Construction of Analytical Solutions to Fractional Differential Equations Using Homotopy Analysis Method

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Abstract— In this paper, we present an algorithm of the homotopy analysis method (HAM) to obtain symbolic approximate solutions for linear and nonlinear differential equations of fractional order. We show that the HAM is different from all analytical methods; it provides us with a simple way to adjust and control the convergence region of the series solution by introducing the auxiliary parameter ℏ, the auxiliary function \( H(t) \), the initial guess \( y_0(t) \) and the auxiliary linear operator \( \mathcal{L} \). Three examples, the fractional oscillation equation, the fractional Riccati equation and the fractional Lane-Emden equation, are tested using the modified algorithm. The obtained results show that the Adomain decomposition method, Variational iteration method and homotopy perturbation method are special cases of homotopy analysis method. The modified algorithm can be widely implemented to solve both ordinary and partial differential equations of fractional order.

Index Terms—Adomian decomposition method, Caputo derivative, Fractional Lane-Emden equation, Fractional oscillation equation, Fractional Riccati equation, Homotopy analysis method.

I. INTRODUCTION

Fractional differential equations have gained importance and popularity during the past three decades or so, mainly due to its demonstrated applications in numerous seemingly diverse fields of science and engineering. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives, and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow. The differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena [1-8]. This is because of the fact that the realistic modeling of a physical phenomenon does not depend only on the instant time, but also on the history of the previous time which can also be successfully achieved by using fractional calculus. Most nonlinear fractional equations do not have exact analytic solutions, so approximation and numerical techniques must be used. The Adomain decomposition method (ADM) [9-14], the homotopy perturbation method (HPM) [15-25], the variational iteration method (VIM) [26-28] and other methods have been used to provide analytical approximation to linear and nonlinear problems. However, the convergence region of the corresponding results is rather small as shown in [9-28]. The homotopy analysis method (HAM) is proposed first by Liao [29-33] for solving linear and nonlinear differential and integral equations. Different from perturbation techniques; the HAM doesn't depend upon any small or large parameter. This method has been successfully applied to solve many types of nonlinear differential equations, such as projectile motion with the quadratic resistance law [34], Klein-Gordon equation [35], solitary waves with discontinuity [36], the generalized Hirota-Satsuma coupled KdV equation [37], heat radiation equations [38], MHD flows of an Oldroyd 8-constant fluid [39], Vakhnenko equation [40], unsteady boundary-layer flows [41]. Recently, S. Chen and Z. Zhang [42] used the HAM to solve fractional KdV-Burgers-Kuramoto equation, Cang and his co-authors [43] constructed a series solution of non-linear Riccati differential equations with fractional order using HAM. They proved that the Adomian decomposition method is a special case of HAM, and we can adjust and control the convergence region of solution series by choosing the auxiliary parameter ℏ close to zero.

The objective of the present paper is to modify the HAM to provide symbolic approximate solutions for linear and nonlinear differential equations of fractional order. Our modification is implemented on the fractional oscillation equation, the fractional Riccati equation and the fractional Lane-Emden equation. By choosing suitable values of the auxiliary parameter ℏ, the auxiliary function \( H(t) \), the initial guess \( y_0(t) \) and the auxiliary linear operator \( \mathcal{L} \), we can adjust and control the convergence region of solution series. Moreover, we illustrated for several examples that the Adomain decomposition, Variational iteration and homotopy perturbation solutions are special cases of homotopy analysis solution.

II. DEFINITIONS

For the concept of fractional derivative we will adopt Caputo’s definition [7] which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the

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initial conditions are given in terms of the field variables and their integer order which is the case in most physical processes [4].

Definition 1. A real function \( f(x) \), \( x > 0 \), is said to be in the space \( C_\mu \), \( \mu \in \mathbb{R} \) if there exists a real number \( p > \mu \), such that \( f(x) = x^p f_1(x) \), where \( f_1(x) \in C[0,\infty) \) and it is said to be in the space \( C^n_\mu \) iff \( f^{(n)}(x) \in C_\mu \), \( n \in \mathbb{N} \).

Definition 2. The Riemann-Liouville fractional integral operator of order \( \alpha > 0 \), of a function \( f(x) \in C_\mu \), \( \mu \geq -1 \), is defined as:

\[
J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt ,
\]

(1) \( \alpha > 0 \), \( x > 0 \),

\[
J^0 f(x) = f(x).
\]

(2) Properties of the operator \( J^\alpha \) can be found in [5-8], we mention only the following:

For \( f \in C_\mu \), \( \mu \geq -1 \), \( \alpha, \beta \geq 0 \) and \( \gamma \geq -1 \)

\[
J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) = J^\beta J^\alpha f(x),
\]

(3) \( J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma} \).

(4)

Definition 3. The fractional derivative of \( f(x) \) in the Caputo sense is defined as:

\[
D^\alpha_x f(x) = J^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt ,
\]

(5) for \( n-1 < \alpha \leq n \), \( n \in \mathbb{N} \), \( x > 0 \), \( f \in C^n_\mu \).

Lemma 1. If \( n-1 < \alpha \leq n \), \( n \in \mathbb{N} \) and \( f \in C^n_\mu \), \( \mu \geq -1 \), then

\[
D^\alpha_x J^\alpha f(x) = f(x),
\]

(6)

\[
J^\alpha D^\alpha_x f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!} , \quad x > 0 .
\]

(7)

III. HOMOTOPY ANALYSIS METHOD

The principles of the HAM and its applicability for various kinds of differential equations are given in [29-43]. For convenience of the reader, we will present a review of the HAM [29, 38]. To achieve our goal, we consider the nonlinear differential equation.

\[
N[y(t)] = 0 \quad , \quad t \geq 0 ,
\]

(8) where \( N \) is a nonlinear differential operator, and \( y(t) \) is unknown function of the independent variable \( t \).

A. Zeroth-order Deformation Equation

Liao [29] constructs the so-called zeroth-order deformation equation:

\[
(1-q)\mathcal{L}[\phi(t; q) - y_0(t)] = qhH(t)N[\phi(t; q)] ,
\]

(9) where \( q \in [0,1] \) is an embedding parameter, \( h \neq 0 \) is an auxiliary parameter, \( H(t) \neq 0 \) is an auxiliary function, \( \mathcal{L} \) is an auxiliary linear operator, \( N \) is nonlinear differential operator, \( \phi(t; q) \) is an unknown function, and \( y_0(t) \) is an initial guess of \( y(t) \), which satisfies the initial conditions. It should be emphasized that one has great freedom to choose the initial guess \( y_0(t) \), the auxiliary linear operator \( \mathcal{L} \), the auxiliary parameter \( h \) and the auxiliary function \( H(t) \). According to the auxiliary linear operator and the suitable initial conditions, when \( q = 0 \), we have

\[
\phi(t; 0) = y_0(t) ,
\]

(10) and when \( q = 1 \), since \( h \neq 0 \) and \( H(t) \neq 0 \), the zeroth-order deformation equation (9) is equivalent to (8), hence

\[
\phi(t; 1) = y(t) .
\]

(11)

Thus, as \( q \) increasing from 0 to 1, the solution \( \phi(t; q) \) various from \( y_0(t) \) to \( y(t) \).

Define

\[
y_m(t) = \left. \frac{1}{m!} \frac{\partial^m \phi(t; q)}{\partial q^m} \right|_{q=0} .
\]

(12) Expanding \( \phi(t; q) \) in a Taylor series with respect to the embedding parameter \( q \), by using (10) and (12), we have:

\[
\phi(t; q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t)q^m .
\]

(13) Assume that the auxiliary parameter \( h \), the auxiliary function \( H(t) \), the initial approximation \( y_0(t) \) and the auxiliary linear operator \( \mathcal{L} \) are properly chosen so that the series (13) converges at \( q = 1 \). Then at \( q = 1 \), from (11), the series solution (13) becomes

\[
y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) .
\]

(14)

B. High-order Deformation Equation

Define the vector:

\[
\bar{y}_n = (y_0(t), y_1(t), y_2(t), ..., y_n(t)) .
\]

(15) Differentiating equation (9) \( m \)-times with respect to embedding parameter \( q \), then setting \( q = 0 \) and dividing them by \( m! \), we have, using (12), the so-called \( m \)-th order deformation equation

\[
\mathcal{L}[y_m(t) - \chi_{m} y_{m-1}(t)] = hH(t) R_{m,0}(y_{m-1}(t) , m = 1,2, ..., n ,
\]

(16)
where
\[
R_m(\tilde{y}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}N[\phi(t; q)]}{\partial q^{m-1}} \bigg|_{q=0},
\]
(17)
and
\[
\chi_m = \begin{cases} 
0, & m \leq 1 \\
1, & m > 1
\end{cases}
\]
(18)
The so-called \(m\)-th order deformation equation (16) is a linear which can be easily solved using Mathematica package.

IV. AN ALGORITHM OF HOMOTOPY ANALYSIS METHOD

The HAM has been extended in [42,43] to solve some fractional differential equations. They use the auxiliary linear operator \(\mathcal{L}\) to be \(d/dt\) or \(D_t^{\alpha}\), where \(0 < \alpha \leq 1\). Also they fix the auxiliary function \(H(t)\) to be 1. In this section, we present an efficient algorithm of the HAM. This algorithm can be established based on the assumptions that the nonlinear operator can involve fractional derivatives, the auxiliary function \(H(t)\) can be freely selected and the auxiliary linear operator \(\mathcal{L}\) can be considered as \(\mathcal{L} = D_t^{\beta}\), \(\beta > 0\). To illustrate its basic ideas, we consider the following fractional initial value problem
\[
N_a[y(t)] = 0, \quad t \geq 0,
\]
(19)
where \(N_a\) is a nonlinear differential operator that may involves fractional derivatives, where the highest order derivative is \(n\), subject to the initial conditions
\[
y^{(k)}(t) = c_k, \quad k = 0,1,2,\ldots,n-1.
\]
(20)
The so-called zeroth-order deformation equation can be defined as
\[
(1 - q)D_t^{\beta}[\phi(t; q) - y_0(t)] = qH(t)N[\phi(t; q)],
\]
\[
\beta > 0,
\]
subject to the initial conditions
\[
\phi^{(k)}(0; q) = c_k, \quad k = 0,1,2,\ldots,n-1.
\]
(22)
Obviously, when \(q = 0\), since \(y_0(t)\) satisfies the initial conditions (20) and \(\mathcal{L} = D_t^{\beta}\), \(\beta > 0\), we have
\[
\phi(t; 0) = y_0(t),
\]
(23)
and, the so-called \(m\)-th order deformation equation can be constructed as
\[
D_t^{\beta}[y_m(t) - \chi_m y_{m-1}(t)] = hH(t)R_m(\tilde{y}_{m-1}(t)),
\]
\[
\beta > 0,
\]
subject to the initial conditions
\[
y_m^{(k)}(0) = 0, \quad k = 0,1,2,\ldots,n-1.
\]
(25)

Operating \(J^\beta\), \(\beta > 0\) on both sides of (24) gives the \(m\)-th order deformation equation in the form:
\[
y_m(t) = \chi_m y_{m-1}(t) + h J^\beta[H(t)R_m(\tilde{y}_{m-1}(t))].
\]
(26)

V. EXAMPLES

In this section we employ our algorithm of the homotopy analysis method to find out series solutions for some fractional initial value problems.

**Example 1.** Consider the composite fractional oscillation equation [18]
\[
\frac{d^2 y}{d t^2} - a D_t^\alpha y(t) - b y(t) - 8 = 0, \quad t \geq 0,
\]
\[
n - 1 < \alpha \leq n, n = 1,2
\]
subject to the initial conditions
\[
y(0) = 0, \quad y'(0) = 0.
\]
(28)
First, if we set \(a = 1, b = -1\), then the equation (27) becomes linear differential equation of second order which has the following exact solution
\[
y(t) = 8 - 8 e^{-t/2}\left[\cos\frac{\sqrt{3}}{2} t + \frac{1}{\sqrt{3}} \sin\frac{\sqrt{3}}{2} t\right].
\]
(29)
Since, \(N_a[y(t)] = \frac{d^2 y}{d t^2} - a D_t^\alpha y(t) - b y(t) - 8\), according to (12) and (17), we have
\[
R_m(\tilde{y}_{m-1}) = y_m^{(n)}(t) - a D_t^\alpha y_{m-1}(t) - b y_{m-1}(t) - 8(1 - \chi_m).
\]
(30)
If we take the auxiliary function \(H(t) = 1\), and the parameter \(\beta = 2\), then the auxiliary linear operator becomes
\[
\mathcal{L} = d^2/dt^2,
\]
(31)
and
\[
y_m(t) = \chi_m y_{m-1}(t) + h J^\beta[H(t)R_m(\tilde{y}_{m-1}(t))],
\]
subject to the initial conditions
\[
y_m(0) = y'_m(0) = 0.
\]
(32)
(33)
(I) If we choose the initial guess approximation
\[
y_0(t) = 0,
\]
(34)
then we have
\[
y_1 = -4 h t^2,
\]
(35)
\[ y_2 = -4h(1 + h)t^2 + \frac{h^2b}{3}t^4 + \frac{8h^2a}{\Gamma(5 - \alpha)} t^{4-a}, \]

\[ y_3 = -4h(1 + h)^2t^2 + \frac{2h^2(1 + h)b}{3}t^4 - \frac{h^3b^2}{90} t^6 + \frac{16h^2(1 + h)a}{\Gamma(5 - \alpha)} t^{4-a} \]

\[ + \frac{16h^3ab}{\Gamma(7 - \alpha)} t^{6-a} - \frac{16h^3ab}{\Gamma(7 - \alpha)} t^{6-a} \]

\[ - \frac{8h^3a^2}{\Gamma(7 - 2\alpha)} t^{6-2a}, \]

Moreover, if we set the auxiliary parameter \( h = -1 \), then the result is the Variational iteration solution obtained by Momani and Odibat in [18] and the same homotopy perturbation solution obtained by Odibat in [21].

**(II)** If we take the initial guess

\[ y_0 = 4t^2, \]  

then we will get the approximations

\[ y_1 = -\frac{hb}{3}t^4 + \frac{8ha}{\Gamma(5 - \alpha)} t^{4-a}, \]

\[ y_2 = \frac{h(1 + h)b}{3}t^4 + \frac{h^2b^2}{90} t^6 - \frac{8h(1 + h)a}{\Gamma(5 - \alpha)} t^{4-a} \]

\[ + \frac{16h^2ab}{\Gamma(7 - \alpha)} t^{6-a} + \frac{8h^2a^2}{\Gamma(7 - 2\alpha)} t^{6-2a}, \]

\[ y_3 = -\frac{h(1 + h)^2b}{3}t^4 + \frac{11h^2(1 + h)b^2}{90} t^6 - \frac{h^3b^3}{35(6)^4} t^8 - \frac{8h(1 + h)^2a}{\Gamma(5 - \alpha)} t^{4-a} \]

\[ + \frac{32h^2(1 + h)ab}{\Gamma(7 - \alpha)} t^{6-a} + \frac{16h^3(1 + h)a^2}{\Gamma(7 - 2\alpha)} t^{6-2a} \]

\[ - \frac{24h^3ab^2}{\Gamma(9 - \alpha)} t^{8-a} - \frac{24h^3a^2b}{\Gamma(9 - 2\alpha)} t^{8-2a} \]

\[ - \frac{8h^3a^3}{\Gamma(9 - 3\alpha)} t^{8-3a}, \]

Here, if we put \( h = -1 \) in (37), then we will get the Adomian decomposition solution obtained by Momani and Odibat in [18].

**(III)** If we take the initial guess

\[ y_0 = 3t^2, \]  

then we will get the approximations

\[ y_1 = -ht^2 - \frac{hb}{4}t^4 - \frac{6ha}{\Gamma(5 - \alpha)} t^{4-a}, \]

\[ y_2 = -ht^2 - \frac{hb}{4}t^4 - \frac{6ha}{\Gamma(5 - \alpha)} t^{4-a} \]

**VI** If we replace \( H(t) = 1 \) by \( H(t) = t^{1/2} \) in (III), then we have

\[ y_1 = -\frac{8h}{15}t^2 - \frac{4hb}{21}t^2 \]

\[ - \frac{24ha}{3} \]

\[ - \frac{24ha}{(9 - 2\alpha)(7 - 2\alpha)\Gamma(3 - \alpha)} t^{3-a}, \]

\[ - \frac{24ha}{(9 - 2\alpha)(7 - 2\alpha)\Gamma(3 - \alpha)} t^{3-a} + \frac{h^2}{(4)^{1+a}} t^{3-2a} [...]. \]

By means of the so-called \( h \)-curves [29], Fig.1 shows that the valid region of \( h \) is the horizontal line segment. Thus, the valid regions of \( h \) for the HAM of Eq. (27) at different values of \( \alpha \) are shown in Fig.1.

Fig. 2. (A) shows the approximate solutions for Eq. (27) obtained for different values of \( \alpha \), \( h \), \( H(t) \) and \( y_0(t) \) using HAM. Fig. 2.a shows that the best solution results when we use the initial guess \( y_0 = 3t^2 \). Fig. 2.b shows that the best solution results when we use the auxiliary parameter \( h = -0.7 \). Fig. 2.c shows that the best solution results when we use the auxiliary function \( H(t) = 1 \). In Fig. 2.d-f, we compare the approximate solutions for different values of \( h \) and \( H(t) \). As shown in Fig. 2.a-f, we can observe that by choosing a proper value of the auxiliary parameter \( h \), the auxiliary function \( H(t) \), the auxiliary linear operator \( \mathcal{L} \) and the initial guess \( y_0(t) \) we can adjust and control convergence region of the series solutions. Moreover, we can observe that convergence region increases as \( h \) goes to left end point of the valid region of \( h \), but this may decreases the agreement with the exact solution. Choosing a suitable auxiliary function \( H(t) \) with the left end point of the valid region of \( h \) scrips the disagreement with the exact solution. Fig. 3, show the ‘residual error’ for 15th order approximation for Eq. (27) at \( \alpha = 1 \) obtained for different values of \( h \) and \( H(t) \).

![Fig.1](image-url)
Example 2. Consider the fractional Riccati equation [17]

\[ D_\\alpha^2 y(t) + y^2(t) - 1 = 0, \quad 0 < \alpha \leq 1, \quad t \geq 0, \]  

subject to the initial condition

\[ y(0) = 0. \]

According to (17), we have

\[ R_m(y_{m-1}) = D_\\alpha^m y_{m-1}(t) + \sum_{i=0}^{m-1} y_i(t) y_{m-1-i}(t) - (1 - \chi_m), \]

If we choose the initial guess approximation

\[ y_0(t) = t, \]

Fig. 2. Approximate solutions for Eq. (27)
the parameter $\beta = 1$ and $H(t) = 1$, then according to (26) we find

$$y_1 = -\hbar \left[ t - \frac{t^3}{3} - \frac{t^{2-a}}{\Gamma(3-a)} \right],$$

$$y_2 = -\hbar \left[ t - \frac{t^3}{3} - \frac{t^{2-a}}{\Gamma(3-a)} \right]$$

$$-\hbar^2 \left[ \frac{2t^3}{3} - \frac{2t^5}{15} + \frac{t^{2-a}}{\Gamma(3-a)} - \frac{2t^{4-a}}{\Gamma(4-a)} - \frac{t^{3-2a}}{\Gamma(4-2a)} \right],$$

$$y_3 = -\hbar \left[ t - \frac{t^3}{3} - \frac{t^{2-a}}{\Gamma(3-a)} \right]$$

$$-\hbar^2 \left[ \frac{4t^3}{3} - \frac{4t^5}{15} + \frac{t^{2-a}}{\Gamma(3-a)} - \frac{2t^{3-2a}}{\Gamma(4-2a)} \right]$$

$$-\frac{6}{\Gamma(3-a)} + \frac{6}{\Gamma(4-a)} - \frac{2t(5-a)}{\Gamma(4-a)} \frac{t^{4-a}}{\Gamma(5-a)} + \frac{1}{\Gamma(5-a)} \frac{4t^{4-a}}{\Gamma(6-a)} \frac{t^{5-2a}}{\Gamma(6-2a)} \frac{t^{3-2a}}{\Gamma(4-2a)} \frac{t^{4-3a}}{\Gamma(5-3a)} \right].$$

Now, if we take $\hbar = -1$, then we have the homotopy perturbation solution obtained by Odibat and Momani in [17].

(I) If we chose the initial guess approximation

$$y_0(t) = t^{1/2},$$

the auxiliary linear operator

$$L = D_t^\beta, \quad \beta > 0,$$

and the auxiliary function

$$H(t) = t^\gamma, \quad \gamma > -1,$$

then (26) gives

$$y_1 = h t^{\beta + \gamma} \left[ \Gamma(1 + \gamma) \frac{t^{1-\gamma}}{\Gamma(1 + \gamma + \beta)} + \Gamma(2 + \gamma) t \right]$$

$$+ \frac{\sqrt{\pi} \Gamma(\frac{3}{2} + \gamma - \alpha)}{2 \Gamma\left(\frac{3}{2} - a\right) \Gamma\left(\frac{3}{2} + \gamma - \alpha + \beta\right)} t^{\frac{1}{2} - a},$$

$$y_2 = h t^{\beta + \gamma} \left[ \Gamma(1 + \gamma) \frac{t^{1-\gamma}}{\Gamma(1 + \gamma + \beta)} + \Gamma(2 + \gamma) t \right]$$

$$+ \frac{\sqrt{\pi} \Gamma(\frac{3}{2} + \gamma - \alpha)}{2 \Gamma\left(\frac{3}{2} - a\right) \Gamma\left(\frac{3}{2} + \gamma - \alpha + \beta\right)} t^{\frac{1}{2} - a} \frac{h^2 t^{2\beta + 2\gamma - 1}}{2},$$

and so on. As pointed above, the valid region of $\hbar$ is a horizontal line segment. Thus, the valid region of $\hbar$ for the HAM solution of Eq. (40) at $\alpha = 0.5, \gamma = -0.5, \beta = 1$ is $-1.15 < \hbar < -0.1$ as shown in Fig. 4.

Fig. 5.(a)~(d) show the approximate solution for Eq. (40) obtained for different values of $\hbar, \beta, H(t)$ and $y_0(t)$ using HAM. As observed in Fig. 2.(a)~(f), we can notice that the convergence region can be adjusted and controlled by choosing proper values of the auxiliary parameter $\hbar$, the auxiliary function $H(t)$, the auxiliary linear operator $L$ and the initial guess $y_0(t)$.

(II) If we chose the initial guess approximation

$$y_0(t) = t^{1/2},$$

the auxiliary linear operator

$$L = D_t^\beta, \quad \beta > 0,$$

and the auxiliary function

$$H(t) = t^\gamma, \quad \gamma > -1,$$
Example 3. Consider the Lane-Emden fractional differential equation \[ D_\alpha^\alpha y + \frac{2}{t} y' + y^3 - (6 + t^6) = 0, \quad t \geq 0, \] (47) subject to initial conditions \[ y(0) = 0, \quad y'(0) = 0. \] (48) Hence, according to (17), we have \[ R_m(y_{m-1}) = D_\alpha^\alpha y_{m-1}(t) + \frac{2}{t} y'_{m-1}(t) \] (49) \[ + \sum_{i=0}^{m-1} y_{m-1-i}(t) \sum_{j=0}^{i} y_j(t) y_{i-j}(t) \] \[ - (6 + t^6)(1 - \chi_m). \] In view of the modified homotopy analysis method, if we set \[ y_0 = 0, \quad H(t) = 1, \quad \beta = 2, \] (50) then the mth-order deformation equation (26) gives \[ y_1 = -h^2 \left[ 3 + \frac{t^6}{56} \right]. \] (51)
choose a proper value for the auxiliary parameter. Methods, such as ADM and HPM, in this method we can emphasize our belief that the method is a reliable technique to find approximate solutions for many problems. The work of the HAM which introduces an efficient tool for solving linear and nonlinear differential equations of fractional order. In this work, we carefully proposed an efficient algorithm of the HAM which introduces an efficient tool for solving linear and nonlinear differential equations of fractional order. The modified algorithm has been successfully implemented to solve other nonlinear problems in fractional calculus field.

Fig. 7. Approximate solutions for Eq. (47)

\[ y_2 = -h t^2 \left[ 3 + \frac{t^6}{56} \right] \]

\[ - \frac{6h^2 t^{4-a}}{\Gamma(11-a)} \left[ 120t^6 + \frac{\Gamma(\alpha-4)}{\Gamma(\alpha-10)} + h t^2(1176 + t^6) \right], \]

and so on. Moreover, if we replace the initial guess \( y_0 = 0 \) by \( y_0 = t^2 \), then we have \( y_m = 0, \forall m \geq 1 \), hence, \( y(t) = t^2 \) is the exact solution.

The valid region of \( h \) for the HAM solution of Eq. (47) at \( \alpha = 2 \) is \(-0.6 < h < -0.15\), at \( \alpha = 1.75 \) is \(-0.6 < h < -0.15\) and at \( \alpha = 1.5 \) is \(-0.6 < h < -0.15\) as shown in Fig. 6.

Fig. 7(a) shows the approximate solution for Eq. (47) at \( \alpha = 2 \) obtained for different values of \( h \) using HAM. As the previous examples, the convergence region of the series solution increases as \( h \) goes to the left end point of its valid region. Fig. 7(b) shows the HAM solution for Eq. (47) at different values of \( \alpha \) obtained for \( h = -0.35 \).

VI. DISCUSSION AND CONCLUSIONS

In this work, we carefully proposed an efficient algorithm of the HAM which introduces an efficient tool for solving linear and nonlinear differential equations of fractional order. The modified algorithm has been successfully implemented to find approximate solutions for many problems. The work emphasized our belief that the method is a reliable technique to handle nonlinear differential equations of fractional order. As an advantage of this method over the other analytical methods, such as ADM and HPM, in this method we can choose a proper value for the auxiliary parameter \( h \), the auxiliary function \( H(t) \), the auxiliary linear operator \( \mathcal{L} \) and the initial guess \( y_0 \) to adjust and control convergence region of the series solutions.

There are some important points to make here. First, we can observe that the convergence region of the series solution increases as \( h \) tends to zero. Second, choosing a suitable auxiliary function \( H(t) \) or initial approximation \( y_0(t) \) may accelerate the rapid convergence of the series solution and may increase the agreement with the exact solution. Third, for a certain value of \( h \), choosing a suitable auxiliary linear operator \( \mathcal{L} = D_t^\beta \), \( \beta > 0 \) may increase the convergence region. Finally, generally speaking, the proposed approach can be further implemented to solve other nonlinear problems in fractional calculus field.

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