Sensitivity Analysis for Random Fuzzy Portfolio Selection Model with Investor's Subjectivity

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***Abstract*—This paper focuses on the proposition of a portfolio selection problem considering an investor's subjectivity and the sensitivity analysis for the change of subjectivity. Since this proposed problem is formulated as a random fuzzy programming problem due to both randomness and subjectivity presented by fuzzy numbers, it is not well-defined. Therefore, introducing Sharpe ratio which is one of important performance measures of portfolio models, the main problem is transformed into the standard fuzzy programming problem. Furthermore, using the sensitivity analysis for fuzziness, the analytical optimal portfolio with the sensitivity factor is obtained.**

***Index Terms*—Portfolio selection problem, Random fuzzy programming, Sensitivity analysis, Analytical solution method.**

I. Introduction

Since Markowitz's outstanding study [20], Portfolio selection problem has been one of standard and most important problems in investment and financial research fields. It has been central to research activity in the real financial field and numerous researchers have contributed to the development of modern portfolio theory (cf. Elton and Gruber [3], Luenberger [19]), and many researchers have proposed several types of portfolio models which are extended Markowitz mean-variance model; mean-absolute deviation model (Konno [15], Konno, et al. [16]), safety-first model [3], Value at Risk (VaR) and conditional Value at Risk (cVaR) model (Rockafellar and Uryasev [21]), etc.. As a result, nowadays it is common practice to extend these classical economic models of financial investment to various types of portfolio models because investors correspond to present complex markets. In practice, many researchers have been trying different mathematical approaches to develop the theory of portfolio model.

Particularly, the performance measure of portfolio is one of the most important factors in theoretical and practical investment. Particularly, Sharpe ration is most standard measure proposed by Sharpe [22] and it has also been central to research activity in ranking portfolio performances and mutual fund management, often called passive management. The Sharpe ratio has its principal advantage that it is directly computable from any observed series of returns without need for additional information surrounding the source of profitability. In most previous researches, it has been treated as only random variables, and the expected returns and variances also have been assumed to be fixed values. However, investors receive effective or ineffective information from the real world and ambiguous factors usually exist in it. Furthermore, investors often have the subjective prediction for future markets which are not derived from the statistical analysis of historical data, but their long-term experiences. Then, even if investors hold a lot of information from the investment field, it is difficult that the present or future random distribution of each asset is strictly set. Consequently, we need to consider not only random conditions but also ambiguous and subjective conditions for portfolio selection problems.

As recent studies in the sense of mathematical programming, some researchers have proposed various types of portfolio models under randomness and fuzziness. These problems presented by probabilities and possibilities are generally called stochastic programming problems and fuzzy programming problems, respectively, and there are some basic studies using the fuzzy programming approach to treat ambiguous factors (Inuiguchi and Ramik [11], Leon, et al. [17], Tanaka and Guo [23], Tanaka, et al. [24], Vercher et al. [26], Watada [27]) as well as stochastic and goal programming approaches. Furthermore, some researchers have proposed mathematical programming problems with both randomness and fuzziness as fuzzy random variables (for instance, Katagiri et al. [13, 14]). In these studies [13, 14], fuzzy random variables were related with the ambiguity of the realization of a random variable and dealt with a fuzzy number that the center value occurs according to a random variable. On the other hand, future returns may be dealt with random variables derived from the statistical analysis, whose parameters are assumed to be fuzzy numbers due to the decision maker's subjectivity, i.e., random fuzzy variables which Liu (Liu [18]) defined. There are a few studies of random fuzzy programming problem (Hasuike et al. [7, 8], Katagiri et al. [12], Huang [9]). Most recently, Hasuike et al. [8] proposed several portfolio selection models including random fuzzy variables and developed the analytical solution method.

However, in [8], the random distribution of each asset is assumed to be a normal distribution. From some practical studies with respect to the present practical market, it is not clear that price movements of assets occur according to normal distributions. Therefore, in this paper we consider a random fuzzy portfolio selection problem with general uncertainty distributions. However, since the proposed model is not formulated as a well-defined problem due to fuzziness, we need to set some certain optimization criterion so as to transform into well-defined problems. In this paper, introducing the Sharpe ratio and fuzzy goals, we transform the main problem into the deterministic standard

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(Advance online publication: 19 August 2010)

mathematical programming problem, and develop an efficient solution method to find a global optimal solution of deterministic equivalent problem. Furthermore, we consider the sensitivity analysis in order to deal with investor's subjectivity. Sensitivity analysis in fuzzy linear programming (FLP) problem with crisp parameters and soft constraints was considered first by Hamacher et al. [5] and later on by many others, e.g. Tanaka et al. [25], and Fulle'r [4]. Sensitivity analysis for fuzzy linear fractional programming problem (FLFP) was studied by Dutta et al. [2].

This paper is organized in the following way. In Section 2, we introduce mathematical concepts of random fuzzy variables and parameters used in this paper. In Section 3, we propose a random fuzzy portfolio selection problem maximizing the Sharpe ratio. Then, we transform the main problem into the deterministic mathematical programming. In Section 4, in order to perform the sensitivity analysis, we provide a simple numerical example. Finally, in Section 5, we conclude this paper.

II. Mathematical Definition and Notation

Until now, there are many studies of portfolio selection problems whose future returns are assumed to be random variables or fuzzy numbers. However, there are few studies treated as random fuzzy variables. Therefore, first of all, we introduce a definition of random fuzzy variables proposed by Liu [18] as follows.

Definition 1 (Liu [18])
A random fuzzy variable is a function ξ from a collection of random variables R to $[0,1]$. An n-dimensional random fuzzy vector $\boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_n)$ is an n-tuple of random fuzzy variables $\xi_1, \xi_2, ..., \xi_n$.

That is, a random fuzzy variable is a fuzzy set defined on a universal set of random variables. Furthermore, the following random fuzzy arithmetic definition is introduced.

Definition 2 (Liu [18])
Let $\xi_1, \xi_2, ..., \xi_n$ be random fuzzy variables, and $f : R^n \to R$ be a continuous function. Then, $\xi = f(\xi_1, \xi_2, ..., \xi_n)$ is a random fuzzy variable on the product possibility space $(\Theta, P(\Theta), \mathrm{Pos})$, defined as

$$\xi(\theta_1, \theta_2, ..., \theta_n) = f(\xi_1(\theta_1), \xi_2(\theta_2), ..., \xi_n(\theta_n))$$

for all $(\theta_1, \theta_2, ..., \theta_n) \in \Theta$.

From these definitions, the following theorem is derived.

Theorem 1(Liu[18])
Let ξ_i be random fuzzy variables with membership functions μ_i, $i = 1, 2, ..., n$, respectively, and $f : R^n \to R$ be a continuous function. Then, $\xi = f(\xi_1, \xi_2, ..., \xi_n)$ is a random fuzzy variable whose membership function is

$$\mu(\eta) = \sup_{\eta_i \in R_i, 1 \le i \le n} \left\{ \min_{1 \le i \le n} \mu_i(\eta_i) \middle| \eta = f(\eta_1, \eta_2, ..., \eta_n) \right\}$$

for all $\eta \in R$, where

$$R = \left\{ f(\eta_1, \eta_2, ..., \eta_n) \middle| \eta_i \in R_i,\ i = 1, 2, ..., n \right\}.$$

Using this random fuzzy variable, we consider the random fuzzy portfolio selection problem. Notation of parameters used in this paper is as follows:

x_j : Budgeting allocation to the j th financial asset

$\bar{\tilde{r}}_j$: Future return of the j th financial asset assumed to be a random fuzzy variable, whose fuzzy expected value is $\tilde{m}_j$ and variance-covariance matrix is $\mathbf{V}$, respectively.

r_f : Risk free rate which is constant

$\hat{b}_j$: Limited upper value of each budgeting to the j th financial asset

n : Total number of assets

In the study of Hasuike and Katagiri [6], all fuzzy expected returns $\tilde{m}_j$ were assumed to be interval values. In this paper, in order to consider more practical cases, all fuzzy expected returns are characterized by general membership functions, and we particularly focus on these α-cuts presented by $\tilde{\boldsymbol{m}}_\alpha = \left[\boldsymbol{m}_L^\alpha, \boldsymbol{m}_U^\alpha\right]$. Furthermore, we assume that all fuzzy expected values for any satisfaction grades have the same variance-covariance matrix $\mathbf{V}$ derived from the statistical analysis to these interval values, and so $\mathbf{V}$ is assumed to be constant.

III. Formulation of Portfolio Selection Problem with Random Fuzzy Returns

The previous studies on random and fuzzy portfolio selection problems often have considered standard mean-variance model or safety first models introducing probability or fuzzy chance constraints based on modern portfolio theories (e.g. Hasuike et al. [8]). However, there are few studies maximizing the performance measures such as Sharpe ration for fuzzy portfolio selection problems. Therefore, in this paper, we propose the new model maximizing the fuzzy Sharpe ratio for the random fuzzy portfolio selection problem.

First, we deal with the following portfolio selection problem involving the random fuzzy variable based on the standard asset allocation problem to maximize total future returns.

$$\begin{aligned} &\text{Maximize} \quad \sum_{j=1}^{n} \bar{\tilde{r}}_j x_j \\ &\text{subject to} \quad \sum_{j=1}^{n} x_j = 1, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1, 2, \cdots, n \end{aligned} \tag{1}$$

In [8], we consider several models and solution approaches based on standard safety-first models of portfolio selection problems. However, we must assume that each return occurs according to the normal distributions in the sense of randomness in order to solve the previous models analytically. This assumption is a little restricted. Therefore, in this paper, we do not assume certain random distributions for future returns. Alternatively, we introduce the following portfolio model maximizing the Sharpe ratio, which is the most standard performance measures in the investment field, under particular satisfaction level h.

$$\begin{array}{ll} \text{Maximize} & \dfrac{\tilde{\boldsymbol{m}}_h{}^t \boldsymbol{x} - r_f}{\sqrt{\boldsymbol{x}^t \mathbf{V} \boldsymbol{x}}} \\ \text{subject to} & \sum_{j=1}^{n} x_j = 1, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1,2,\cdots,n \end{array} \tag{2}$$

Subsequently, we assume that $\tilde{\boldsymbol{m}}_h{}^t \boldsymbol{x} - r_f \ge 0$ to any satisfaction levels of fuzzy numbers because investors basically select only the risk free financial asset in the case $\tilde{\boldsymbol{m}}_h{}^t \boldsymbol{x} - r_f < 0$. Under this assumption, the main problem is equivalently transformed into the following problem:

$$\begin{array}{ll} \text{Minimize} & \dfrac{\sqrt{\boldsymbol{x}^t \mathbf{V} \boldsymbol{x}}}{\tilde{\boldsymbol{m}}_h{}^t \boldsymbol{x} - r_f} \\ \text{subject to} & \sum_{j=1}^{n} x_j = 1, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1,2,\cdots,n \end{array} \tag{3}$$

Then, introducing parameter $t = \dfrac{1}{\tilde{\boldsymbol{m}}_h{}^t \boldsymbol{x} - r_f}$, this fractional programming problem is transformed into the following nonlinear programming problem:

$$\begin{array}{ll} \text{Minimize} & \sqrt{\boldsymbol{x}^t \mathbf{V} \boldsymbol{x}} \\ \text{subject to} & \left(\tilde{\boldsymbol{m}}_h{}^t \boldsymbol{x} - r_f\right) t \cong 1, \\ & \sum_{j=1}^{n} x_j = 1, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1,2,\cdots,n \end{array} \tag{4}$$

Furthermore, since we set parameter $y_j = t x_j$ and obtain that minimizing $\sqrt{\boldsymbol{x}^t \mathbf{V} \boldsymbol{x}}$ is equivalent to minimizing $\boldsymbol{x}^t \mathbf{V} \boldsymbol{x}$ due to the positive definite matrix $\mathbf{V}$, problem (4) is equivalently transformed into the following quadratic programming problem:

$$\begin{array}{ll} \text{Minimize} & \boldsymbol{y}^t \mathbf{V} \boldsymbol{y} \\ \text{subject to} & \tilde{\boldsymbol{m}}_h{}^t \boldsymbol{y} - r_f t \cong 1, \\ & \sum_{j=1}^{n} y_j = t, \quad 0 \le y_j \le \hat{b}_j t, \quad j = 1,2,\cdots,n \end{array} \tag{5}$$

If expected values and variances do not include fuzziness, i.e., all m_j are fixed values, fuzzy constraint $\tilde{\boldsymbol{m}}_h{}^t \boldsymbol{y} - r_f t \cong 1$ is transformed into deterministic linear constraint $\boldsymbol{m}_h{}^t \boldsymbol{y} - r_f t = 1$, and so main problem (2) is equivalent to a quadratic programming problem similar to the probability maximization model due to $\mathbf{V}$ is the positive definite matrix. Therefore, we can analytically solve the problem not including fuzziness using previous solution approaches (e.g. [10]).

If expected values $\tilde{m}_j$ include fuzziness, the main problem is a fuzzy programming problem, and so we need to set some criterion for fuzziness. In general, investors have the target values for the total future return and the variance in order to obtain the stable and higher total future return. Each investor sets the original goal according to her or his subjectivity for the current market. In this paper, in order to present this investor's subjectivity for these goals, we introduce the following fuzzy goals for the total variance and deviation between total portfolio rate and risk-free rate based on Hamacher et al. [5]:

$$\begin{array}{l} \mu_\sigma(\omega) = \begin{cases} 1 - \dfrac{t_\sigma}{p_\sigma} & \boldsymbol{x}^t \mathbf{V} \boldsymbol{x} - t_\sigma = \sigma_G, \ t_\sigma \ge 0 \\ 1 & \boldsymbol{x}^t \mathbf{V} \boldsymbol{x} - t_\sigma \le \sigma_G \end{cases} \\ \mu_f(\omega) = \begin{cases} 1 - \dfrac{\left|t_f\right|}{p_f} & \tilde{\boldsymbol{m}}_h{}^t \boldsymbol{y} - r_f t = 1 + t_f, \\ 1 & \tilde{\boldsymbol{m}}_h{}^t \boldsymbol{y} - r_f t = 1 \end{cases} \end{array} \tag{6}$$

where σ_G is the aspiration level for the total variance. Then, p_σ and p_f are maximally acceptable violations of σ_G and $\tilde{\boldsymbol{m}}_h{}^t \boldsymbol{y} - r_f t - 1$. Using this linear membership functions for these target values, problem (5) is transformed into the following problem extending studies (e.g. [2]):

$$\begin{array}{ll} \text{Maximize} & \lambda \\ \text{subject to} & \lambda p_\sigma + t_\sigma \le p_\sigma, \\ & \boldsymbol{y}^t \mathbf{V} \boldsymbol{y} - t_\sigma \le \sigma_G, \\ & -(1-\lambda) p_f \le \tilde{\boldsymbol{m}}^t \boldsymbol{y} - r_f t - 1, \\ & \tilde{\boldsymbol{m}}_h{}^t \boldsymbol{y} - r_f t - 1 \le (1-\lambda) p_f, \ \tilde{\boldsymbol{m}}_h \in \left[\boldsymbol{m}_L^h, \boldsymbol{m}_U^h\right] \\ & \sum_{j=1}^{n} y_j = t, \quad 0 \le y_j \le \hat{b}_j t, \quad j = 1,2,\cdots,n \\ & \lambda, \ t_\sigma \ge 0 \end{array} \tag{7}$$

Furthermore, considering the interval $\left[\boldsymbol{m}_L^h, \boldsymbol{m}_U^h\right]$, problem (7) is transformed into the following deterministic equivalent problem:

$$\begin{array}{ll} \text{Maximize} & \lambda \\ \text{subject to} & \lambda p_\sigma + t_\sigma \le p_\sigma, \\ & \boldsymbol{y}^t \mathbf{V} \boldsymbol{y} - t_\sigma \le \sigma_G, \\ & -(1-\lambda) p_f \le \left(\boldsymbol{m}_L^h\right)^t \boldsymbol{y} - r_f t - 1, \\ & \left(\boldsymbol{m}_U^h\right)^t \boldsymbol{y} - r_f t - 1 \le (1-\lambda) p_f, \\ & \sum_{j=1}^{n} y_j = t, \quad 0 \le y_j \le \hat{b}_j t, \quad j = 1,2,\cdots,n \\ & \lambda, \ t_\sigma \ge 0 \end{array} \tag{8}$$

This problem is a deterministic quadratic programming problem, and so we analytically solve this problem by using the standard nonlinear programming problem.

IV. SENSITIVITY ANALYSIS

For the sensitivity analysis, we consider the following cases for problem (8) using the approach discussed in Hamacher et al. [5]:

(i) $\lambda_{\max} = 0$, if the constraints are strongly violated.

(ii) $\lambda_{\max} = 1$, if the constraints are satisfied in the crisp sense.

(iii) $0 < \lambda_{\max} < 1$, $\lambda_{\max}$ increases monotonously from 0 to 1.

In order to perform the sensitivity analysis on problem (8), we introduce the following simple numerical example. Table 1 shows that we assume four decision variables. In this numerical example, fuzzy mean values are characterized by symmetric triangle fuzzy numbers presented by $\langle \bar{m}_j, \alpha_j \rangle$ where $\bar{m}_j$ is the center value and α_j is the spread. Then, $\mathbf{V}$ is assumed to be a symmetric positive definite matrix satisfying $\sigma_{ij} = 0$. Let initial values of parameters be $r_f = 0.01,\ \sigma_G = 0.3$. Using the membership functions (6), we calculate the following three cases with respect to parameters p_σ, p_f, and h.

(a) Case 1: We fix $p_f = 3$ and $h = 0.8$. In the case $p_\sigma = 0.1, 0.2$, we solve problem (8) and obtain each optimal portfolio in Table 2. All optimal portfolios have same values, and the difference of the optimal values λ is 0.004 which is very small. Therefore, it is restrictive that changing parameter p_σ operates the sensitivity of portfolio performance.

(b) Case 2: We fix $p_\sigma = 0.2$ and $h = 0.8$. In the case $p_f = 1, 2, 3$, we solve problem (8) and obtain each optimal portfolio in Table 3.

TABLE 1. EXPECTED VALUES WITH INTERVALS AND VARIANCES

	x_1	x_2	x_3	x_4
$\tilde{m}_j$	<0.09,0.08>	<0.16,0.1>	<0.17,0.02>	<0.24,0.04>
σ_j^2	0.161	0.255	0.283	0.438

TABLE 2. OPTIMAL PORTFOLIO WITH $p_f = 3$

p_σ	λ	x_1	x_2	x_3	x_4
0.1	0.752	0.2022	0.2594	0.2805	0.2579
0.2	0.756	0.2022	0.2594	0.2805	0.2579

TABLE 3. OPTIMAL PORTFOLIO WITH $p_\sigma = 0.2$

p_f	λ	x_1	x_2	x_3	x_4
1	0.313	0.2022	0.2594	0.2805	0.2579
2	0.640	0.2022	0.2594	0.2805	0.2579
3	0.756	0.2022	0.2594	0.2805	0.2579

TABLE 4. OPTIMAL PORTFOLIO WITH $p_f = 3, p_\sigma = 0.2$

h	λ	x_1	x_2	x_3	x_4
1	0.763	0.2284	0.2704	0.2599	0.2413
0.8	0.756	0.2022	0.2594	0.2805	0.2579
0.5	0.747	0.1506	0.2377	0.3212	0.2905

In Table 3, particularly, the difference of optimal values λ between $p_f = 1$ and $p_f = 3$ is 0.443 and very large. Therefore, the relation between p_f and λ is stronger than that of p_σ.

(c) Case 3: We fix $p_f = 3$ and $p_\sigma = 0.2$. In the case $h = 1, 0.8, 0.5$, we solve problem (8) and obtain each optimal portfolio in Table 4. The differences of the optimal values λ among three confidence levels of h are very small, but each optimal portfolio is largely changed. Therefore, differences of confidence levels h strongly relate to the composition of optimal portfolios.

V. CONCLUSION

In this paper, we have considered a portfolio selection problem maximizing the Shrape ratio with fuzzy numbers. By performing equivalent transformations, the main problem has been equivalent to a quadratic programming problem, and we have presented that the proposed model can be solved analytically. Furthermore, with the help of simple numerical example, we have performed the sensitivity analysis for the proposed model. However, this sensitivity analysis is restricted to the small-scale example. Therefore, in the future, we will consider larger-scale sensitivity analysis of portfolio performance. Furthermore, we will show the analytical sensitivity analysis such as the study of Ali [1].

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