \int_0^\infty Q_t dt \). By integration of (5), we get

\[
V^\beta u(x) := \int_0^\infty Q_t u(x) dt = \int_0^\infty u(\Phi(t, x)) \kappa(dt)
\]

for all \( u \in \mathcal{B} \) and \( x \in E \).

**Definition 2.3** A positive measurable function \( l \in \mathcal{F} \) is called \( Q \)-Liapunov function for \( \Phi \) if for any \( x \in E \)

(i) The function \( t \to Q_t l(x) \) is decreasing,

(ii) \( \lim_{t \to 0} Q_t l(x) = l(x) \).

We denote by \( L^\beta \) the cone of such functions.

Let \( \text{Im}(V^\beta) := \{ V^\beta u : u \in \mathcal{F}, V^\beta u \in \mathcal{F} \} \). It is clear to see that \( \text{Im}(V^\beta) \subset L^\beta \). If we instead \( Q \) by the deterministic semigroup \( H \) associated to \( \Phi \) then each function \( l \in \mathcal{F} \) satisfying (i) and (ii) is called classical Liapunov function for \( \Phi \).

Let \( \Phi \) be SDS and \( \beta \) be in \( \mathcal{K} \). A \( \beta \)-exit law associated to \( \Phi \) is a measurable function \( f : [0, \infty[ \times E \to [0, \infty] \) which satisfies:

\[
\int_0^\infty f(t, \Phi(s, x)) \beta_t(dr) = f(s + t, x)
\]

for all \( s, t > 0 \) and \( x \in E \). The functional equation (7) is called \( \beta \)-exit equation. By (5) and the notation \( f_s(x) := f(t, x) \), (7) is equivalent to

\[
Q_s f_t(x) = f_{s+t}(x), \quad s, t > 0, x \in E
\]

For example, for \( u \in \mathcal{F}_+ \), the function \( (t, x) \to Q_t u(x) \) is a \( \beta \)-exit law for \( \Phi \) whenever it is finite. This follows immediately from the semigroup property of \( Q \). Two \( \beta \)-exit laws \( f \) and \( \psi \) are said to be equivalent if \( f_t = \psi_t \), \( \lambda \)-a.e.

**Lemma 2.4** Let \( \beta \in \mathcal{K} \). Then

\[
\beta_{s+t} = \beta_s \ast \beta_t, \quad s, t > 0
\]

and

\[
\beta_t = -\beta_t' \ast \kappa, \quad t > 0
\]

where \( \beta_t' := \frac{d}{dt} \beta_t \) and \( \kappa = \int_0^\infty \beta_t dt \).

Proof. Let \( \beta \in \mathcal{K} \). Since \( \mathcal{L}(\beta_t)(r) = \exp(-tf(r)) \), then by differentiation with respect to \( t \) under the integral sign, we obtain

\[
\mathcal{L}(\beta_t) = \frac{d}{dt} \mathcal{L}(\beta_t)(r) = -f(r) \exp(-tf(r)); \quad t, r > 0
\]

Let \( s, t, r > 0 \), we get

\[
\mathcal{L}(\beta_s \ast \beta_t)(r) = \mathcal{L}(\beta_s)(r) \mathcal{L}(\beta_t)(r)
\]

\[
= -f(r) \exp(-sf(r)) \exp(-tf(r))
\]

\[
= -f(r) e^{-(s+t)f(r)}
\]

\[
= \mathcal{L}(\beta_{s+t})(r)
\]

Moreover, since \( \mathcal{L}(\kappa)(r) = \frac{1}{f(r)} \) (cf. [2], Proposition 14.1) we have

\[
\mathcal{L}(-\beta_s \ast \kappa)(r) = -\mathcal{L}(\beta_s)(r) \mathcal{L}(\kappa)(r)
\]

\[
= f(r) \exp(-sf(r)) \frac{1}{f(r)}
\]

\[
= \mathcal{L}(\beta_t)(r)
\]

We deduce (9) and (10) by the injectivity of Laplace transform.

**3 Representation in terms of \( \beta \)-exit laws**

**Proposition 3.1** Let \( \Phi \) be a SDS and let \( f = (f_t)_{t > 0} \) be a \( \beta \)-exit law such that \( l(x) := \int_0^\infty f_t(x) dt < \infty \). Then \( l \) is \( Q \)-Liapunov function, moreover

\[
f_t(x) = -\frac{\partial}{\partial t} Q_t l(x), \quad t > 0, x \in E
\]

Proof. By Fubini’s Theorem and (8) we get for all \( x \in E \)

\[
Q_t l(x) = \int_0^t Q_s f_s(x) ds = \int_0^t f_s(x) ds.
\]

Therefore, \( Q_t l \) is finite since \( \int_0^\infty f_t dt < \infty \) and

\[
Q_t l(x) = \int_0^t f_s(x) ds, \quad t > 0, x \in E
\]

Now from (12), we easily deduce that \( l \) is \( Q \)-Liapunov function. Moreover, by (12) again we have for \( r, t > 0 \)

\[
\frac{1}{r} (Q_{r+t} l - Q_t l) = -\frac{1}{r} \int_t^{r+t} f_s ds
\]

Hence we obtain (11).
Let $\mathcal{R}^L$ be the cone of functions $u := \int_0^\infty f_x dt$ such that $f$ is an exit law for $\Phi$ and $u$ is finite. From Proposition 3.1, it follows that

$$\text{Im}(V^\beta) \subset \mathcal{R}^L \subset L^\beta.$$ 

But, the converse is not true in general, i.e. elements of $L^\beta$ are not necessary on the form $u = \int_0^\infty f_x ds$ for some $Q$-exit laws $f$. As it is observed in many papers related to this problem (cf. [6, 7, 8, 9, 10, 12, 13, 14, 15]), we need some finiteness assumptions, in order to represent the $Q$-Liapunov functions in terms of the $\beta$-exit laws of $\Phi$. In what follows, elements $u$ of $L^\beta$ for which there exists a $v \in \mathcal{F}_+$ such that $u(\Phi(t, x)) \leq v(x)$ for each $t \geq 0$ and each $x \in E$ will be called $\beta$-Liapunov functions. This means that $u$ is bounded on each trajectory of $\Phi$.

**Theorem 3.2** Let $\Phi$ be a SDS, $\beta$ in $\mathcal{K}$ and let $l$ be an associated $\beta$-Liapunov function, then the function $f$ defined by

$$f_t(x) = -\int_0^\infty l(\Phi(s, x)) \beta_t(ds), \quad t \geq 0, \quad x \in E$$

is an exit law for $\Phi$.

Proof. Let $\beta$ be in $\mathcal{K}$ and let $l$ be a $\beta$-Liapunov function. Since $l \circ \Phi_t \leq v$ for each $t \geq 0$ and $\beta_t([0, \infty]) = 1$, it follows that

$$Q_l(x) = \int_0^\infty l(\Phi(r, x)) \beta_t(dr) \leq v(x).$$

Hence $Q_l$ is a finite function. Now, since $l \circ \Phi_t \leq v$ again and the total variation of $\beta_t$ is finite, the following function is well defined

$$f_t(x) := -\int_0^\infty l(\Phi(r, x)) \beta_t'(dr), \quad t > 0, \quad x \in E,$$ and the differentiation with respect to $t$ under the integral sign is justified in $Q_l$. We may define

$$f_t(x) = -\frac{\partial}{\partial t} Q_l(x), \quad t > 0, \quad x \in E \quad (14)$$

Now, since $t \to Q_l(x)$ is decreasing, (14) allows us to conclude that $f_t \geq 0$ for all $t > 0$. Moreover, by Fubini Theorem’s, (5) and (9), we have

$$Q_tf_s(x) = \int_0^\infty f_t(\Phi(m, x)) \beta_t(dm)$$

$$= -\int_0^\infty \int_0^\infty l(\Phi(r, \Phi(m, x))) \beta_t'(dr) \beta_t(dm)$$

$$= -\int_0^\infty \int_0^\infty l(\Phi(r + m, x)) \beta_t'(dr) \beta_t(dm)$$

$$= -\int_0^\infty l(\Phi(r, x)) (\beta_t' \beta_t)(dr)$$

$$= -\int_0^\infty l(\Phi(r, x)) \beta_t'(dr)$$

It follows that $f$ is a $Q$-exit law.

**Remarks 3.3** In [14] under the condition $\lim_{s \to \infty} Q_l = 0$, we proved the representation given above by (17) of $\eta^\beta$-Liapunov function defined by the fractional power subordinator of order $\alpha \in [0, 1]$ in terms of $\eta^\beta$-exit law.

Now we may obtain under the same condition the representation for all subordinator in $\mathcal{K}$. Indeed, from (14) it is easy to see that

$$Q_l(x) - Q_s(x) = \int_s^\infty f_r(dr), \quad s, t > 0, \quad x \in E \quad (15)$$

then, by letting $s \to \infty$ in (15), we deduce that $r \mapsto f_r(x)$ is integrable at $\infty$ and

$$Q_l(x) = \int_0^\infty f_r(dr), \quad t > 0, \quad x \in E \quad (16)$$

we conclude by letting $t \to 0$ in (16).

In fact in Theorem 3.4 we prove that condition $\lim_{s \to \infty} Q_l = 0$, is not necessary to get the representation of $\beta$-Liapunov functions in terms of $\beta$-exit law where $\beta$ is a Bochner subordinator in the class $\mathcal{K}$.

**Theorem 3.4** Let $\Phi$ be a SDS and let $\beta$ in $\mathcal{K}$. For each $\beta$-Liapunov function $l$, there exists a unique (up to equivalence) $\beta$-exit law $f = (f_t)_{t > 0}$ for $\Phi$ such that

$$l(x) = \int_0^\infty f_t(x) dt, \quad x \in E \quad (17)$$

Moreover, $f$ is explicitly given by

$$f_t(x) = -\int_0^\infty l(\Phi(s, x)) \partial_t \beta_t(ds), \quad t > 0, \quad x \in E \quad (18)$$

Proof. Let $\beta$ be in $\mathcal{K}$ and let $l$ be a $\beta$-Liapunov function. By Theorem 3.2 we may define

$$f_t(x) = -\partial_t Q_l(x), \quad t > 0, \quad x \in E.$$ 

By Fubini’s Theorem, (5), (10) and (9) we have for fixed $s, t > 0$

$$Q_{s+t} = \int_0^\infty H_r(Q_s l) \beta_t(dr)$$

$$= -\int_0^\infty H_r(Q_s l) (\beta_t^\prime \kappa)(dr)$$

$$= -\int_0^\infty \int_0^\infty H_{r+t} Q_s l (\beta_t^\prime dr) \kappa(d\ell)$$

$$= -\int_0^\infty \int_0^\infty H_r Q_s l (\beta_t^\prime dr) \beta_t(d\ell) dq$$

$$= -\int_0^\infty \int_0^\infty H_r Q_s l (\beta_t^\prime \beta_t)(dr) dq$$

$$= -\int_0^\infty \int_0^\infty H_r Q_s l (\beta_t^\prime \beta_t^\prime)(dr) dq$$

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for all \( x \in E \). Let \( \mathcal{S}^\beta \) be the cone of finite functions on the form (22). From (5) and (22) again, we deduce that
\[
\text{Im}(V^\beta) \subset \mathcal{S}^\beta \subset \mathcal{R}^\beta.
\]

3. We consider the function \( g_t \) be the density of \( \eta_t^2 \). It is easy to see that \( g_t \) is a \( Q \)-exit law. Furthermore it is known that \( \lim_{t \to 0} g_t(x) = 0 \) for each \( x \in R \). Hence
\[
u := \int_0^\infty g_t \, dt \in R^2 \setminus S^2.
\]
(14) Example 2.7.2. Under some regular assumption we prove that \( \mathcal{S}^\beta = \mathcal{R}^\beta \). Similar results of this problem are obtained in other contexts in [1].

4. Let \( \Phi \) be a SDS and let \( \beta \) be in \( K \). A \( \beta \)-Liapunov function \( l \) is said satisfies (C) if \( s \to [l(\Phi(t, x)) - l(\Phi(s + t, x))] \) is \( \nu \) integrable for all \( x \in E \) and \( r > 0 \) where \( \nu \) is the parameter of the associated Bernstein function given in (4).

5. Let \( \Phi \) be a SDS and let \( \beta \) be in \( K \) with bounded associated Bernstein function. Then condition (C) is fulfilled for each \( \beta \)-Liapunov function.

**Theorem 3.7** Let \( \Phi \) be a SDS and let \( \beta \) be in \( K \). Then each \( \beta \)-Liapunov function \( l \) such that (C) holds, admits the integral representation
\[
l(x) = \int_0^\infty \varphi_t (x) \kappa (dt), \quad x \in E
\]
where
\[
\varphi_t(x) := \int_0^\infty \left[ l(\Phi(t, x)) - l(\Phi(s + t, x)) \right] \nu (ds).
\]

Proof. Let \( \beta \) be in \( K \) and let \( l \) be a \( \beta \)-Liapunov function satisfying (C). Then by Theorem 3.4, there exist a unique \( \beta \)-exit law \( f_t \) such that \( l(x) = \int_0^\infty f_t(x) \, dt \). By (18), we get
\[
f_{t+s}(x) = - \int_0^\infty Q_t l(\Phi(r, x)) \beta'_r (dr) = - \frac{\partial}{\partial s} Q_t l(x)
\]
(24)
On the other hand, since \( \beta_t(\cdot, 0, \cdot) = 1 \) and the differentiation with respect to \( t \) under integral sign is justified in \( \beta_t \), then \( \int_0^\infty \beta'_t (dt) = 0 \). Therefore, we have
\[
\frac{\partial}{\partial t} Q_t u(x) = \int_0^\infty (u(\Phi(s, x)) - u(x)) \beta'_s (ds), \quad t > 0, \quad x \in E.
\]
Now since \( \text{C} \) holds, then for each \( t > 0 \) and \( x \in E \), the following function is well defined
\[
\varphi_t(x) := \int_0^\infty \left[ l(\Phi(t, x)) - l(\Phi(s + t, x)) \right] \nu (ds).
\]
By letting \( s \to 0, \) (24) and ([17], p. 265), we get
\[
f_t(x) = \int_0^\infty \left( Q_t l(x) - Q_t l(\Phi(r, x)) \right) \nu (dr), \quad t > 0, \quad x \in E.
\]
It follows from (5) that \( f_t = \int_0^\infty \varphi_s \beta_t (ds) \) and we conclude by the well definition of \( \kappa \) to get (23).
4 Applications

1. One-sided stable subordinator: Let $\eta^\alpha$ be the one-sided stable subordinator of order $\alpha \in [0,1]$, i.e. the unique convolution semigroup $\eta^\alpha = (\eta^\alpha_t)_{t>0}$ on $[0,\infty]$ such that for each $t > 0$, the Laplace Transform $\mathcal{L}(\eta^\alpha_t)(r) = \exp(-tr^\alpha)$ for $r > 0$. Moreover, following ([17], p. 263), the measure $\eta^\alpha_t$ has a density, denoted by $g^\alpha_t$, with respect to $\lambda$. If we consider $\alpha = \frac{1}{2}$, then the subordinator $\eta^\frac{1}{2}$ is called the Inverse Gaussian subordinator (cf. [3], p. 869). In this case (cf. [18], p. 268)

$$g^\frac{1}{2}_t(s) := 1_{[0,\infty)}(s) \frac{1}{\sqrt{4\pi}} t s^{-\frac{3}{2}} \exp(-\frac{t^2}{4s}), \quad t > 0.$$ 

Following (cf. [3], p. 869), for each $\alpha \in [0,1]$, $\eta^\alpha \in \mathcal{K}$. Let $\Phi$ be a SDS and let $l$ be a $\eta^\alpha$-Liapunov function. Following Theorem 3.4, in the special case if $\alpha = \frac{1}{2}$, $l$ is on the form

$$l(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \varphi_l(x) t^{\alpha-1} dt, \quad x \in E,$$

where

$$\varphi_l(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{l(\Phi(t,x)) - l(\Phi(s+t,x))}{s^{\alpha+1}} ds.$$

2. Gamma subordinator: The $\Gamma$-subordinator $\gamma := (\gamma_t)_{t>0}$ is given by $\gamma_t := h_t \cdot \lambda$ where

$$h_t(s) := 1_{[0,\infty)}(s) \frac{1}{\Gamma(t)} s^{t-1} \exp(-s), \quad t > 0.$$ 

In this case $\kappa := \int_0^\infty \gamma_t dt = d \cdot \lambda$ where

$$d(t) := \exp(-t) \int_0^\infty \frac{1}{\Gamma(s)} t^{s-1} ds.$$ 

Moreover $\gamma \in \mathcal{K}$ (cf. [3], p. 874). Let $\Phi$ be a SDS, by application of Theorem 3.4, each $\Gamma$-Liapunov function admits the integral representation

$$l(x) = \int_0^\infty \int_0^\infty l(\Phi(s+t,x)) s^{t-1} \frac{1}{\Gamma(t)} \left( \frac{l'(t)}{\Gamma(t)} - \log s \right) e^{-s} ds dt,$$

for all $t > 0$ and $x \in E$. Moreover, if (C) holds then by Theorem 3.7 each $\Gamma$-Liapunov function $l$ admits the integral representation

$$l(x) = \int_0^\infty \varphi_l(x) k(t) dt, \quad x \in E,$$

where

$$\varphi_l(x) = \int_0^\infty \left( l(\Phi(s+t,x)) - l(\Phi(t,x)) \right) s^{-1} \exp(-s) ds.$$

3. Compound Poisson subordinator: Let $q$ be an arbitrary probability measure on $[0,\infty]$. With $q_j := \{q\}^j$ such that $q_0 \equiv \varepsilon_0$ and fixed $c > 0$, the following semigroup (cf. [3], p. 870)

$$\tau_t := e^{-ct} \sum_{j=0}^{\infty} \frac{(ct)^j}{j!} q_j, \quad t > 0,$$

is called Compound Poisson subordinator. Moreover, the Bernstein function associated to $\tau := (\tau_t)_{t>0}$ which is bounded is given by $f(r) = c\mathcal{L}(\varepsilon_0 - q)(r)$, $r > 0$. Note that $\tau \in \mathcal{K}$. For $q = \varepsilon_1$, we obtain the Poisson subordinator with jump $c$. In particular, if we consider the Poisson subordinator with jump $1$ by Theorem 3.7 and Remark 3.6.4 each $\tau$-Liapunov function $l$ is on the form

$$l(x) = \sum_{n=0}^{n=\infty} f_n(x),$$

where

$$f_n(x) = l(\Phi(t,x)) - l(\Phi(t+1,x)), \quad t > 0.$$

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References


