The Numerical Solution of the Exterior Boundary Value Problems for the Helmholtz's Equation for the Pseudosphere

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Abstract—**In this paper, the global Galerkin method is used to numerically solve the exterior Neumann and Dirichlet problems for the Helmholtz equation for the Pseudosphere in three dimensions based on Jones' modified integral equation approach. Warnapala and Morgan have used this method for the Oval of Cassini and obtained good results. Theoretical and computational details of the method for small values of k for the pseudosphere are presented.**

Index Terms—**Helmholtz Equation, Pseudosphere**

I. Introduction

The Helmholtz equation can be considered as a mathematical model of solving partial differential equations in space and time. The Helmholtz equation is given by

$$\Delta u + k^2 u = 0, Im\ k \geq 0,$$

where k is the wave number. The integral equation approach is widely recognized as the best approach for solving exterior problems for the Helmholtz's equation because of the non-uniqueness issue. Therefore to overcome the non-uniqueness problem arising in integral equations for the exterior boundary-value problems for the Helmholtz's equation, Jones [8] suggested adding a series of outgoing waves to the free-space fundamental solution. Jost used this method for the Maxwell equations of electromagnetic scattering for the sphere with an explicit coefficient choice [9]. Here we use Jones' modified integral equation approach, where we solve the exterior Dirichlet and Neumann problems for the modified integral equation, using the same global Galerkin method used by Lin [13]. We restrict to the region formed by the pseudosphere. Betrami found a shape, analogous to a sphere, but with a surface that obeys Lobachevsky's geometry. This is called a "pseudosphere", and it can be thought of as the opposite of a sphere or as a sphere of imaginary radius. The surface of a pseudosphere behaves according to the rules of hyperbolic geometry. Tarius and Sausset worked to provide a framework for building periodic boundary conditions on the pseudosphere [17]. Some theoretical work was done by Criado and Alamo on Thomas rotation of the pseudosphere corresponding to the space of relativistic velocities [7]. Up to date there are no numerical results for the Pseudoshpere for the Helmholtz equation with Neumann and Dirichlet boundary conditions.

When the surface and the boundary functions are sufficiently smooth, our method leads to quite small linear systems and converges quickly.

II. Definitions

It can be shown that (Colton and Kress) that the Helmholtz integral equation is uniquely solvable if k is not an eigenvalue for the corresponding interior Helmholtz equation. For the interior eigenvalues the homogeneous integral equation has non-trivial solutions. Therefore it is necessary to develop integral equations which are uniquely solvable for all frequencies k.

Let S be a closed bounded surface in $\Re^3$ and assume it belongs to the class of C^2. Let D_-, D_+, denote the interior and exterior of S respectively. The exterior Dirichlet problem for the Helmholtz's equation is given by

$$\Delta u\,(A) + k^2 u(A) = 0, A = (x, y, z) \in D_+, Im\ k \geq 0 \quad (1)$$
$$u(p) = f(p), p \in S$$

for the exterior Neumann problem the boundary condition changes to

$$\frac{u(P)}{\partial v_p} = f(p), p \in S$$

with f a given function and u satisfying the Sommerfeld radiation condition

$$u = O\left(\frac{1}{r}\right), \left(\frac{\partial}{\partial r} - ik\right)u = O\left(\frac{1}{r}\right) \text{ as } r_{qA} = |A| \to \infty. \quad (2)$$

Theoretical Framework of the Boundary Value Problems

The exterior Dirichlet and Neumann problems will be written as integral equations. We represented the solution as a modified double layer and single layer potentials respectively, based on the modified fundamental solution. (See [6]).

$$u(A) = \int_S u(q) \frac{\partial\left(\frac{e^{irk}}{4\pi r} + x(A,q)\right)}{\partial v_q} d\sigma_q \text{ with } A \in D_+ \quad (3a)$$
$$\text{where } r = |A - q|$$

$$u(A) = \int_S u(q)\Psi(A,q)d\sigma_q \text{ with } A \in D_+, \quad (3b)$$
$$r_{qA} = |A - q|\ .$$

The series of radiating waves is given by

$$\chi(A,q) = ik \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm} h_n^{(1)}(k|A|) Y_n^m \quad (4)$$
$$\left(\frac{A}{|A|}\right) h_n^{(1)}(k|q|)\overline{Y_n^m}\left(\frac{q}{|q|}\right).$$

Here $h_n^{(1)}$ denote the spherical Hankel function of the first kind and of order n, $Y_n^m, n = -m, \dots m$ are the linearly independent spherical harmonics of order m given by

$$Y_n^m(\phi, \theta) = \left(\frac{1}{2\pi}\left(m + \frac{1}{2}\right)\frac{(m-n)!}{(m+n)!}\right)^{\frac{1}{2}} p_n^m(\cos\theta)e^{im\phi}.$$

As in [6], here we assume that D₋ (pseudosphere) to be a connected domain containing the origin and we choose a ball B of radius R and center at the origin such that $\bar{B} \subset D_-$. On the coefficients a_{nm} we imposed the condition that the series χ (p, q) is uniformly convergent in p and in q in any region $|p|, |q| \geq R + \epsilon, \epsilon > 0$, and that the series can be two times differentiated term by term with respect to any of the variables with the resulting series being uniformly convergent.

We also assumed that the series χ is a solution to the Helmholtz equation satisfying the Sommerfeld radiation condition for $|p|, |q| > R$.

By letting A tend to a point $p \in S$, we obtain the following integral equations

$$-2\pi\mu(p) + \int_S \mu(q)\frac{\partial\Psi(p,q)}{\partial vq}d\sigma_q = -4\pi f(p), p \in S \quad (5a)$$

$$2\pi\mu(p) + \int_S \mu(q)\frac{\partial\Psi(p,q)}{\partial vq}d\sigma_q = 4\pi f(p), p \in S \quad (5b)$$

$$\text{where } \Psi = \frac{-e^{ikr}qp}{r} - 4\pi\chi(p,q).$$

We denote the above integral equations by

$$-2\pi\mu + K\mu = -4\pi f \text{ where in the Dirichlet case} \quad (6)$$

$$K\mu(p) = \int_S \mu(q)\frac{\partial}{\partial v_q}\left(\frac{-e^{ikr}qp}{r} - 4\pi\chi(p,q)\right)d\sigma_q \text{ and}$$

$$K = \int_S \mu(q)\frac{\partial}{\partial v_q}\left(\frac{-e^{ikr}qp}{r} - 4\pi\chi(p,q)\right)d\sigma_q \text{ in the}$$

Neumann case.

By the assumptions on the series χ (p, q) the kernel $\frac{\partial\chi(p,q)}{\partial v_q}$ is continuous on $S \times S$, and hence K is compact from $C(S)$ to $C(S)$ and $L^2(S)$ to $L^2(S)$. Kleinman and Roach [11] gave an explicit form of the coefficient a_{mn} that minimizes the upper bound on the spectral radius. If B is the exterior of a sphere radius R with center at the origin then the optimal coefficient for the Dirichlet and the Neumann problems was given by

$$(*) \; a_{nm} = -\frac{1}{2}\left(\frac{j_n(kR)}{h_n^1(kR)} + \frac{j_n^1(kR)}{h_n^{(1)\prime}(kR)}\right) \text{ for } n = 0, 1, 2, \dots \text{ and}$$

$$m = -n, \dots n.$$

This choice of the coefficient minimizes the condition number, and (6) is uniquely solvable for the pseudosphere. This coefficient was given for spherical regions, but the pseudosphere is a hyperbolic region. Therefore we restricted our pseudosphere is such a way that the hyperbolic part (the tails) are minimized. Cohl worked on the Laplace equation for hyperboloid spaces such as pseudospheres. He computed the Laplace-Beltrami operators and used it for solving the Laplace's equation for a radially symmetric solution [5].

Warnapala and Morgan used this method with the same coefficient choice a_{nm} and obtained good convergence results for the oval of Cassini for small wave numbers k.

III. SMOOTHNESS OF THE INTEGRAL OPERATOR K

Smoothness results of the double layer and single layer operators were proven by Lin [13, 14]. We know that the series χ can be differentiated term by term with respect to

any of the variables and that the resulting series is uniformly convergent. So the second derivative of the series is continuous on $\Re^3\backslash B$ where $B = \{x : |x| \leq R\}$. Furthermore the series χ is a solution to the Helmholtz equation satisfying the Sommerfeld radiation condition for $|x|, |y| > R$, when $B = \{x : |x| \leq R\}$ is contained in D.

By (Theorem 3.5 [6]) any two times continuously differentiable solution of the Helmholtz's equation is analytic and analytic functions are infinitely differentiable. So the series χ (p, q) is infinitely differentiable with respect to any of the variables p, q. Furthermore it is evident that if μ is bounded and integrable and $S \in C^l$ then $\int_S \chi$ (p, q) μ (q) $d\sigma_q \in C^l$ (S) and $\int_S \frac{\partial\chi(p,q)}{\partial v_q}\mu(q)d\sigma_q \in C^l(S)$.

IV. THE FRAMEWORK OF THE GALERKIN METHOD

The variable of integration in (6) was changed, converting it to a new integral equation defined on the unit sphere. The Galerkin method was applied to this new equation, using spherical polynomials to define the approximating subspaces.

$$m : U \to_{onto}^{1-1} S, \text{ where } m \text{ is at least differentiable.}$$
By changing the variable of integration on (6) we obtained the new equation over U,

$$-2\pi\hat{\mu} + \hat{K}\hat{\mu} = -4\pi\hat{f}, \hat{f} \in C(U). \quad (7)$$

The notation "^" denotes the change of variable from S to U. The operator $(-2\pi + \hat{K})^{-1}$ exists and is bounded on $C(U)$ and $L^2(U)$. Let $X = L^2(U)$, $\alpha = -2\pi$, and let an approximating subspace of spherical polynomials of degree $\leq N$ is denoted by X_N. The dimension of X_N is $d_N = (N + 1)^2$: and we let $\{h_1, \dots h_d\}$ denote the basis of spherical harmonics. Galerkin's method for solving (7) for the Dirichlet boundary conditions is given by

$$(-2\pi + P_N\hat{K})\widehat{\mu_N} = -4\pi P_N\hat{f}. \quad (8a)$$

The solution is given by $\widehat{\mu_N} = \sum_{j=1}^d \alpha_j h_j$

$$-2\pi\alpha_i(h_i, h_i) + \sum_{j=1}^d \alpha_j\left(\hat{K}h_j, h_i\right) = -4\pi(\hat{f}, h_i), \quad (9a)$$
$$i = 1, \dots d.$$

For the Neumann boundary conditions this can be written as
$$(-2\pi + P_N\hat{K})\widehat{\mu_N} = -4\pi P_N\hat{f} \quad (8b)$$
The solution is given by $\widehat{\mu_N} = \sum_{j=1}^d \alpha_j h_j$

$$-2\pi\alpha_i(h_i, h_i) + \sum_{j=1}^d \alpha_j\left(\hat{K}h_j, h_i\right) = -4\pi(\hat{f}, h_i), \quad (9b)$$
$$i = 1, \dots d$$

The convergence of μ_N to μ in $L^2(S)$ is straightforward. We know from previous literature that $P_N\hat{\mu} \to \hat{\mu}$ for all $\hat{\mu} \in L^2(U)$. From standard results it follows that $\|\hat{K} - P_N\hat{K}\| \to 0$ and we can obtain the desired convergence. (Also see [2]). Using the smoothness results of the integral operator K from section III, and following the same proof as in [1], we can prove the following theorems.

Theorem 4.1

Assume that $f \in C^{l,\lambda}(S), S \in C^{l+1,\lambda}$ $(S \in C^2 \text{ for } l = 0)$ and that the mapping m satisfies (7) for some $l \geq 0$. Then for all sufficiently large N, the inverses $(-2\pi + P_N\hat{K})^{-1}$ exist and are uniformly bounded and $\|\mu - \mu_N\| \leq \frac{c}{N^{l+\lambda'}}$ where $0 < \lambda' < \lambda$ is arbitrary. The constant c depends on l, μ and λ'.

Convergence in *C(U)*. To prove uniform convergence of $\widehat{\mu_N}$ to $\hat{\mu}$ is slightly more difficult. The main problem is that there is $\hat{\mu}$ in *C(U)* for which $P_N\hat{\mu}$ does not converge to $\hat{\mu}$. Convergence for all $\hat{\mu}$ would imply uniform boundedness of $\|P_N\|$.

Theorem 4.2

Assume that $S \in C^2$ and that *m* satisfies (7) with $l = 0$. Then considering $\widehat{K}$ as an operator on *C (U)*,
$$\left\|\widehat{K} - P_N\widehat{K}\right\| \to 0 \text{ as } N \to \infty . \qquad (10)$$
This implies the existence and uniform boundedness on *C(U)* of $(-2\pi + P_N\widehat{K})^{-1}$ for all sufficiently large *N*. Let $(-2\pi + \widehat{K})\hat{\mu} = -4\pi\hat{f}$ and $(-2\pi + P_N\widehat{K})\widehat{\mu_N} = -4\pi P_N\hat{f}$. If $f \in C^{0,\lambda}(S), \lambda > \frac{1}{2}$, then $\widehat{\mu_N}$ converges uniformly to $\hat{\mu}$. Moreover, if $S \in C^{l+1,\lambda}$ ($S \in C^2$ for $l = 0$) and $f \in C^{l,\lambda}(S), l + \lambda > \frac{1}{2}$, then $\|\mu - \mu_N\|_\infty \leq \frac{c}{N^{l+\lambda'-\frac{1}{2}}}$ with $0 < \lambda' < \lambda$. The constant *c* depends on f, l, λ'.

The Approximation of True Solutions for the Dirichlet Problem

Given μ_N an approximate solution of (6), we defined the approximate solution μ_N of (1) using the integral (5a).
$$u_N(A) = \int_S u_N(q) \frac{\partial}{\partial\nu_q}\left(\frac{e^{ikr_{qA}}}{4\pi r_{qA}} + \chi(A,q)\right) d\sigma_q, A \in D_+ \quad (11)$$
To show the convergence of $u_N(A)$ we used the following lemma.

Lemma 4.3
$$sup_{\{A \in K\}}\int_S \left|\frac{\partial}{\partial\nu_q}\left(\frac{e^{ikr_{qA}}}{4\pi r_{qA}} + \chi(A,q)\right)\right| d\sigma_q < \infty, \quad (12)$$
where *K* is any compact subset of *D*, from Warnapala and Morgan [20]. Similarly we can prove Lemma 4.3 for the Neumann problem.

Implementation of the Galerkin Method for the Dirichlet Problem

The coefficients $(\widehat{K}h_j, h_i)$ are fourfold integrals with a singular integrand. Because the Galerkin coefficients $(\widehat{K}h_j, h_i)$ depends only on the surface *S*, we calculate them separately for $N \leq N_{max}$. The following derivation was done for the Dirichlet problem; similar manipulations were done for the Neumann boundary conditions.

To decrease the effect of the singularity in computing $\widehat{K}h_j(\hat{p})$ in the Dirichlet case, we used the identity
$$\int_S -\frac{\partial}{\partial\nu_q}\frac{1}{r_{qp}} d\sigma_q = 2\pi, p \in S \text{ where } r_{qp} = |p - q|$$
to write
$$\widehat{K}h_j(\hat{p}) = \int_U -h_j(\hat{q})\frac{\partial}{\partial\nu_q}\left(\frac{\widehat{e^{ikr_{qp}} - 1}}{r_{qp}} + 4\pi\chi(p,q)\right)$$
$$|J(\hat{q})|d\sigma_{\hat{q}} + 2\pi h_j(\hat{p}) +$$
$$\int_U \left(-h_j(\hat{q}) - h_j(\hat{p})\right)\frac{\widehat{\partial}}{\partial\nu_q}\frac{1}{r_{qp}}|J(\hat{q})|d\sigma_{\hat{q}}$$
The integrands are bounded at $\hat{q} = \hat{p}$, where $J(\hat{q})$ is the Jacobian.

V. NUMERICAL EXAMPLES/EXPERIMENTAL SURFACE

In this section, several numerical examples are presented. The true solution is given by
$$u(x,y,z) = \frac{e^{ikr}}{r}.$$
The pseudosphere is analogous to the sphere in the same sense that is has constant Gauss curvature. The parametric equation for the pseudosphere is given by
$$f(x,y) = \left[\cos x \sin y, \sin x \sin y, \cos y + c\, ln(tan(\frac{y}{2}))\right]$$

where *x* varies from zero to π and *y* varies from zero to 2π. Balazs and Voros worked on the topic of Chaos on the pseudosphere, where they offered a diagnosis for chaoticity in quantum systems [3]. A pseudosphere has a radius *r* and the curvature of $\frac{-1}{r^2}$. The surface area of the pseudosphere is the same as of a sphere but the volume is half of that of a sphere. For analysis it is important to realize that the pseudosphere is simply connected and is of infinite extent. The convergence results for the Neumann problem was better than for the Dirichlet problem, thus confirming the fact that our method converges appropriately.

In our tables *NINTI* are the interior nodes for calculating $\widehat{K}h_j$, *NINTE* are the exterior nodes needed for calculating $(\widehat{K}h_j, h_i)$ and *NINT* are the nodes for calculating u_N. *NDEG* denotes the degree of the approximate spherical harmonics, recall that the number *d* of basic functions equals to $(NDEG + 1)^2$. In most cases we only added a few terms from the series. According to Jones [8] this is sufficient to remove the corresponding interior Dirichlet and interior Neumann eigenvalues and obtain unique solutions at the same time.
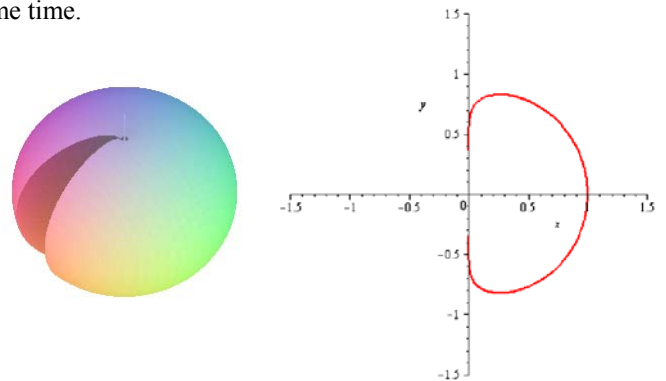

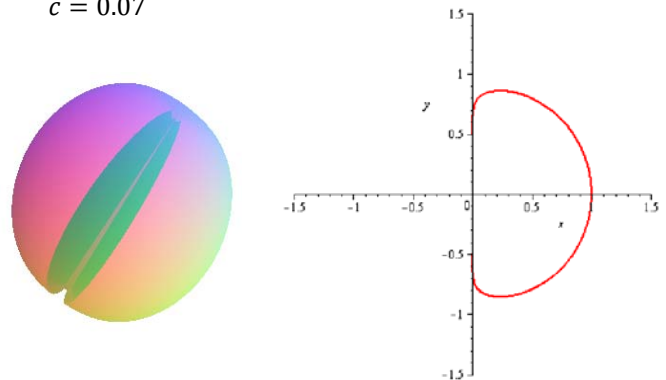Fig. 1. Let *S* be the pseudosphere for different values of *c*.
$c = 0.07$

Fig. 2. Let *S* be the pseudosphere for different values of *c*.
$c = 0.055$

For the radius of the pseudosphere we used the following formula:

$$r = \sqrt{1 + 2c \cos y \, ln(tan(y/2)) + c^2(ln(tan(y/2)))^2}$$

The following numerical results were obtained for the Dirichlet problem.

TABLE I: For $k = 1, c = 0.055, NDEG = 7, NINTI = 16, NINTE = 8$ and true solution u.

point	absolute error
(2,3,4)	2.054×10^{-3}
(12,4,11)	2.21×10^{-4}
(10,11,12)	7.585×10^{-5}
(20,21,22)	3.075×10^{-5}

In all the examples we used the coefficient (*).

TABLE II: For $k = 1, c = 0.06, NDEG = 7, NINTI = 16, NINE = 8$ and true solution u.

point	absolute error
(2,3,4)	2.306×10^{-3}
(12,4,11)	2.97×10^{-4}
(10,11,12)	4.749×10^{-5}
(20,21,22)	4.368×10^{-5}

When we added 10 terms from the series (in the above tables only 5 terms were added from the series), we obtained the following TABLE III.

TABLE III: For $k = 1, c = 0.055, NDEG = 7, NINTI = 16, NINTE = 8$ and true solution u.

point	absolute error
(2,3,4)	2.05×10^{-3}
(12,4,11)	2.193×10^{-4}
(10,11,12)	7.403×10^{-5}
(20,21,22)	3.144×10^{-5}

As you can see from the above TABLE III and the results were mostly better compared to the TABLE I. Now we increased the integration nodes, and obtained the following table.

TABLE IV: For $k = 1, c = 0.055, NDEG = 7, NINTI = 32, NINTE = 8$ and true solution u.

point	absolute error
(2,3,4)	1.945×10^{-3}
(12,4,11)	2.193×10^{-4}
(10,11,12)	2.193×10^{-5}
(20,21,22)	4.535×10^{-5}

We can see from the above table that when you increase the integration nodes, the accuracy improves. Thus we can add more than five terms and still obtain good results if we increase the number of integration nodes. But as more terms and increasing of integration nodes, increases the CPU time considerably (this will be discussed later more extensively), we added only a few terms, only five in other cases of pseudosphere. In the next table we changed the wave number k.

TABLE V: For $k = 0.01, c = 0.005, NDEG = 7, NINTI = 16, NINTE$ 8 and true solution u.

point	absolute error
(2,3,4)	3.633×10^{-4}
(12,4,11)	9.789×10^{-5}
(10,11,12)	1.413×10^{-4}
(20,21,22)	1.006×10^{-4}

From the above tables, we see that for the points away from the boundary there is much greater accuracy than for points near the boundary. This is because the integrand is more singular at points near the boundary. We use $NINTI = 16$ in calculating the Galerkin coefficients $(\widehat{K}h_j, h_i)$. The errors are printed in the column absolute error.

TABLE VI: For $k = 5, c = 0.005, NDEG = 7, NINTI = 16, NINTE = 8$ and true solution u.

point	absolute error
(2,3,4)	1.814×10^{-2}
(12,4,11)	2.075×10^{-3}
(10,11,12)	9.00×10^{-4}
(20,21,22)	1.632×10^{-4}

From the above tables we see that to obtain similar accuracy as in the previous tables we might need to increase the integration nodes. This is due to the following fact: the kernel function involves $\sin kr$ and $\cos kr$, and these trigonometric functions are much more oscillatory when k becomes large. Therefore in this case we must increase the integration nodes to achieve the same accuracy.

Remark:
We picked $NINTE < NINTI$, because the integrand of $(h_i, \widehat{K}h_j)$ is smoother than the integrand of $\widehat{K}h_j$. We also picked $NINTE \geq (NDEG + 1)$. Now we changed to the Neumann condition and the following tables are for the Neumann problem.

TABLE VII: For $k = 1, c = 0.001, NDEG = 7, NINTI = 32, NINTE = 16$ and true solution u.

point	absolute error
(20,21,22)	3.646×10^{-6}
(10,11,12)	3.748×10^{-6}
(12,4,11)	2.537×10^{-6}
(2,3,4)	5.987×10^{-5}

TABLE VIII: For $k = 1$, $c = 0.01$, $NDEG = 7$, $NINTI = 32$, $NINTE = 16$ and true solution u.

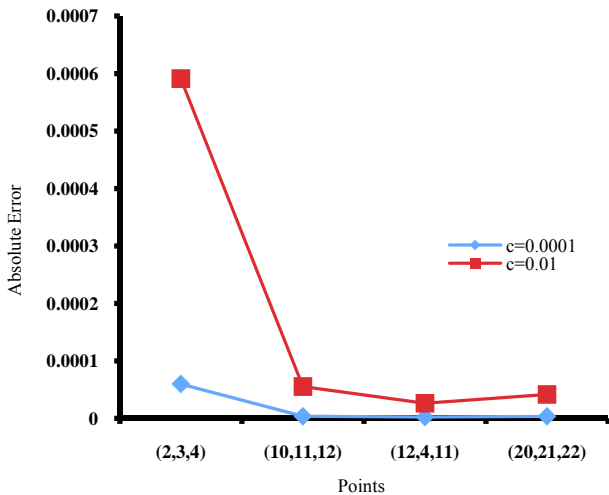| point	absolute error
(20,21,22)	7.931×10^{-6}
(10,11,12)	3.934×10^{-6}
(12,4,11)	2.712×10^{-5}
(2,3,4)	3.176×10^{-4}

Fig. 3. Absolute Error at k = 1.

When c is changed, we obtained new shapes of pseudospheres. When we increased the nodes we obtained better results for relatively larger c values.

TABLE IX: For $k = 5$, $c = 0.001$, $NDEG = 7$, $NINTI = 32$, $NINTE = 16$ and true solution u.

point	absolute error
(16,18,20)	6.87×10^{-6}
(10,11,12)	7.593×10^{-6}
(5,6,7)	4.875×10^{-5}
(2,3,4)	3.393×10^{-4}

Fig. 4. Absolute Error at k = 5.

As you see in the above two tables, still the accuracy is quite good for the two different pseudospheres.

Final Remarks:

Few terms from the infinite series were added in all our numerical experiments. This is because in numerical calculations it is inefficient to add the full series. So we allow only a finite number of the coefficients a_{nm} to be different from zero.

According to Jones [8], this is sufficient to ensure uniqueness for the modified integral equations in a finite range of wave numbers k. In practical applications, one is usually concerned with a finite range of k so this is not a serious draw back. In order to use a large amount of nodes we need a considerably high amount of CPU time. From the above examples, we see that the error is affected by the boundary S, $NINTI$, $NINTE$, boundary data and k and c.

As the value of the c decreases the hyperbolic nature of the pseudosphere for the Dirichlet problem increases and the rate of convergence decreases. See table XI.

TABLE X: For $k = 1$, $c = 0.01$, $NDEG = 7$, $NINTI = 32$, $NINTE = 16$ and true solution u.

point	absolute error
(20,21,22)	4.175×10^{-5}
(10,11,12)	4.545×10^{-5}
(12,4,11)	2.641×10^{-5}
(2,3,4)	5.912×10^{-4}

TABLE XI: For $k = 1$, $NDEG = 7$, $NINTI = 8$, $NINTE = 16$ and true solution u.

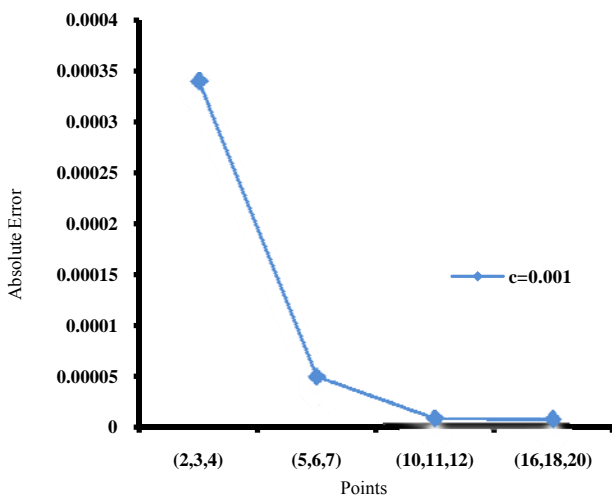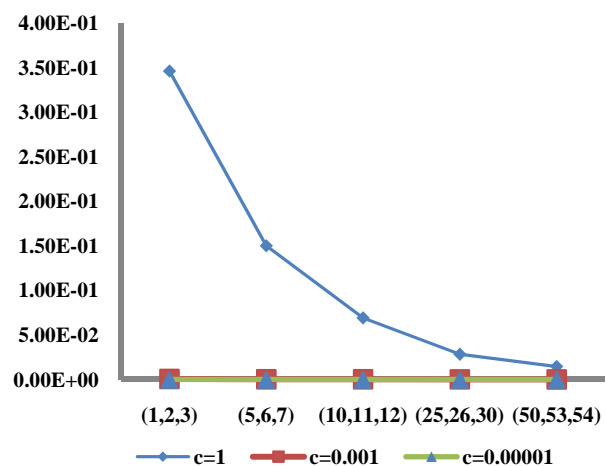| point	c = 1	c = 0.001	c = 0.00001
(2,3,4)	3.46E-01	5.73E-04	9.10E-06
(5,6,7)	1.50E-01	1.91E-04	3.22E-06
(10,11,12)	6.90E-02	1.06E-04	1.79E-06
(25,26,27)	2.82E-02	4.26E-05	6.41E-07
(50,53,54)	1.45E-02	2.26E-05	4.09E-07

Fig. 5. Absolute Error for varying c.

This is not surprising considering that a_{nm} originally defined by Jones for a constant radius R.

The role of k is more significant for ill-behaved boundary shapes. If we want to obtain more accuracy, we must increase the number of integration nodes for calculating the Galerkin coefficients $(\widehat{K}h_j, h_i)$. The cost of calculating the Galerkin coefficients is high.

Some of the increased cost comes from the complex number calculations, which is an intrinsic property of the Helmholtz equation. Furthermore any integration method is affected by k, due to the oscillatory behavior of the fundamental solution $\frac{e^{ikr}}{r}$. Also the CPU time depends on the number of terms added from the series.

In order to eliminate more interior Neumann and Dirichlet eigenvalues we need a more powerful computer which would decrease the CPU time considerably. We also see that for the shapes of pseudosphere with small c values the convergence results are quite good, even though the coefficient choice that was used was originally designed for spherical regions.

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