An Interval Analytic Method for a Nonlinear Boundary Value Problem Describing Transport Process

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Abstract—**In recent times, the methods of interval analysis have been successfully employed to establish existence results for the solution of initial value problems. In this paper, we extend the methods to establish existence of solution for a special boundary value problem. A particular case of this problem describes equations arising in transport process.**
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1 Introduction

In this paper we consider the rather special type of boundary value problem:

$$\left.\begin{array}{l} x'(t) = f(t,x,y), \quad x(a) = x_a \\ \qquad\qquad\qquad\qquad\qquad\qquad , t \in I \\ -y'(t) = g(t,x,y), \quad y(b) = y_b \end{array}\right\} \qquad (1)$$

where $f, g \in C^{1,2}(I \times I\!R^2, I\!R)$ and $I = [a, b]$. This is a generalized form of an equation arising in the transport process of different types of particles moving in opposite direction within a rod of finite length when subjected to certain fluxes. Because of the importance of this problem and its application in many other physical phenomena, other authors in [4], [9], have earlier studied it using other methods which include resistive condition of monotonicity. Here we develop an interval analytic method which entails the construction of an interval operator. With this interval operator, existence results are established using interval fixed point theory. The method of interval analysis developed in this paper renders the assumption of monotonicity of the two functions f and g appearing in the equation given in H_2 of [4] unnecessary.

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The basic objects of interval analysis used in this paper are the closed, non-empty and bounded real intervals with notations, arithmetic and analytic operations as given in [1], [2], [3], [5], [6], [7], and [8]. Readers who are not familiar with the subject may refer to these references.

Let the following assumptions hold true:

H_0: Let there exist functions
$u, v, \sigma, \tau \in C^1(I, I\!R)$
such that
$u(t) \le v(t), \quad \sigma(t) \le \tau(t), \quad t \in I$
and
$u(a) \le x_a \le v(a); \quad \sigma(b) \le y_b \le \tau(b)$

H_1: $u'(t) \le f(t,\xi,y) + f_1(t,\xi,y)(u-\xi)$
$v'(t) \ge f(t,\xi,y) - f_1(t,\xi,y)(\xi - v)$

H_2: $-\sigma'(t) \le g(t,x,\eta) + g_2(t,x,\eta)(\sigma-\eta)$
$-\tau'(t) \ge g(t,x,\eta) - g_2(t,x,\eta)(\eta-\tau)$
for all functions $x, y, \xi, \eta \in C^1(I, I\!R)$ such that $u \le \xi \le v$, $u \le x \le v$, $\sigma \le \eta \le \tau$, $\sigma \le y \le \tau$ where the subscripts $1, 2$ denote partial differentiation with respect to x and y respectively.

2 Interval Majorant of Solution

In this section we establish a new Lemma which extends the notion of lower and upper solutions for the boundary value problem (1) and also some of the results of Lemma 1 of Lakshmikantham and Pachpatte [4], without the monotone nondecreasing condition imposed on the functions f in its third argument and g in its second argument.

Lemma 2.1
Suppose that the assumptions $H_0 - H_2$ above are true. Then the solution $(x(t), y(t))$ of the boundary value problem (1) satisfies

$$(u(t), \sigma(t)) \le (x(t), y(t)) \le (v(t), \tau(t)), \; t \in I$$

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where the inequality is taken componentwise.

Proof:
To prove this lemma we shall show that

$$u(t) \leq x(t) \leq v(t) \quad \text{and} \quad \sigma(t) \leq y(t) \leq \tau(t), \quad t \in I.$$

Let the interval functions F_1 and G_2 be natural interval extensions of f_1 and g_2 respectively. Then for intervals X, Y in $I\!R$ with $x \in X$ and $y \in Y$

$$f_1(t, x, y) \in F_1(t, X, Y)$$

and

$$g_2(t, x, y) \in G_2(t, X, Y)$$

Define m on I by

$$m(t) = x(t) - u(t)$$

then

$$\begin{aligned} m(a) &\geq 0 \text{ and} \\ m'(t) &= x'(t) - u'(t) \\ &\geq f(t, x, y) - f(t, \xi, y) - f_1(t, \xi, y)(u - \xi) \\ &= -\{f(t, \xi, y) - f(t, x, y)\} - f_1(t, \xi, y)(u - \xi) \\ &= -f_1(x, \gamma, y)(\xi - x) - f_1(t, \xi, y)(u - \xi) \\ &\geq -|F_1(t, X, .Y)|m(t) = Lm(t) \end{aligned}$$

From this we have

$$m(t) \geq m(a)e^{L(t-a)} \geq 0$$

which implies

$$x(t) \geq u(t), \quad t \in I \tag{2}$$

If we also define n by

$$n(t) = v(t) - x(t)$$

we have

$$\begin{aligned} n'(t) &= v'(t) - x'(t) \\ &\geq f(t, \xi, y) - f_1(t, \xi, y)(\xi - v) - f(t, x, y) \\ &= -f_1(t, \xi, y)(\xi - v) - \{f(t, x, y) - f(t, \xi, y)\} \\ &= -f_1(t, \xi, y)(\xi - v) - f_1(t, \theta, y)(x - \xi) \\ &\geq |F_1(t, X, Y)|(v - x) = -Ln(t). \end{aligned}$$

which gives

$$n(t) \geq n(a)e^{-L(t-a)} \geq 0$$

and this implies

$$v(t) \geq x(t) \tag{3}$$

Combination of (2) and (3) yields the first result.

Next we prove that

$$\sigma(t) \leq y(t) \leq \tau(t)$$

Define p by

$$p(t) = y(t) - \sigma(t), \quad p(b) \geq 0$$

$$\begin{aligned} p'(t) &= y'(t) - \sigma'(t) \\ &\leq -g(t, x, y) + g(t, x, \eta) + g_2(t, x, \eta)(\sigma - \eta) \\ &\leq |G_2(t, X, Y)|(\eta - y) + |G_2(t, X, Y)|(\sigma - \eta) \\ &= |G_2(t, X, Y)|(\sigma - y) = -Mp(t). \end{aligned}$$

this gives

$$p(b) \leq p(t)e^{-M(b-t)}$$

which implies that $0 \leq p(t)$. and thus,

$$\sigma(t) \leq y(t) \tag{4}$$

Similarly, with

$$q(t) = \tau(t) - y(t), \quad q(b) \geq 0$$

$$\begin{aligned} q'(t) &= \tau'(t) - y'(t) \\ &\leq g_2(t, x, \eta)(\eta - \tau) - g(t, x, \eta) + g(t, x, y) \\ &\leq |G_2(t, X, Y)|(\eta - \tau) + |G_2(t, X, Y)|(y - \eta) \\ &= |G_2(t, X, Y)|(\tau - y) = -Mq(t) \end{aligned}$$

This yields

$$q(t) \geq q(b)e^{M(b-t)} \geq 0$$

which implies that

$$\tau(t) \geq y(t) \tag{5}$$

From (4) and (5) we obtain the desired result.

3 Existence of Solution

Here we prove the existence of interval sequences $\{X_n\}$ and $\{Y_n\}$ majorising the solutions x, y of equation (1) and show that these sequences converge to limits $X(t)$, $Y(t)$ which also contain these solutions. However, before then we give some results which will be needed to establish the theorem.

Lemma 3.1 [8]

If X and Y are intervals, then

$$X \subseteq Y$$

if, and only if,

$$|m(Y) - m(X)| \leq \frac{1}{2}\{w(Y) - w(X)\} \quad (6)$$

Theorem 3.1 [6]

If P is an inclusion monotonic interval operator majorant of a real operator p and if

$$P(Y_0) \subseteq Y_0 \quad (7)$$

Then, the sequence $\{Y_n\}$ of intervals defined by

$$Y_{n+1} = P(Y_n), \quad n = 0, 1, 2, \ldots \quad (8)$$

has the following properties

(i) $Y_{k+1} \subseteq Y_k$, ; $k = 0, 1, 2, \ldots$

(ii) For every $a \leq t \leq b$, the limit

$$Y(t) = \bigcap_{k=0}^{\infty} Y_k(t) \quad (9)$$

exists as an interval function and

$$Y(t) \subseteq Y_k(t), \;\; k = 0, 1, 2, \ldots.$$

(iii) any solution of the operator equation

$$y(t) = p(y)(t) \quad (10)$$

such that

$$y(t) \in Y_0(t) \;\; \forall\, t \in [a, b]$$

satisfies

$$y(t) \in Y_k(t) \;\; \forall\, k \;\; \text{and} \;\; y(t) \in Y(t) \;\; \forall\, t \in [a, b]$$

(iv) if there exists a real number c such that $0 \leq c \leq 1$ for which $X \subseteq Y_0$ implies

$$\sup_t w(P(X(t))) \leq c \sup_t w(X(t))$$

then the operator equation (10) has the unique solution $y(t)$ given by (9)

Theorem 3.2
Suppose that the hypothesis $H_0 - H_2$ above hold true. Then there exist sequences $\{X_n(t)\}$ and $\{Y_n(t)\}$ of interval functions with initial interval functions $X_0(t) = [u(t), v(t)]$ and $Y_0(t) = [\sigma(t), \tau(t)]$ such that the limits

$$X(t) = \lim_{n\to\infty} X_n(t)$$

and

$$Y(t) = \lim_{n\to\infty} Y_n(t)$$

exist as interval functions on I. Moreover, the limits X, Y of these interval sequences majorise the solution $x(t)$ and $y(t)$ of the boundary value problem (1).

Proof:
The solution of the boundary value problem is equivalent to

$$x(t) = x_a + \int_a^t f(s, x(s), y(s))ds, \quad t \in I$$

and

$$y(t) = y_b + \int_t^b g(s, x(s), y(s))ds, \quad t \in I$$

Considering interval extensions of the functions $f(t, x, y)$ and $g(t, x, y)$, respectfully, of the form;

$$\begin{aligned} F(t, X(t), Y(t)) &= f(t, m(X), m(Y)) + F_1(t, X, Y) \\ &\quad (X - w(X)) + F_2(t, X, Y)(Y - w(Y)) \\ &\text{and} \\ G(t, X(t), Y(t)) &= g(t, m(X), m(Y)) + G_1(t, X, Y) \\ &\quad (X - w(X)) + G_2(t, X, Y)(Y - w(Y)) \end{aligned}$$

where $F_1(t, X, Y)$, $F_2(t, X, Y)$, $G_1(t, X, Y)$ and $G_2(t, X, Y)$ are natural interval extensions of the functions $f_x(t, x, y)$, $f_y(t, x, y)$, $g_x(t, x, y)$ and $g_y(t, x, y)$ respectively, we have

$$\begin{aligned} x(t) \in x_a &+ \int_a^t f(s, m(X(s)), m(Y(s)))ds \\ &+ \int_a^t F_1(s, X(s), Y(s))(X(s) - m(X(s))) \\ &+ \int_a^t F_2(s, X(s), Y(s))(Y(s) - m(Y(s)))ds \end{aligned} \quad (11)$$

and

$$y(t) \in y_b + \int_t^b g(s, m(X(s)), m(Y(s)))ds$$

$$+\int_t^b G_1(s,X(s),Y(s))(X(s)-m(X(s)))ds$$

$$+\int_t^b G_2(s,X(s),Y(s))(Y(s)-m(Y(s)))ds \qquad (12)$$

Let

$$P(X(t),Y(t)) = x_a + \int_a^t f(s,m(X(s)),m(Y(s)))ds$$

$$+\frac{1}{2}\int_a^t |F_1(s,X(s),Y(s)))|w(X(s))[-1,1]ds$$

$$+\frac{1}{2}\int_a^t |F_2(s,X(s),Y(s))|w(Y(s))[-1,1]ds \qquad (13)$$

and

$$\Phi(X(t),Y(t)) = y_b + \int_t^b g(s,m(X(s)),m(Y(s)))ds$$

$$+\frac{1}{2}\int_t^b |G_1(s,X(s),Y(s)))|w(X(s))[-1,1]ds$$

$$+\frac{1}{2}\int_t^b |G_2(s,X(s),Y(s))|w(Y(s))[-1,1]ds \qquad (14)$$

where $w(\cdot)$ and $m(\cdot)$ are the width and midpoint of their arguments. Then,

$$x(t)\in P(X(t),Y(t)) \quad \text{and} \quad y(t)\in \Phi(X(t),Y(t)), \quad t\in I$$

Define the interval sequences $\{X_n\}$ and $\{Y_n\}$ by

$$X_{n+1}(t) = P(X_n(t),Y_n(t)), \quad t\in I \qquad (15)$$

and

$$Y_{n+1}(t) = \Phi(X_n(t),Y_n(t)), \quad t\in I \qquad (16)$$

respectively, with

$$X_0(t) = [u(t),v(t)]$$

and

$$Y_0(t) = [\sigma(t),\tau(t)].$$

These will converge to unique limits if

$$P(X_0(t),Y_0(t)) \subseteq X_0(t)$$

and

$$\Phi(X_0(t),Y_0(t)) \subseteq Y_0(t)$$

By (6) of Lemma (3.1) these hold true if

$$|m(X_0(t)) - m(P(X_0(t),Y_0(t)))|$$

$$\le \frac{1}{2}\{w(X_0(t)) - w(P(X_0(t),Y_0(t)))\}$$

and

$$|m(Y_0(t)) - m(\Phi(X_0(t),Y_0(t)))|$$

$$\le \frac{1}{2}\{w(Y_0(t)) - w(\Phi(X_0(t),Y_0(t)))\}$$

where $m(X_0(t))$ and $m(Y_0(t))$ are the mid-points of the intervals $X_0(t)$ and $Y_0(t)$ respectively. From the mid-points $m(P)$, $m(\Phi)$ of the interval operators $P(X_0(t),Y_0(t))$ and $\Phi(X_0(t),Y_0(t))$ respectively, we have

$$\begin{aligned}
m(P) &= x_a + \int_a^t f(x,m(X_0,m(Y_0))ds \\
&\ge u(t) - u(t) + u(a) \\
&\quad + \int_a^t f(s,m(X_0(s)),m(Y_0(s)))ds. \\
&= u(t) - \int_a^t \{u'(s) - f(s,m(X_0),m(Y_0))\}ds \\
&\ge u(t) - \int_a^t |F_1(s,X_0,Y_0)|(u-\xi)ds \\
&\quad - \int_a^t \{f(s,\xi,y) - f(x,m(X_0),m(Y_0))\}ds \\
&\ge u(t) - \int_a^t |F_1(s,X_0,Y_0)|(u-m(X_0))ds \\
&\quad - \int_a^t f_2(s,m(X_0),\gamma)(y-m(Y_0))ds \\
&\ge u(t) + \frac{1}{2}\int_a^t |F_1(s,X_0,Y_0)|w(X_0)ds \\
&\quad + \frac{1}{2}\int_a^t |F_2(s,X_0,Y_0)|w(Y_0)ds
\end{aligned}$$

that is

$$m(P) \ge u(t) + \frac{1}{2}w(P) \qquad (17)$$

where $w(P)$ is the width of P.
Also

$$\begin{aligned}
m(P) &= x_a + \int_a^t f(s,m(X_0),m(Y_0))ds \\
&\le v(t) - v(t) + v(a) \\
&\quad + \int_a^t f(s,m(X_0),m(Y_0))ds \\
&= v(t) - \int_a^t v'(s)ds \\
&\quad + \int_a^t f(s,m(X_0),m(Y_0))ds \\
&\le v(t) + \int_a^t |F_1(t,X_0,Y_0)|(\xi - v)ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_a^t \{f(s,m(X_0,m(Y_0))-f(s,\xi,y)\}ds \\
\leq\ & v(t)+\int_a^t |F_1(s,X_0,Y_0)|(m(X_0)-v)ds \\
&+\int_a^t |F_2(t,X_0,Y_0)|(m(Y_0)-\tau)ds \\
\leq\ & v(t)-\frac{1}{2}\int_a^t |F_1(s,X_0,Y_0)|w(X_0)ds \\
&-\frac{1}{2}\int_a^t |F_2(s,X_0,Y_0)|w(Y_0)ds
\end{aligned}
$$

that is

$$m(P) \leq v(t)-\frac{1}{2}w(P) \tag{18}$$

(13) and (17) give

$$|m(X_0(t))-m(P)| \leq \frac{1}{2}\{w(X_0(t))-w(P)\}$$

as required.
Hence

$$P(X_0(t),Y_0(t)) \subseteq X_0(t).$$

Similarly

$$
\begin{aligned}
m(\Phi) &= y_b+\int_t^b g(s,m(X_0),m(Y_0))ds \\
&\geq \sigma(t)-\sigma(t)+\sigma(b)+\int_t^b g(s,m(X_0),m(Y_0))ds \\
&= \sigma(t)+\int_t^b \{\sigma'(s)+g(s,m(X_0(s)),m(Y_0)))\}ds \\
&\geq \sigma(t)-\int_t^b |G_2(s,X_0,Y_0)|(\sigma-\eta)ds \\
&\quad -\int_t^b \{g(s,x,\eta)-g(s,m(X_0),m(Y_0))\}ds \\
&\geq \sigma(t)-\int_t^b |G_2(s,X_0,Y_0)|(\sigma-m(Y_0))ds \\
&\quad -\int_t^b G_1(s,X_0,Y_0)|(u-m(X_0))ds \\
&\geq \sigma(t)+\frac{1}{2}\int_t^b |G_2(s,X_0,Y_0)|w(Y_0)ds \\
&\quad +\frac{1}{2}\int_t^b G_1(s,X_0,Y_0)|w(X_0)ds
\end{aligned}
$$

i.e.

$$m(\Phi) \geq \sigma(t)+\frac{1}{2}w(\Phi) \tag{19}$$

Also

$$
\begin{aligned}
m(\Phi) &= y_b+\int_t^b g(s,m(X_0(s)),m(Y_0(s)))ds \\
&\leq \tau(t)-\tau(t)+\tau(b)+\int_t^b g(s,m(X_0),m(Y_0))ds \\
&= \tau(t)+\int_t^b \{\tau'(s)+g(s,m(X_0(s)),m(Y_0(s)))\}ds \\
&\leq \tau(t)+\int_t^b \{|G_2(t,X_0,Y_0)|(\eta-\tau)\}ds \\
&\quad +\int_t^b \{g(s,m(X_0),m(Y_0))-g(s,x,\eta)\}ds \\
&\leq \tau(t)+\int_t^b |G_2(s,X_0,Y_0)|(m(Y_0)-\tau)ds \\
&\quad +\int_t^b g(s,\rho,m(Y_0))(m(X_0)-x)ds \\
&\leq \tau(t)+\int_t^b |G_2(s,X_0,Y_0)|(m(Y_0)-\tau)ds \\
&\quad +\int_t^b |G_1(s,X_0,Y_0)|(m(Y_0)-x)\}ds \\
&\leq \tau(t)-\frac{1}{2}\int_t^b |G_1(t,X_0,Y_0)|w(X_0(s))ds \\
&\quad -\frac{1}{2}\int_t^b |G_2(t,X_0,Y_0)|w(Y_0(s))ds
\end{aligned}
$$

i.e.

$$m(\Phi) \leq \tau(t)-\frac{1}{2}w(\Phi). \tag{20}$$

(19) and (20) imply that

$$|m(Y_0)-m(\Phi)| \leq \frac{1}{2}\{w(Y_0)-w(\Phi)\}$$

which, by Lemma (3.1) also implies that

$$\Phi(X_0(t),Y_0(t)) \subseteq Y_0(t).$$

Hence the sequences (15), (16) converge by theorem (3.1) to unique limits $X(t)$, $Y(t)$ respectively, with $x(t) \in X(t)$ and $y(t) \in Y(t)$.

Theorem 3.3
Let the assumptions of Theorem 3.2 hold. Assume further that the natural interval extensions of the partial derivatives of f and g are chosen such that they satisfy $max\left\{\int_a^t\{|F_1(s,X(s),Y(s))|+|F_2(s,X(s),Y(s))|\}ds,\ \int_t^b\{|G_1(s,X(s),Y(s))|+|G_2(s,X(s),Y(s))|\}ds\right\} < 1$. Then the limits of the interval sequences generated in Theorem 3.2 are degenerate and thus coincide with the real valued solution of the b.v.p. (1).

Proof

Let the components of the interval vector function $\mathbf{Z} = (X, Y)$ be the limits of the interval sequences (15) & (16) then,

$$w(\mathbf{Z}) = max\{w(X), w(Y)\} \leq max \left\{\int_a^t \{|F_1(s, X(s), Y(s))| + |F_2(s, X(s), Y(s))|\} w(X(s))ds, \int_t^b \{|G_1(s, X(s), Y(s))| + |G_2(s, X(s), Y(s))|\} w(Y(s))ds\right\}$$

Set $k = max\left\{\int_a^t \{|F_1| + |F_2|\}ds, \int_t^b \{|G_1| + |G_2|\}ds\right\}$, then $\sup_t w(\mathbf{Z}) \leq k \sup_t w(\mathbf{Z})$,

this by the hypothesis implies that $w(\mathbf{Z}) = 0$, therefore $\mathbf{Z}$ is degenerate and hence coincides with the solution (x, y) of the b.v.p. (1).

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