Asymptotic Behavior of Random Maturity American Options

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***Abstract*—We consider a class of American options with random maturity given by the first hitting time of some barrier β. We focus on the convergence of the no-arbitrage price of such a contract towards both the corresponding deterministic price and the perpetual one, when the volatility parameter ε shrinks to zero and the barrier β goes to infinity, respectively. Moreover, the convergence of the corresponding optimal strategies are analyzed. In the American put case, the optimal strategy is shown to be the first hitting time of some barrier $b_{\varepsilon,\beta}$ and it is proven that the price function is a non-increasing convex one solving a certain variational inequality. An explicit expression for the American put price is given as well as its approximation errors towards the corresponding deterministic and perpetual prices. Furthermore, we present several graphical illustrations of the prices's convergence. Which show that the deterministic price for small volatility and the perpetual price for large barrier provide accurate approximations.**

***Index Terms*—American options, random maturity, optimal stopping problem, asymptotic behavior, variational inequality, perpetual American options.**

I. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $B = \{B_t, 0 \leq t < \infty\}$ a standard Brownian motion and $\mathbb{F} \triangleq \{\mathcal{F}_t\}_{t\geq 0}$ the $\mathbb{P}$-completion of the natural filtration of B. To simplify the exposition and to make the results more explicit, we work throughout the paper in the setting of the classical Black-Scholes market model, under the risk-neutral probability, that is the underlying value $S^{\varepsilon,x}_{\cdot}$ is subject to the dynamic:

$$dS^{\varepsilon,x}_t = rS^{\varepsilon,x}_t dt + \varepsilon S^{\varepsilon,x}_t dB_t. \tag{1}$$

where $S^{\varepsilon,x}_0 = x > 0$ is the spot, $r > 0$ is the instantaneous interest rate and $\varepsilon > 0$ is the volatility of $S^{\varepsilon,x}_t$. When $\varepsilon = 0$, the underlying price becomes a function, given by $S^{0,x}_t = xe^{rt}$. The discounted price process $A^{\varepsilon,x}_t \triangleq e^{-rt}S^{\varepsilon,x}_t$ is the $\mathbb{P}$-martingale given by $A^{\varepsilon,x}_t = x\exp\left(\varepsilon B_t - \frac{\varepsilon^2}{2}t\right)$. Let ψ be a nonnegative function on $\mathbb{R}$ satisfying the following Lipchitz condition:

$$(\mathbf{H_1}) \quad \exists M > 0 : \forall x, y \in \mathbb{R}, \ \ |\psi(x) - \psi(y)| \leq M|x - y|$$

Obviously, call and put payoff options satisfy this hypothesis. The price of the so-called American contingent claim with payoff ψ, maturity T and spot x is given by the expression

$$\sup_{\gamma\in\mathcal{T}(0,T)} \mathbb{E}\left[e^{-r\gamma}\psi(S^{\varepsilon,x}_\gamma)\right]$$

where γ runs across the set $\mathcal{T}(0, T)$ of all stopping times of the Brownian filtration such that $\gamma \leq T$ almost surely (a.s.).

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On the subject of pricing American options and solving the associated optimal stopping problems, the reader can consult other references such as: [3], [4], [5], [6], [10], [11], [12], [15] and [16]. Let us state mathematically the definition of random maturity American options.

Definition 1: A random maturity American option (RMAO) is a financial instrument consisting of the following:

(i) An expiration date T which is an a.s. finite stopping time.
(ii) The selection of a stopping time $\gamma \in \mathcal{T}(0, T)$; i.e. such that $0 \leq \gamma \leq T$ a.s.
(iii) A payoff $Y_{\cdot}$ on exercise, where $\{Y_t, t \geq 0\}$ is an $\mathbb{F}$-adapted nonnegative continuous process, which satisfies

$$\mathbb{E}\left[\left(\sup_{t\geq 0} Y_{t\wedge T}\right)^\delta\right] < \infty \quad \text{for some} \quad \delta > 1$$

We denote by (T, Y) the characteristics of the RMAO.

Let us consider a RMAO with characteristics $(T, Y_t = e^{-rt}\psi(S^{\varepsilon,x}_t))$. Since the investor is risk-neutral, traditional arbitrage approach in complete markets may be followed to value an American contingent claim even if the time to maturity is random. The valuation of an American contingent claim is an optimal stopping problem. To obtain the expression for the American contingent claim we have to take the supremum over all possible stopping times of the expected discounted value of the payoff which is written as:(see e.g. [1]).

$$\sup_{\gamma\in\mathcal{T}(0,\infty)} \mathbb{E}[e^{-r(\gamma\wedge T)}\psi(S^{\varepsilon,x}_{\gamma\wedge T})]$$

where $\mathcal{T}(0, \infty)$ denotes the set of all a.s. finite stopping times of the Brownian filtration.

The last expression is then the price at time $t = 0$ of the RMAO with characteristics $(T, Y_t = e^{-rt}\psi(S^{\varepsilon,x}_t))$. The remainder of this paper is organized as follows: In section (II), we introduce the pricing problem of American options with random maturity given by the first hitting time of some barrier β. The optimal stopping time is presented as the first exit time of the underlying from the corresponding continuation region. In Section (III), we study the convergence of the no-arbitrage price of such a contract towards the corresponding deterministic problem when the volatility parameters ε shrinks to zero. Furthermore, the convergence of the corresponding optimal strategy is also provided. Section (IV) is devoted to studying the case of American put. It is shown that the price function is convex, non-increasing, solving a certain variational inequality and has an explicit expression. The optimal strategy, is shown to be the first hitting time by the underlying of some barrier which is

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characterized via an implicit function. A second order Taylor expansion of this barrier is provided and its approximation with the second order expansion is tested numerically. Moreover, we have studied the approximation of the Put price toward the deterministic price and the corresponding error. Numerical table and graphical illustrations show that when the volatility parameter ε is small enough, we get an accurate approximation which is useful for small volatilities market models. In section (V), we focus on the approximation of the random maturity American options toward the perpetual price when the barrier β goes to infinity. The analytical explicit expression for the perpetual price as well as the error of approximation are provided. In section (VI), we turn our attention to the approximation of the random maturity American put toward the corresponding perpetual one. In addition, we present the convergence of the optimal strategy. Finally, we show that when the barrier β is large enough we obtain a good approximation, which is useful for pricing random maturity American puts for large barrier β.

II. The class of random maturity American options

Let $\varepsilon_0 = \sqrt{r} \in]0,1]$ and notice that the interest rate r satisfies

$$r > \frac{\varepsilon_0^2}{2}. \tag{2}$$

We consider a RMAO with characteristics $(T, Y_t = e^{-rt}\psi(S_t^{\varepsilon,x}))$, where the random maturity time $T_\beta^{\varepsilon,x}$ is the first hitting time of the level $\beta > 0$ by the process $S_t^{\varepsilon,x}$, namely

$$T_\beta^{\varepsilon,x} \triangleq \inf\{t > 0; S_t^{\varepsilon,x} \geq \beta\}, \tag{3}$$

with the convention $\inf \emptyset = \infty$, i.e. $T_\beta^{\varepsilon,0} = \infty$.

Note that if $x \geq \beta$, $T_\beta^{\varepsilon,x} = 0$ a.s. and if $x \in [0,\beta[$, the stopping time $T_\beta^{\varepsilon,x}$ is also the first exit time of the process $S_t^{\varepsilon,x}$ from the interval $[0,\beta[$. Set

$$\rho = \log \beta \tag{4}$$

and define for $x \in]0,\beta]$ the process: $x_t^{\varepsilon,x} \triangleq \log S_t^{\varepsilon,x} = \log x + \left(r - \frac{\varepsilon^2}{2}\right)t + \varepsilon B_t$. For $x \in]0,\beta]$ the stopping time $T_\beta^{\varepsilon,x}$ is also the hitting time of the level ρ by the process $x_t^{\varepsilon,x}$:

$$T_\beta^{\varepsilon,x} = \inf\{t > 0 : x_t^{\varepsilon,x} = \rho\} = \inf\{t > 0 : \nu t + B_t = a(\beta)\}. \tag{5}$$

where

$$\nu = \frac{r}{\varepsilon} - \frac{\varepsilon}{2}, \tag{6}$$

$$\forall z > 0,\ a(z) = \frac{1}{\varepsilon}\log\left(\frac{z}{x}\right). \tag{7}$$

Assume that the barrier β satisfies the hypothesis

$$(\mathbf{H_2}) \qquad \beta > K > 1.$$

We have

Lemma 1: For all $x \in]0,\beta[$, there exists a random time $\nabla_\beta^{\varepsilon_0,x}$, finite a.s. on Ω, such that, for all $\varepsilon \in [0,\varepsilon_0]$

$$0 \leq T_\beta^{\varepsilon,x} \leq \nabla_\beta^{\varepsilon_0,x} < \infty \text{ a.s.}$$

Proof: The proof is put in Appendix (A). ■

Assumption (i) of definition (1), is thus realized.

Obviously, the payoff $e^{-rt}\psi(S_t^{\varepsilon,x})$ on exercise satisfies then assumption (iii) of definition (1). Let $v_\beta^\varepsilon(x)$ be the price at time $t = 0$ of the RMAO with characteristics $(T_\beta^{\varepsilon,x}, Y_t = e^{-rt}\psi(S_t^{\varepsilon,x}))$, $x \in [0,\beta]$. We always take

$$\forall x \geq \beta, \qquad v_\beta^\varepsilon(x) = \psi(\beta). \tag{8}$$

For $\varepsilon \neq 0$, the function $v_\beta^\varepsilon(x)$ is the value function of the following optimal stopping problem defined on $[0,\beta]$ by

$$\left(\mathcal{P}_{\varepsilon,\beta}^{am}\right) \quad v_\beta^\varepsilon(x) = \sup_{\gamma \in \mathcal{T}(0,\infty)} \mathbb{E}\left[e^{-r(\gamma \wedge T_\beta^{\varepsilon,x})}\psi(S_{\gamma\wedge T_\beta^{\varepsilon,x}}^{\varepsilon,x})\right].$$

We obviously have the following integrability property:

The family (9)
$\left\{e^{-r\gamma}\psi(S_\gamma^{\varepsilon,x}), \gamma \text{ stopping time } / \gamma \leq T_\beta^{\varepsilon,x} \text{ a.s.}\right\}$
is uniformly integrable for all $x \geq 0$.

Let D be the continuation region given by

$$D \triangleq \{x \in [0,\beta] / v_\beta^\varepsilon(x) > \psi(x)\}.$$

Since $v_\beta^\varepsilon = \psi$ on $\{0,\beta\}$, we have $D \subsetneq [0,\beta[$. It follows that

$$\gamma_\beta^{\varepsilon,x} \leq T_\beta^{\varepsilon,x} < \infty \text{ a.s. for all } x \in D, \tag{10}$$

where

$$\gamma_\beta^{\varepsilon,x} \triangleq \inf\{t > 0; S_t^{\varepsilon,x} \notin D\}. \tag{11}$$

Since hypothesis $(\mathbf{H_1})$ is realized, the stopping time $\gamma_\beta^{\varepsilon,x}$ given by (11) is an optimal stopping time (i.e. for which the supremum in $\left(\mathcal{P}_{\varepsilon,\beta}^{am}\right)$ is achieved). Moreover, (uniqueness) if γ_{opt} is an optimal stopping time, then $\gamma_{opt} \leq \gamma_\beta^{\varepsilon,x}$ a.s. (see e.g. [14]).

III. Asymptotic behavior with respect to the volatility ε

A. Asymptotic values of random maturity American options

For general payoff ψ, we can not compute explicitly v_β^ε for fixed $\varepsilon > 0$. We have, therefore, to proceed by approximation. We will show that the function v_β^ε goes to v_β^0 when the volatility ε shrinks to zero, where v_β^0 is the value function of the asymptotic deterministic optimization problem:

$$\left(\mathcal{P}_{0,\beta}^{am}\right) \qquad v_\beta^0(x) = \sup_{0 \leq s \leq T_\beta^x} e^{-rs}\psi(xe^{rs}), x \in [0,\beta[,$$

where $T_\beta^x \triangleq T_\beta^{0,x} = \frac{\rho - \log(x)}{r} = \frac{1}{r}\log\left(\frac{\beta}{x}\right)$ is the hitting time of the level β by the function $S_t^{0,x}$.

Proposition 1: For all $x \in]0,\beta]$, $\lim_{\varepsilon\to 0} T_\beta^{\varepsilon,x} = T_\beta^x$ a.s.

Proof: Fix $x \in]0,\beta]$. According to lemma (1),, we have $\sup_{0\leq\varepsilon\leq\varepsilon_0} T_\beta^{\varepsilon,x} < \infty$ a.s. On the other hand, $S_{T_\beta^{\varepsilon,x}}^{\varepsilon,x} = \beta$ and $S^{\varepsilon,x}$ converges to $S^{0,x}$ as ε shrinks to 0 uniformly on compact sets. We deduce that all cluster point τ of $T_\beta^{\varepsilon,x}$ verifies $S_\tau^{0,x} = \beta$. Hence, $\tau = \frac{1}{r}\log(\frac{\beta}{x})$. Therefore, $T_\beta^{\varepsilon,x}$ goes to T_β^x a.s. ■

Proposition 2: For all $x \in [0,\beta]$, $\lim_{\varepsilon\to 0} v_\beta^\varepsilon(x) = v_\beta^0(x)$.

Proof: We have $v_\beta^\varepsilon(x) - v_\beta^0(x)| \leq \mathbb{E}\left[\sup_{t\geq 0}\left|e^{-r(t\wedge T_\beta^{\varepsilon,x})}\psi\left(S_{t\wedge T_\beta^{\varepsilon,x}}^{\varepsilon,x}\right) - e^{-r(t\wedge T_\beta^x)}\psi(S_{t\wedge T_\beta^x}^{0,x})\right|\right]$. By a dominated convergence argument, it is enough to show that we have a.s.:

$$\lim_{\varepsilon\to 0}\left|e^{-r(t\wedge T_\beta^{\varepsilon,x})}\psi\left(S_{t\wedge T_\beta^{\varepsilon,x}}^{\varepsilon,x}\right) - e^{-r(t\wedge T_\beta^x)}\psi(S_{t\wedge T_\beta^x}^{0,x})\right| = 0,$$

which results from the uniform convergence a.s. of $S^{\varepsilon,x}$ to $S^{0,x}$ and $T_\beta^{\varepsilon,x}$ to T_β^x as ε tends to 0 on compact sets. ■

B. Asymptotic exercise times of random maturity American options

It is useful to recall that the stopping time $\gamma_\beta^{\varepsilon,x}$ given by (11) is an exercise time of the random maturity American option with characteristics $(T_\beta^{\varepsilon,x}, e^{-rt}\psi(S_t^{\varepsilon,x}))$.
We will show that when ε goes to zero, the exercise time $\gamma_\beta^{\varepsilon,x}$ of the RMAO converges to the deterministic time $\gamma_\beta^x \triangleq \gamma_\beta^{0,x}$ which is a solution of the deterministic optimal stopping problem $\left(\mathcal{P}_{0,\beta}^{am}\right)$. Let h be the (continuous) function defined as

$$h(x) \triangleq \psi(e^x). \tag{12}$$

Equation (8) is equivalent to $v_\beta^\varepsilon(x) = h(\rho)$ for all $x \geq \beta$. For any $\mathbb{F}$-stopping time γ, set for $x \in [0,\beta[$, $v_\beta^{\varepsilon,\gamma}(x) = \mathbb{E}\left[e^{-r(\gamma\wedge T_\beta^{\varepsilon,x})} h\left(x_{\gamma\wedge T_\beta^{\varepsilon,x}}^{\varepsilon,x}\right)\right]$, then we have:

$$\left(\mathcal{P}_{\varepsilon,\beta}^{am}\right) \qquad v_\beta^\varepsilon(x) = \sup_{\gamma\in\mathcal{T}(0,\infty)} v_\beta^{\varepsilon,\gamma}(x), x \in [0,\beta[$$

Define Γ_β^ε as $\Gamma_\beta^\varepsilon \triangleq \{z \leq \rho : v_\beta^\varepsilon(e^z) = h(z)\}$. Then the stopping time $\gamma_\beta^{\varepsilon,x}$ given by (11) is also the time of the first entry of the process $x_t^{\varepsilon,x}$ into the set Γ_β^ε; i.e.

$$\gamma_\beta^{\varepsilon,x} = \inf\{t \geq 0 : x_t^{\varepsilon,x} \in \Gamma_\beta^\varepsilon\}$$

The problem $\left(\mathcal{P}_{0,\beta}^{am}\right)$ is equivalently rewritten as

$$\left(\mathcal{P}_{0,\beta}^{am}\right) \qquad v_\beta^0(x) = \sup_{\log(x)\leq y\leq\rho} xe^{-y}h(y), x \in [0,\beta[,$$

where h is given by (12).

Remark 1: The value $v_\beta^0(x)$ is considered as the value of an American option with deterministic underlying asset price $S_t^{0,x}$, deterministic maturity T_β^x and payoff ψ; that is in a market with no risky assets. Set
$\Gamma_\beta^0 = \{z \leq \rho : v_\beta^0(e^z) = h(z)\} = \{x \leq \beta : v_\beta^0(x) = \psi(x)\}$.

Then Γ_β^0 is nonempty since $\rho \in \Gamma_\beta^0$.

Example 1: (1) Case of a call option : $\psi(y) = \max(y - K, 0)$, $K > 0$. Since $\beta > K$, then $\Gamma_\beta^0 = \{\rho\}$.

(2) Case of a put option : $\psi(y) = \max(K - y, 0)$, $K > 0$. Since $\beta > K$, then $\Gamma_\beta^0 =]-\infty, \rho]$.
The set Γ_β^0 is rewritten as $\Gamma_\beta^0 = \{z \leq \rho : \varrho(z) = e^{-z}h(z)\}$, where

$$\varrho(z) = \sup_{z\leq y\leq\rho} e^{-y}h(y). \tag{13}$$

It comes that the deterministic exit time

$$\gamma_\beta^x \triangleq \gamma_\beta^{0,x} = \inf\{t > 0; \log(x) + rt \in \Gamma_\beta^0\}, \tag{14}$$

is a solution of the problem $\left(\mathcal{P}_{0,\beta}^{am}\right)$. We have the following cases

Case 1: Γ_β^0 satisfies the hypothesis

$(\mathbf{H_3})$. $\quad \Gamma_\beta^0$ is reduced to a single point, i.e. $\Gamma_\beta^0 = \{\rho\}$

In this case, the solution γ_β^x of the problem $\left(\mathcal{P}_{0,\beta}^{am}\right)$ is written as:

$$\gamma_\beta^x = \frac{\rho - \log(x)}{r}. \tag{15}$$

While the function $v_\beta^0(x)$ is explicitly given by

$$v_\beta^0(x) = e^{-\rho}\psi(e^\rho)x, \quad x \in [0,\beta[.$$

For the case of a call if $\beta > K$, we get $\gamma_\beta^x = \frac{1}{r}\log\left(\frac{\beta}{x}\right) = T_\beta^x$ and for all $x \in [0,\beta[$, we get $v_\beta^0(x) = \frac{\beta-K}{\beta}x$.

Case 2: Γ_β^0 is not reduced to a single point, then for example for the case of a put option we get $\gamma_\beta^x = 0$ and for all $x \in [0,\beta[$, we get $P_\beta^0(x) = K - x$.

We have the following result which states the convergence of the optimal strategy when the volatility ε shrinks to zero.

Proposition 3: Under hypothesis $(\mathbf{H_3})$, we have for all $x \in [0,\beta]$

$$\gamma_\beta^x = \lim_{\varepsilon\to 0}\gamma_\beta^{\varepsilon,x} \text{ a.s.}$$

Proof: For each compact set C, we have

$$\sup_{x\in C}|v_\beta^\varepsilon(x) - v_\beta^0(x)| \leq$$
$$\mathbb{E}\left[\sup_{x\in C, t\geq 0}\left|e^{-r(t\wedge T_\beta^{\varepsilon,x})}\psi\left(S_{t\wedge T_\beta^{\varepsilon,x}}^{\varepsilon,x}\right) - e^{-r(t\wedge T_\beta^x)}\psi(S_{t\wedge T_\beta^x}^{0,x})\right|\right]$$

using dominated convergence argument as in proposition (2), this shows the uniform convergence of v_β^ε to v_β^0 on compact sets. On the other hand, we have the following lemma the proof of which is put is Appendix (B):

Lemma 2: For each compact set C we have

$$\chi \triangleq \sup_{0\leq\varepsilon\leq\varepsilon_0, x\in C} T_\beta^{\varepsilon,x} < \infty \text{ a.s.}$$

Fix $x \in [0,\beta]$ and let γ be a cluster point of $\gamma_\beta^{\varepsilon,x}$ as ε goes to zero. We have $v_\beta^\varepsilon(S_{\gamma_\beta^{\varepsilon,x}}^{\varepsilon,x}) = \psi(S_{\gamma_\beta^{\varepsilon,x}}^{\varepsilon,x})$ and $S^{\varepsilon,x} \to S^{0,x}$ uniformly on $[0,T]$. Hence $v_\beta^0(S_\gamma^{0,x}) = \psi(S_\gamma^{0,x})$. Finally by hypothesis $(\mathbf{H_3})$, we obtain $S_\gamma^{0,x} = \rho$ and $\gamma = \gamma_\beta^{0,x}$. ■

Let us summarize the asymptotic results in the following theorem:

Theorem 1: Let us fix $x \in]0,\beta[$, then under hypothesis $(\mathbf{H_1})$, $(\mathbf{H_2})$ and $(\mathbf{H_3})$, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} v_\beta^\varepsilon(x) &= v_\beta^0(x) \\
\lim_{\varepsilon\to 0} T_\beta^{\varepsilon,x} &= T_\beta^x \triangleq T_\beta^{0,x} \text{ a.s.} \\
\lim_{\varepsilon\to 0} \gamma_\beta^{\varepsilon,x} &= \gamma_\beta^x \triangleq \gamma_\beta^{0,x} \text{ a.s.}
\end{aligned}$$

We say that the problem $(\mathcal{P}_{\varepsilon,\beta}^{am})$ is asymptotic to $(\mathcal{P}_{0,\beta}^{am})$ as the volatility ε shrinks to zero.

Remark 2: Hypothesis $(\mathbf{H_3})$ is needed only to show the convergence of the optimal strategy.

IV. Asymptotic behavior of random maturity American put with respect to the volatility ε

We show that in the case of American puts, the optimal strategy is the first hitting time of some barrier $b_{\varepsilon,\beta}$. In other words, the problem is reduced to a maximization over $b > 0$ of the Laplace transform of the stopping times $T_\beta^{\varepsilon,x} \wedge \tau^{\varepsilon,x}(b)$, where for $b > 0$, $\tau^{\varepsilon,x}(b)$ is given by

$$\tau^{\varepsilon,x}(b) \triangleq \inf\{t > 0; S_t^{\varepsilon,x} < b\}. \tag{16}$$

Exploiting the explicit formulae in the case of the Brownian motion with linear drift, we will provide an explicit formulae for the random maturity American put price.

A. Explicit formulae for the price of random maturity American put

The initial price $P^{\varepsilon}_{\beta}(x)$ of the random maturity American put option with payoff $\psi(x) = \max(K-x, 0)$, where $K > 0$ is the strike, is given for $0 \le x < \beta$ by $\left(\mathcal{P}^{amput}_{\varepsilon,\beta}\right)$

$$P^{\varepsilon}_{\beta}(x) = \sup_{\gamma \in \mathcal{T}(0,\infty)} \mathbb{E}\left[e^{-r(\gamma \wedge T^{\varepsilon,x}_{\beta})} \max(K - S^{\varepsilon,x}_{\gamma \wedge T^{\varepsilon,x}_{\beta}}, 0)\right].$$

Set $\hat{P}^{\varepsilon}_{\beta}(x) = 0$ for $x \ge \beta$ and for $0 \le x < \beta$:

$$\hat{P}^{\varepsilon}_{\beta}(x) = \tag{17}$$
$$\sup_{b \in]0,\infty[} \mathbb{E}\left[e^{-r(\tau^{\varepsilon,x}(b) \wedge T^{\varepsilon,x}_{\beta})} \max\left(K - S^{\varepsilon,x}_{\tau^{\varepsilon,x}(b) \wedge T^{\varepsilon,x}_{\beta}}, 0\right)\right],$$

For $x \in [0, \beta[$, set $\vartheta(x, b) =$

$$\mathbb{E}\left[e^{-r(\tau^{\varepsilon,x}(b) \wedge T^{\varepsilon,x}_{\beta})} \max(K - S^{\varepsilon,x}_{T^{\varepsilon,x}_{\beta} \wedge \tau^{\varepsilon,x}(b)}, 0)\right]. \tag{18}$$

Since $\tau^{\varepsilon,x}(b) = 0$, a.s. for $0 \le x < b$, then we have $\hat{P}^{\varepsilon}_{\beta}(x) = \max(K - x, 0)$ for all $x \in [0, b[$. Hence, if $0 \le x < \beta$ and $b^* \in]0, x[$ is optimal for the supremum in (17), then

$$\hat{P}^{\varepsilon}_{\beta}(x) = \begin{cases} \max(K - x, 0) & \text{if } 0 \le x < b^* \\ \sup\limits_{b \in [0,x]} \vartheta(x, b) = \vartheta(x, b^*) & \text{if } b^* \le x < \beta. \end{cases}$$

We have the following result which gives the explicit expression of (18)

Proposition 4: If $0 < b \le x < \beta$, then

$$\begin{aligned} \vartheta(x, b) &= \max(K - \beta, 0)\left(\frac{\beta}{x}\right)^{\delta} \frac{b^{1+\delta} - x^{1+\delta}}{b^{1+\delta} - \beta^{1+\delta}} \\ &+ \max(K - b, 0)\left(\frac{b}{x}\right)^{\delta} \frac{x^{1+\delta} - \beta^{1+\delta}}{b^{1+\delta} - \beta^{1+\delta}}. \end{aligned} \tag{19}$$

where δ is given by:

$$\delta = \frac{2r}{\varepsilon^2}. \tag{20}$$

Proof: See Appendix (C). ■

Proposition 5: Under hypothesis $(\mathbf{H_2})$, let $y_{\varepsilon,\beta}$ be the unique zero on $\left[\frac{\beta}{K}, \infty\right[$ of the nonlinear equation

$$\Phi(y) = -\frac{(1+\delta)\beta}{K} + \delta y + y^{-\delta}, \tag{21}$$

and set $b_{\varepsilon,\beta} \triangleq \frac{\beta}{y_{\varepsilon,\beta}}$, then $\hat{P}^{\varepsilon}_{\beta}(x)$ is explicitly given by

$$\hat{P}^{\varepsilon}_{\beta}(x) = \tag{22}$$

$$\begin{cases} K - x & \text{if} 0 \le x < b_{\varepsilon,\beta} \\ \left(\frac{y_{\varepsilon,\beta}}{\beta}\frac{K\delta}{1+\delta} - 1\right)x + \frac{K}{1+\delta}\left(\frac{\beta}{y_{\varepsilon,\beta}}\right)^{\delta} x^{-\delta} & \text{if} b_{\varepsilon,\beta} \le x < \beta \\ 0 & \text{if} x \ge \beta \end{cases}$$

Proof: See Appendix (D). ■

B. Variational inequality

Suppose that hypothesis $(\mathbf{H_2})$ is realized and let

$$\mathcal{H} \triangleq \mathcal{C}([0,\infty[) \cap \mathcal{C}^1((0,\infty)\backslash\{\beta\}) \cap \mathcal{C}^2((0,\infty)\backslash\{b,\beta\})$$

Find $b \in]0, K[$ and a convex function V decreasing on $[0, \beta]$ in the space $\mathcal{H}$ and such that

$$\frac{\varepsilon^2}{2}x^2V'' + rxV' - rV = 0 \text{ if } b < x < \beta. \tag{23}$$
$$\frac{\varepsilon^2}{2}x^2V'' + rxV' - rV < 0 \text{ if } 0 < x < b. \tag{24}$$
$$V(x) > \max(K - x, 0) \text{ if } b < x < \beta. \tag{25}$$
$$V(x) = K - x \quad \text{if } 0 \le x \le b. \tag{26}$$
$$V(x) = 0 \quad \text{if } x \ge \beta. \tag{27}$$

Proposition 6: Let $y_{\varepsilon,\beta}$ be the unique zero of equation (21) on $[\frac{\beta}{K}, \infty[$ and set $b_{\varepsilon,\beta} \triangleq \frac{\beta}{y_{\varepsilon,\beta}}$. Under hypothesis $(\mathbf{H_2})$ and for ε small enough, the pair $(b_{\varepsilon,\beta}, \hat{P}^{\varepsilon}_{\beta})$ solves the variational inequality (23)-(27), where $\hat{P}^{\varepsilon}_{\beta}$ is given by equation (22).

Proof: The proof is put in Appendix (E). ■

Theorem 2: Suppose that hypothesis $(\mathbf{H_2})$ is realized and let $y_{\varepsilon,\beta}$ be the unique zero of the function Φ given by equation (21) on $[\frac{\beta}{K}, \infty[$, and $b_{\varepsilon,\beta} \triangleq \frac{\beta}{y_{\varepsilon,\beta}}$. If $b_{\varepsilon,\beta} \in]0, K[$ and V solve the variational inequality (23)-(27), then for ε small enough we have

$$\forall x \ge 0, \quad V(x) = \hat{P}^{\varepsilon}_{\beta}(x) = P^{\varepsilon}_{\beta}(x). \tag{28}$$

Moreover, for all $x \in]b_{\varepsilon,\beta}, \beta[$, the stopping time $\tau^{\varepsilon,x}(b_{\varepsilon,\beta})$ given by equation (16) is optimal for the problem $(\mathcal{P}^{am}_{\varepsilon,\beta})$, i.e.

$$\hat{P}^{\varepsilon}_{\beta}(x) = P^{\varepsilon}_{\beta}(x) = \tag{29}$$
$$\mathbb{E}\left[e^{-r(\tau^{\varepsilon,x}(b_{\varepsilon,\beta}) \wedge T^{\varepsilon,x}_{\beta})} \max(K - S^{\varepsilon,x}_{\tau^{\varepsilon,x}(b_{\varepsilon,\beta}) \wedge T^{\varepsilon,x}_{\beta}}, 0)\right].$$

For all $x \in]b_{\varepsilon,\beta}, \beta[$, the optimal strategy of the random maturity American put is then the first hitting time of the barrier $b_{\varepsilon,\beta} > 0$ by the underlying price $S^{\varepsilon,x}$.

Proof: In light of proposition (6), we have $V = \hat{P}^{\varepsilon}_{\beta}$. Let us show that $V = P^{\varepsilon}_{\beta}$. Since $V(x) = P^{\varepsilon}_{\beta}(x) = 0$ for all $x \ge \beta$ and $V(x) = P^{\varepsilon}_{\beta}(x) = K - x$ for all $x \in [0, b_{\varepsilon,\beta}]$, it remains to show that $V(x) = P^{\varepsilon}_{\beta}(x)$ for all $x \in]b_{\varepsilon,\beta}, \beta[$. Let $x \in]b_{\varepsilon,\beta}, \beta[$ and applying Itô's formula to the discounted process $\{e^{-rt}V(S^{\varepsilon,x}_t), t \ge 0\}$, we get taking account of (1), (23) and (24) for every stopping time γ:

$$\begin{aligned} & V(x) - e^{-r(\gamma \wedge T^{\varepsilon,x}_{\beta})} V\left(S^{\varepsilon,x}_{\gamma \wedge T^{\varepsilon,x}_{\beta}}\right) \\ &+ \varepsilon \int_0^{\gamma \wedge T^{\varepsilon,x}_{\beta}} e^{-rt} \left(\xi V'(\xi)\right)|_{\xi = S^{\varepsilon,x}_t} dB_t. \\ &= -\int_0^{\gamma \wedge T^{\varepsilon,x}_{\beta}} e^{-rt} \left(\frac{\varepsilon^2}{2}\xi^2 V''(\xi) + r\xi V'(\xi) - rV(\xi)\right)\Big|_{\xi = S^{\varepsilon,x}_t} \\ &\ge 0. \text{ a.s.} \end{aligned} \tag{30}$$

Since the function V is convex and non-increasing, we have

$$0 \le -\xi V'(\xi) \le V(0) - V(\xi) \le V(0), \quad \text{for } 0 < \xi < \infty$$

Thus, we obtain $\mathbb{E}\left[\int_0^{\infty} e^{-2rt}(S^{\varepsilon,x}_t V'(S^{\varepsilon,x}_t))^2 dt\right] \le \frac{V^2(0)}{2r} < \infty$. After application of the expectation in both sides of the inequality (30), we obtain

$$V(x) \ge \mathbb{E}\left[e^{-r(\gamma \wedge T^{\varepsilon,x}_{\beta})} V(S^{\varepsilon,x}_{\gamma \wedge T^{\varepsilon,x}_{\beta}})\right].$$

This yields taking account of (25), (26) and the inequality $S^{\varepsilon,x}_{\gamma \wedge T^{\varepsilon,x}_{\beta}} \le \beta$ a.s. for all stopping time γ:

$$V(x) \ge \mathbb{E}\left[e^{-r(\gamma \wedge T^{\varepsilon,x}_{\beta})} \max\left(K - S^{\varepsilon,x}_{\gamma \wedge T^{\varepsilon,x}_{\beta}}, 0\right)\right] \tag{31}$$

Hence

$$\forall x \in]b_{\varepsilon,\beta}, \beta[, \qquad V(x) \geq P^{\varepsilon}_{\beta}(x)$$

If $x \in]b_{\varepsilon,\beta}, \beta[$, then for all $t < \tau^{\varepsilon,x}(b_{\varepsilon,\beta}) \wedge T^{\varepsilon,x}_{\beta}$ a.s. we have $S^{\varepsilon,x}_t \in]b_{\varepsilon,\beta}, \beta[$ a.s. Hence, thanks to (23), we have a.s., for all $t < \tau^{\varepsilon,x}(b_{\varepsilon,\beta}) \wedge T^{\varepsilon,x}_{\beta}$:

$$\left(\frac{\varepsilon^2}{2}\xi^2 V''(\xi) + r\xi V'(\xi) - rV(\xi)\right)\Big|_{\xi = S^{\varepsilon,x}_t} = 0.$$

In light of (30), we get:

$$V(x) \quad = \quad \mathbb{E}\left[e^{-r(\tau^{\varepsilon,x}(b_{\varepsilon,\beta})\wedge T^{\varepsilon,x}_{\beta})} V\left(S^{\varepsilon,x}_{\tau^{\varepsilon,x}(b_{\varepsilon,\beta})\wedge T^{\varepsilon,x}_{\beta}}\right)\right].$$

Nevertheless $S^{\varepsilon,x}_{\tau^{\varepsilon,x}(b_{\varepsilon,\beta})\wedge T^{\varepsilon,x}_{\beta}} \in \{b_{\varepsilon,\beta}, \beta\}$ a.s. and (26)-(27) show that $V(x) = \max(K - x, 0)$ on $\{b_{\varepsilon,\beta}, \beta\}$. This implies that

$$\begin{aligned} & V(x) = \\ & \mathbb{E}\left[e^{-r(\tau^{\varepsilon,x}(b_{\varepsilon,\beta})\wedge T^{\varepsilon,x}_{\beta})} \max\left(K - S^{\varepsilon,x}_{\tau^{\varepsilon,x}(b_{\varepsilon,\beta})\wedge T^{\varepsilon,x}_{\beta}}, 0\right)\right] \\ & \geq \quad \sup_{\gamma \in \mathcal{T}(0,\infty)} \mathbb{E}\left[e^{-r(\gamma \wedge T^{\varepsilon,x}_{\beta})} \max\left(K - S^{\varepsilon,x}_{\gamma\wedge T^{\varepsilon,x}_{\beta}}, 0\right)\right] \\ & = \quad P^{\varepsilon}_{\beta}(x). \end{aligned}$$

Thus $V(x) =$

$$\begin{aligned} & \mathbb{E}\left[e^{-r(\tau^{\varepsilon,x}(b_{\varepsilon,\beta})\wedge T^{\varepsilon,x}_{\beta})} \max\left(K - S^{\varepsilon,x}_{\tau^{\varepsilon,x}(b_{\varepsilon,\beta})\wedge T^{\varepsilon,x}_{\beta}}, 0\right)\right] \\ & = P^{\varepsilon}_{\beta}(x). \end{aligned}$$

The claims of (28) and (29) now follow readily. Finally, since

$$\forall x \leq b_{\varepsilon,\beta}, v^{\varepsilon}_{\beta}(x) = \psi(x) \text{ and } \forall b_{\varepsilon,\beta} < x < \beta, v^{\varepsilon}_{\beta}(x) > \psi(x) \tag{32}$$

and the stopping time $\gamma^{\varepsilon,x}_{\beta}$ given by equation (11) is optimal for the problem $\left(\mathcal{P}^{am}_{\varepsilon,\beta}\right)$, we conclude using definition of $\gamma^{\varepsilon,x}_{\beta}$ and equation (32) that

$$\gamma^{\varepsilon,x}_{\beta} = \tau^{\varepsilon,x}(b_{\varepsilon,\beta}). \tag{33}$$

■

C. Numerical results

In order to obtain numerical results, we take as model's parameters:

$$r = 0.1, \quad \beta = 150, \quad K = 100.$$

Set

$$\phi(\varepsilon, y) = y^{-\frac{2r}{\varepsilon^2}} + \frac{2r}{\varepsilon^2} y - \left(1 + \frac{2r}{\varepsilon^2}\right)\frac{\beta}{K}. \tag{34}$$

For all $\varepsilon > 0$, there exists a unique $y_{\varepsilon,\beta} > \frac{\beta}{K}$ such that $\phi(\varepsilon, y_{\varepsilon,\beta}) = 0$. We propose now to find the second order expansion, with respect to ε, of the function $y_{\varepsilon,\beta}$ in a neighborhood of $\varepsilon = 0$; i.e. to find $a_0 > 0$ $a_1 \geq 0$ and $a_2 \geq 0$ such that $\lim_{\varepsilon\to 0} \phi(\varepsilon, a_0 + a_1\varepsilon + a_2\varepsilon^2 + o(\varepsilon^2)) = 0$. We find easily that $a_0 = \frac{\beta}{K}$, $a_1 = 0$ and $a_2 = \frac{\beta}{2rK}$. Thus

$$y_{\varepsilon,\beta} \quad = \quad \frac{\beta}{K} + \frac{\beta}{2rK}\varepsilon^2 + o(\varepsilon^2). \tag{35}$$

$$b_{\varepsilon,\beta} \quad = \quad \frac{\beta}{y_{\varepsilon,\beta}} = K - \frac{K}{2r}\varepsilon^2 + o(\varepsilon^2). \tag{36}$$

We make the approximation results in evidence and study the error of the approximation of P^{ε}_{β} by the corresponding deterministic price P^0_{β}. The error is measured by the infinite norm on $[0, \beta]$:

$$e^{\varepsilon}_{\beta} \triangleq ||P^{\varepsilon}_{\beta} - P^0_{\beta}||_{\infty} \triangleq \sup_{x\in[0,\beta]} |P^{\varepsilon}_{\beta}(x) - P^0_{\beta}(x)|. \tag{37}$$

TABLE I
THE APPROXIMATION ERROR OF THE PUT PRICE P^{ε}_{β} BY P^0_{β}.

ε	e^{ε}_{β}	ε	e^{ε}_{β}
0.4	33.8676	0.07	1.7590
0.3	23.2143	0.06	1.2992
0.2	12.3191	0.05	0.9080
0.1	3.5044	0.04	0.5820
0.09	2.8626	0.03	0.3283

V. ASYMPTOTIC BEHAVIOR WITH RESPECT TO THE BARRIER β

Let $v^{\varepsilon}_{\infty}(x)$ be the value of the random maturity American option with payoff ψ, underlying asset $S^{\varepsilon,x}$ and maturity $T^{\varepsilon,x}_{\beta}$. It can be computed as the optimal expected reward in the problem of optimal stopping

$$\left(\mathcal{P}^{am}_{\varepsilon,\infty}\right) \qquad v^{\varepsilon}_{\infty}(x) = \sup_{\gamma\in\mathcal{T}(0,\infty)} \mathbb{E}\left[e^{-r\gamma}\psi(S^{\varepsilon,x}_{\gamma})\right].$$

We aim to prove that $v^{\varepsilon}_{\beta}(x)$ goes to $v^{\varepsilon}_{\infty}(x)$ as β goes to infinity and we precise the associated error of convergence. Let D_{∞} be the continuation region given by

$$D_{\infty} \triangleq \{s > 0; v^{\varepsilon}_{\infty}(s) > \psi(s)\}.$$

Then the stopping time

$$\gamma^{\varepsilon,x}_{\infty} \triangleq \inf\{t > 0; S^{\varepsilon,x}_t \notin D_{\infty}\},$$

is optimal since hypothesis $(\mathbf{H_1})$ is realized (see [14]). Let

$$\begin{aligned} k \quad &\triangleq \quad \inf\{x \geq 0; \psi(x) = 0\}, \\ b_{\varepsilon,\infty} \quad &\triangleq \quad \sup\{x \geq 0; v^{\varepsilon}_{\infty}(x) = \psi(x)\}, \end{aligned}$$

with the convention that $\inf(\emptyset) = \infty$ and assume that the function ψ satisfies

$(\mathbf{H_3})$

ψ is convex, non-increasing and differentiable on $[0, k]$.

The hypothesis $(\mathbf{H_3})$ is satisfied in the put case for example. We have the following result:

Proposition 7: Let $\varepsilon \in]0, \sqrt{r}[$. The real number $b_{\varepsilon,\infty}$ is the solution of the first order differential equation

$$z\psi'(z) + \delta\psi(z) = 0,$$

in the interval $]0, k[$ and the expression of $v^{\varepsilon}_{\infty}(x)$ is given by:

$$v^{\varepsilon}_{\infty}(x) = \begin{cases} \psi(x) & \text{if } x \leq b_{\varepsilon,\infty} \\ \psi(x)\left(\frac{x}{b_{\varepsilon,\infty}}\right)^{-\delta} & \text{if } x > b_{\varepsilon,\infty} \end{cases}$$

where δ is given by (20).

Proof: The proof is put in Appendix (F). ■

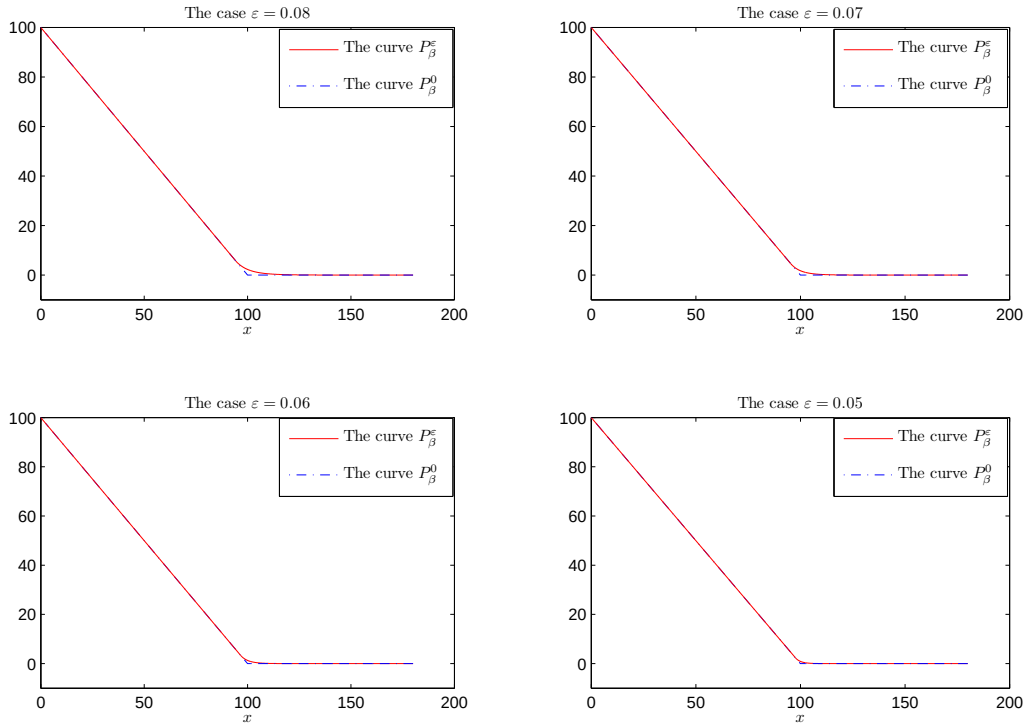

Fig. 1. The curves $x \mapsto P^{\varepsilon}_{\beta}$ and $x \mapsto P^{0}_{\beta}$ for $\varepsilon = 0.05, 0.06, 0.07, 0.08$.

Proposition 8: There exist some constants $\kappa_0 > 0$ and $\kappa_1 < 0$ such that, for all $\beta > b_{\varepsilon,\infty}$ and $x \in]b_{\varepsilon,\infty}, \beta[$:

$$\begin{aligned} 0 &\leq v^{\varepsilon}_{\infty}(x) - v^{\varepsilon}_{\beta}(x) \\ &\leq 2MC(\beta - b_{\varepsilon,\infty}) \left(\frac{x}{b_{\varepsilon,\infty}}\right)^{\frac{1}{2}} \frac{\left(\frac{\beta}{b_{\varepsilon,\infty}}\right)^{-\kappa_1}}{\left(\frac{\beta}{b_{\varepsilon,\infty}}\right)^{\frac{2\kappa_0}{\varepsilon}} - 1} \triangleq g(\beta), \end{aligned}$$

and as β goes to infinity, we have:

$$g(\beta) \sim \beta \to \infty \frac{2MC\sqrt{x}}{b_{\varepsilon,\infty}^{\kappa_1 + \frac{1}{2}}} \beta^{\kappa_1},$$

In particular

$$\lim_{\beta \to +\infty} v^{\varepsilon}_{\beta}(x) = v^{\varepsilon}_{\infty}(x). \tag{38}$$

Proof: Suppose that $\beta > b_{\varepsilon,\infty}$. For $x \in]b_{\varepsilon,\beta}, \beta[$, we have $S^{\varepsilon,x}_{T^{\varepsilon,x}_{\beta}} = \beta$ and $v^{\varepsilon,x}_{\gamma^{\varepsilon,x}_{\infty}} = b_{\varepsilon,\infty}$. Therefore,

$$\begin{aligned} & 0 \leq v^{\varepsilon}_{\infty}(x) - v^{\varepsilon}_{\beta}(x) \\ \leq\ & \mathbb{E}\left[e^{-r\gamma^{\varepsilon,x}_{\infty}} \psi(S^{\varepsilon,x}_{\gamma^{\varepsilon,x}_{\infty}}) - e^{-r(\gamma^{\varepsilon,x}_{\infty} \wedge T^{\varepsilon,x}_{\beta})} \psi(S^{\varepsilon,x}_{\gamma^{\varepsilon,x}_{\infty} \wedge T^{\varepsilon,x}_{\beta}})\right] \\ =\ & \mathbb{E}\left[1_{\{T^{\varepsilon,x}_{\beta} < \gamma^{\varepsilon,x}_{\infty}\}} \left(e^{-r\gamma^{\varepsilon,x}_{\infty}} \psi(b_{\varepsilon,\infty}) - e^{-rT^{\varepsilon,x}_{\beta}} \psi(\beta)\right)\right] \\ \leq\ & \mathbb{E}\left[1_{\{T^{\varepsilon,x}_{\beta} < \gamma^{\varepsilon,x}_{\infty}\}} e^{-rT^{\varepsilon,x}_{\beta}} \left(\psi(b_{\varepsilon,\infty}) - \psi(\beta)\right)\right] \\ \leq\ & M(\beta - b_{\varepsilon,\infty}) \mathbb{E}\left[1_{\{T^{\varepsilon,x}_{\beta} < \gamma^{\varepsilon,x}_{\infty}\}} e^{-rT^{\varepsilon,x}_{\beta}}\right] \text{ by } (\mathbf{H_1}) \\ \leq\ & M(\beta - b_{\varepsilon,\infty}) f(\beta), \end{aligned}$$

with

$$f(\beta) \triangleq E\left[1_{\{T^{\varepsilon,x}_{\beta} < \gamma^{\varepsilon,x}_{\infty}\}} e^{-rT^{\varepsilon,x}_{\beta}}\right].$$

The following lemma, the proof of which is put in Appendix (G) is focused on the computation of $f(\beta)$:

Lemma 3: For all $x \in]b_{\varepsilon,\beta}, \beta[$, we have

$$f(\beta) = 2C \left(\frac{x}{b_{\varepsilon,\infty}}\right)^{\frac{1}{2}} \frac{\left(\frac{\beta}{b_{\varepsilon,\infty}}\right)^{\frac{\kappa_0}{\varepsilon} - \frac{1}{2}}}{\left(\frac{\beta}{b_{\varepsilon,\infty}}\right)^{\frac{2\kappa_0}{\varepsilon}} - 1},$$

where C is a positive constant and $\kappa_0 = \sqrt{\frac{\varepsilon^2}{4} + 2r}$. Set

$$\begin{aligned} g(\beta) &\triangleq M(\beta - b_{\varepsilon,\infty}) f(\beta) \\ &= 2MC(\beta - b_{\varepsilon,\infty}) \left(\frac{x}{b_{\varepsilon,\infty}}\right)^{\frac{1}{2}} \frac{\left(\frac{\beta}{b_{\varepsilon,\infty}}\right)^{\frac{\kappa_0}{\varepsilon} - \frac{1}{2}}}{\left(\frac{\beta}{b_{\varepsilon,\infty}}\right)^{\frac{2\kappa_0}{\varepsilon}} - 1} \\ &\sim_{\beta \to \infty} 2MC \left(\frac{x}{l}\right)^{\frac{1}{2}} \left(\frac{\beta}{l}\right)^{\kappa_1} = \frac{2MC\sqrt{x}}{l^{\kappa_1 + \frac{1}{2}}} \beta^{\kappa_1}, \end{aligned}$$

with $\kappa_1 \triangleq \frac{1}{2} - \frac{\kappa_0}{\varepsilon} < 0$. This implies (38). ■

VI. Approximation of the Random Maturity American Put with the Corresponding Perpetual One

The barrier β is supposed to satisfy the hypothesis $(\mathbf{H_2})$. Recall that $P^{\varepsilon}_{\beta}(x)$ denotes the value of the random maturity put with strike K, underlying asset $S^{\varepsilon,x}$ and maturity $T^{\varepsilon,x}_{\beta}$. It is computed as the optimal expected reward in the problem of optimal stopping $\left(\mathcal{P}^{amput}_{\varepsilon,\beta}\right)$ defined in subsection (IV-A). We also consider the corresponding perpetual put option with strike K and underlying asset $S^{\varepsilon,x}$ with value $P^{\varepsilon}_{\infty}(x)$, which is the value function of the following optimal stopping problem

$$\left(\mathcal{P}^{amput}_{\varepsilon,\infty}\right) \quad P^{\varepsilon}_{\infty}(x) = \sup_{\gamma \in \mathcal{T}(0,\infty)} E\left[e^{-r\gamma} \max(K - S^{\varepsilon,x}_{\gamma}, 0)\right]$$

We have (see e.g. [13]):

$$P_{\infty}^{\varepsilon}(x) = \begin{cases} K - x & \text{if } x \leq b_{\varepsilon,\infty} \\ (K - b_{\varepsilon,\infty})\left(\frac{x}{b_{\varepsilon,\infty}}\right)^{-\delta} & \text{if } x > b_{\varepsilon,\infty} \end{cases} . \quad (39)$$

where

$$b_{\varepsilon,\infty} = \frac{K\delta}{1+\delta}, \quad (40)$$

and δ is given by (20). The stopping time

$$\gamma_{\infty}^{\varepsilon,x} \triangleq \tau^{\varepsilon,x}(b_{\varepsilon,\infty})$$

where $\tau^{\varepsilon,x}(z)$ is given by equation (16) is optimal for the problem $\left(\mathcal{P}_{\varepsilon,\infty}^{amput}\right)$.

Lemma 4: We have

$$\lim_{\beta\to\infty} b_{\varepsilon,\beta} = b_{\varepsilon,\infty}. \quad (41)$$

where $b_{\varepsilon,\infty}$ is given by (40).

Proof: The proof is put in appendix (H). ■

We also have:

Lemma 5: For all $x > 0$, we have

$$\lim_{\beta\to\infty} \gamma_{\beta}^{\varepsilon,x} = \gamma_{\infty}^{\varepsilon,x}. \quad (42)$$

Proof: The proof is put in appendix (I). ■

Finally, we have

Lemma 6:

$$\lim_{\beta\to\infty} P_{\beta}^{\varepsilon}(x) = P_{\infty}^{\varepsilon}(x), \quad \forall x \geq 0. \quad (43)$$

Proof: Taking account of (41), we obtain the convergence (43) of the put price P_{β}^{ε} given by equation (22) to the perpetual price P_{∞}^{ε} given by equation (39) as $\beta \to \infty$. ■

In light of lemma (4), lemma (5) and lemma (6), we say that the problem $\left(\mathcal{P}_{\varepsilon,\beta}^{amput}\right)$ is asymptotic to the problem $\left(\mathcal{P}_{\varepsilon,\infty}^{amput}\right)$ as β goes to infinity.

A. Numerical results

To obtain numerical results, we take as model's parameters: $r = 0.1$, $\varepsilon = 0.3$, $K = 100$. We make the approximation results in evidence and study the error of approximation of P_{β}^{ε} by the perpetual price P_{∞}^{ε}. The error of approximation is measured by the infinite norm on $[0, \beta]$:

$$e(\beta) \triangleq ||P_{\beta}^{0.3} - P_{\infty}^{0.3}||_{\infty}.$$

Figure (2) illustrates the convergence of $P_{\beta}^{\varepsilon}(x)$ to $P_{\infty}^{\varepsilon}(x)$ as β goes to infinity.

TABLE II
THE APPROXIMATION ERROR $e(\beta)$.

β	y_{∞}^{ε}	$e(\beta)$	β	y_{∞}^{ε}	$e(\beta)$
300	53.8949	6.8490	2500	52.6453	0.6494
500	53.0471	3.8827	3000	52.6409	0.5303
800	52.7840	2.3032	3500	52.6383	0.4468
1000	52.7265	1.7974	4000	52.6366	0.3852
1500	52.6718	1.1455	4500	52.6355	0.3380
2000	52.6535	0.8321	5000	52.6347	0.3006

VII. CONCLUDING REMARKS

In this paper we investigated a model for pricing American options, where the maturity is given by the first hitting time of a barrier by the underlying asset. We showed the convergence of the no-arbitrage price of such American options towards the corresponding deterministic and perpetual ones when the volatility parameter ε shrinks to zero and the barrier β goes to infinity, respectively.

It is known that volatility refers to the degree to which financial prices fluctuate. Small volatility means that the returns (i.e. the relative price changes) fluctuate over a tight range of outcomes. During the option's life time, there are always periods of small fluctuations. So we can use the limit price on periods of small volatilities or until maturity for the case of small volatility market models.

Finally, although there is an explicit expression for the random maturity American put price, the advantages of approximating (subject to tolerated errors) this price as the volatility ε goes to zero or the barrier β goes to infinity, is that the limit price does not depend respectively on ε and on β.

APPENDIX A
PROOF OF LEMMA (1)

Let $x \in]0, \beta[$ and $\varepsilon \in [0, \varepsilon_0]$. Since $r > \frac{\varepsilon^2}{2}$ using (2), the function $y_t^{\varepsilon,x} \triangleq \log(x) + (r - \frac{\varepsilon^2}{2})t$ hits the level ρ at the date of time:

$$\lambda_{\beta}^{\varepsilon,x} \triangleq \frac{\rho - \log(x)}{r - \frac{\varepsilon^2}{2}}. \quad (44)$$

Let:

$$\nabla_{\beta}^{\varepsilon,x} = \inf\{t > \lambda_{\beta}^{\varepsilon,x} : B_t = 0\}. \quad (45)$$

Then obviously we have $T_{\beta}^{\varepsilon,x} \leq \nabla_{\beta}^{\varepsilon,x} \leq \nabla_{\beta}^{\varepsilon_0,x} < \infty$ a.s.. The lemma is then proved.

APPENDIX B
PROOF OF LEMMA (2)

According to lemma (1), we have $\lambda_{\beta}^{\varepsilon,x} \geq \varsigma(\beta) \triangleq \inf_{x\in C} \frac{\rho - \log(x)}{r - \frac{\varepsilon_0^2}{2}} < \infty$, where $\lambda_{\beta}^{\varepsilon,x}$ is given by (44). It comes that $\nabla_{\beta}^{\varepsilon_0,x} \leq \nabla(\beta) \triangleq \inf\{t > \varsigma(\beta) : B_t = 0\}$, where $\nabla_{\beta}^{\varepsilon_0,x}$ is given by (45). Hence $\chi \triangleq \sup_{0\leq\varepsilon\leq\varepsilon_0, x\in C} T_{\beta}^{\varepsilon,x} \leq \nabla(\beta) < \infty$ a.s.

APPENDIX C
PROOF OF PROPOSITION (4)

For $x \in [0, \beta[$, we have:

$$\begin{aligned} \vartheta(x,b) &= \max(K-\beta, 0)\mathbb{E}\left[e^{-rT_{\beta}^{\varepsilon,x}}\mathbf{1}_{\{T_{\beta}^{\varepsilon,x} \leq \tau^{\varepsilon,x}(b)\}}\right] \\ &+ \max(K-b, 0)\mathbb{E}\left[e^{-r\tau^{\varepsilon,x}(b)}\mathbf{1}_{\{\tau^{\varepsilon,x}(b) \leq T_{\beta}^{\varepsilon,x}\}}\right]. \end{aligned}$$

Set

$$X_t \triangleq B_t + \nu t, \quad (46)$$

where ν is given by (6), $\mu \triangleq \frac{r}{\varepsilon} + \frac{\varepsilon}{2}$ and for any stochastic process Z and any level $z > 0$, denote by

$$R_z(Z) \triangleq \inf\{t > 0; Z_t = z\}, \quad (47)$$

the (first) hitting time of the level z by the process Z. Then, since $b \leq x < \beta$, it comes that $a(b) \leq 0 < a(\beta)$, where $a(z)$

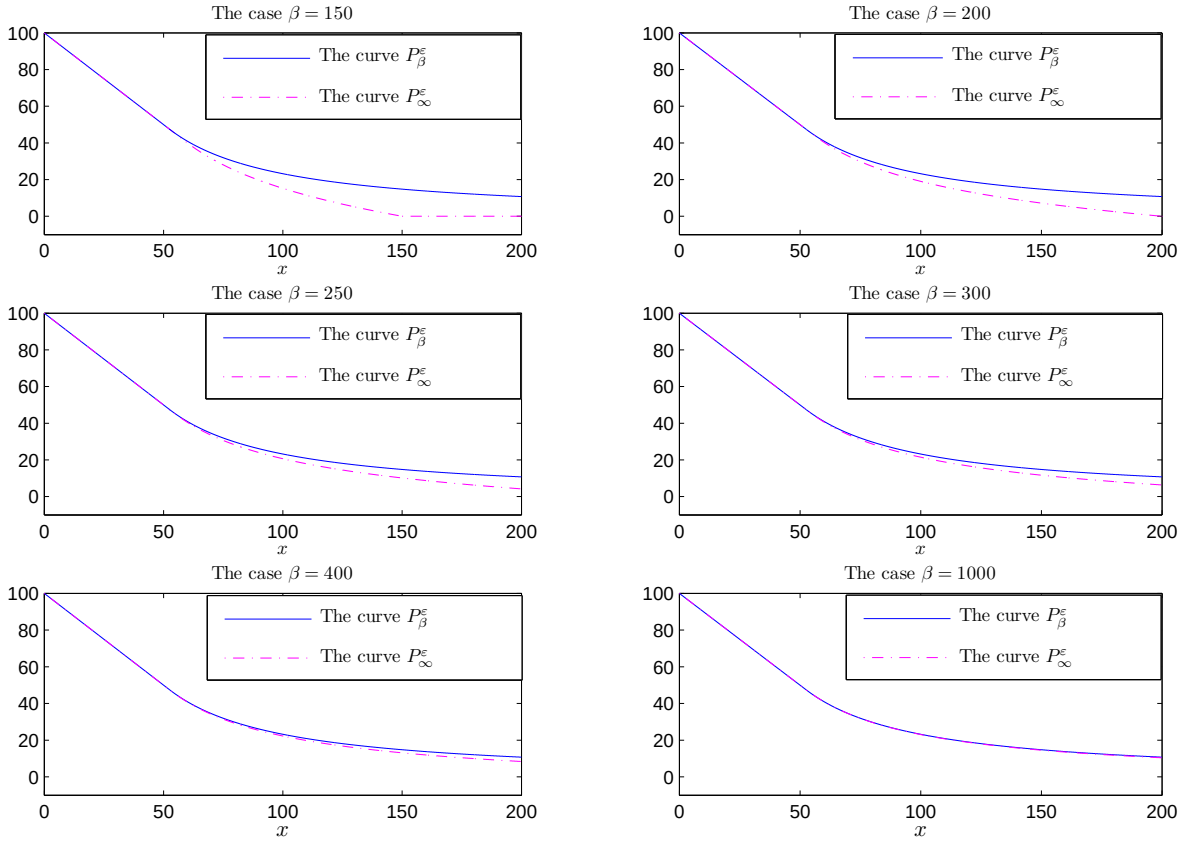

Fig. 2. The curves $x \mapsto P^{\varepsilon}_{\beta}(x)$ and $x \mapsto P^{\varepsilon}_{\infty}(x)$ for $\varepsilon = 0.3$ and $\beta = 150, 200, 250, 300, 400, 1000$.

is given by (7), $T^{\varepsilon,x}_{\beta} = R_{a(\beta)}(X)$ and $T^{\varepsilon,x}_{b} = R_{a(b)}(X)$. It follows that (see [7]):

$$\begin{aligned}\mathbb{E}\left[e^{-rT^{\varepsilon,x}_{\beta}}\mathbf{1}_{\{T^{\varepsilon,x}_{\beta}\leq\tau^{\varepsilon,x}(b)\}}\right] &= \mathbb{E}\left[e^{-r(R_{a(\beta)}(X)\wedge R_{a(b)}(X))}\right.\\ &\times\ \left.1_{\{R_{a(\beta)}(X)\wedge R_{a(b)}(X)=R_{a(\beta)}(X)\}}\right]\\ &= e^{\nu a(\beta)}\frac{\sinh[-\mu a(b)]}{\sinh[\mu(a(\beta)-a(b))]} = \left(\frac{\beta}{x}\right)^{\delta}\frac{b^{1+\delta}-x^{1+\delta}}{b^{1+\delta}-\beta^{1+\delta}},\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[e^{-r\tau^{\varepsilon,x}(b)}\mathbf{1}_{\{\tau^{\varepsilon,x}(b)\leq T^{\varepsilon,x}_{\beta}\}}\right] &= \mathbb{E}\left[e^{-r(R_{a(\beta)}(X)\wedge R_{a(b)}(X))}\right.\\ &\times\ \left.1_{\{R_{a(\beta)}(X)\wedge R_{a(b)}(X)=R_{a(b)}(X)\}}\right]\\ &= e^{\nu a(b)}\frac{\sinh[\mu a(\beta)]}{\sinh[\mu(a(\beta)-a(b))]} = \left(\frac{b}{x}\right)^{\delta}\frac{x^{1+\delta}-\beta^{1+\delta}}{b^{1+\delta}-\beta^{1+\delta}}.\end{aligned}$$

The equation (19) is thus proved.

Appendix D
Proof of Proposition (5)

It is enough to show the equality for $0 \leq x < \beta$. If $0 \leq x < b_{\varepsilon,\beta}$, then $\hat{P}^{\varepsilon}_{\beta}(x) = \max(K-x,0) = K-x$, since $x < b_{\varepsilon,\beta} < K$. Now, suppose $b_{\varepsilon,\beta} \leq x < \beta$, then since $\beta > K$, $\vartheta(x,b)$ given by equation (18) becomes

$$\vartheta(x,b) = \max(K-b,0)\left(\frac{b}{x}\right)^{\delta}\frac{x^{1+\delta}-\beta^{1+\delta}}{b^{1+\delta}-\beta^{1+\delta}}. \quad (48)$$

The maximum of $\vartheta(x,b)$ over b on $[0,x]$, is attained on $[0, K\wedge x]$, since $\vartheta(x,b) = 0$ if $b > K$. For all $b \in]0, K\wedge x]$, we have $\frac{\partial\vartheta}{\partial b}(x,b) = \left(\frac{b}{x}\right)^{\delta}\frac{\beta^{1+\delta}-x^{1+\delta}}{(b^{1+\delta}-\beta^{1+\delta})^2}\times\left[-(1+\delta)\beta^{1+\delta}\right.$ $\left.+\frac{K\delta}{b}\beta^{1+\delta}+Kb^{\delta}\right]$. Moreover, set $y = \frac{\beta}{b}$, then $y > \frac{\beta}{K\wedge x}$. Introducing y in the last expression, we get:

$$\frac{\partial\vartheta}{\partial b}(x,b) = \left(\frac{\beta b}{x}\right)^{\delta}\frac{\beta^{1+\delta}-x^{1+\delta}}{K(b^{1+\delta}-\beta^{1+\delta})^2}\Phi(y),$$

where Φ is given by equation (21). The function Φ is continuous and increasing on $[\frac{\beta}{K},\infty[$ with $\lim_{y\to\infty}\Phi(y) = \infty$, and $\Phi(\frac{\beta}{K}) = \frac{\beta}{K}\left[-1+\left(\frac{\beta}{K}\right)^{-1-\delta}\right] < 0$. Hence, there exists a unique zero $y_{\varepsilon,\beta} = \frac{\beta}{b_{\varepsilon,\beta}} > \frac{\beta}{K}$, such that $\Phi(y_{\varepsilon,\beta}) = 0$. Consequently, we obtain the following cases:

- If $x \geq K > b_{\varepsilon,\beta}$, the maximum of the function $b \mapsto \vartheta(x,b)$ on $[0,x]$ is achieved on $[0,K]$ and the function $\vartheta(x,b)$ admits a unique absolute maximum on $[0,K]$ achieved at the point $b_{\varepsilon,\beta} = \frac{\beta}{y_{\varepsilon,\beta}} \in]0,K[$, where $y_{\varepsilon,\beta}$ is the unique solution of the equation (21).
- If $b_{\varepsilon,\beta} \leq x < K$, then $\frac{\beta}{K} < \frac{\beta}{x} \leq y_{\varepsilon,\beta}$ and the function $b \mapsto \vartheta(x,b)$ attains its absolute maximum on $[0,x]$ at the point $b_{\varepsilon,\beta}$.

Hence, for all $x \in [b_{\varepsilon,\beta},\beta[$, we replace b by $\frac{\beta}{y_{\varepsilon,\beta}}$ in the expression (48) to get $\hat{P}^{\varepsilon}_{\beta}(x) = A_0x + B_0x^{-\delta}$, where

$$A_0 = \frac{\frac{K}{b_{\varepsilon,\beta}}-1}{1-\left(\frac{\beta}{b_{\varepsilon,\beta}}\right)^{1+\delta}} = \left(\frac{Ky_{\varepsilon,\beta}}{\beta}-1\right)\frac{y^{-\delta}_{\varepsilon,\beta}}{y^{-\delta}_{\varepsilon,\beta}-y_{\varepsilon,\beta}},$$

$$B_0 = \frac{(b_{\varepsilon,\beta}-K){b_{\varepsilon,\beta}}^{\delta}\beta^{1+\delta}}{{b_{\varepsilon,\beta}}^{1+\delta}-\beta^{1+\delta}} = \frac{(\frac{\beta}{y_{\varepsilon,\beta}}-K)\beta^{\delta}y_{\varepsilon,\beta}^{-\delta}}{y_{\varepsilon,\beta}^{-1-\delta}-1}.$$

Since $y_{\varepsilon,\beta}^{-\delta} = \frac{\beta(1+\delta)}{K} - \delta y_{\varepsilon,\beta}$, we obtain

$$A_0 = \frac{y_{\varepsilon,\beta}}{\beta}\frac{K\delta}{1+\delta} - 1, B_0 = \frac{K\beta^{\delta}}{1+\delta}\left[\frac{\beta(1+\delta)}{K} - \delta y_{\varepsilon,\beta}\right]. \quad (49)$$

Therefore, we finally get (22).

Appendix E
Proof of proposition (6)

We just have to show that $\hat{P}_{\beta}^{\varepsilon}$ satisfies (23), (24) and (25). The general solution $V \in \mathcal{H}$ of (23) takes the form

$$V(x) = A_1 x + B_1 x^{-\delta}$$

for suitable real constants A_1 and B_1. We show that $A_1 = A_0$ and $B_1 = B_0$, where A_0 and B_0 are given by equation (49). Indeed, the continuity of V at the points $x = b$ and $x = \beta$ and the continuity of V' at the point $x = b$ provides the relations

$$\left\{\begin{array}{lcl} V(\beta) & = & A_1\beta + B_1\beta^{-\delta} = 0 \\ V(b) & = & A_1 b + B_1 b^{-\delta} = K - b \\ bV'(b) & = & A_1 b - \delta B_1 b^{-\delta} = -b \end{array}\right.$$

It follows that

$$A_1\beta + B_1\beta^{-\delta} = 0, B_1 b^{-\delta} = \frac{K}{1+\delta} \text{ and } (A_1+1)b = \frac{K\delta}{1+\delta}. \quad (50)$$

Let $y = \frac{\beta}{b}$, then equation (50) leads to $\Phi(y) = 0$, where Φ is given by (21). Hence, $b = \frac{\beta}{y_{\varepsilon,\beta}} \triangleq b_{\varepsilon,\beta}$, where $y_{\varepsilon,\beta}$ is the unique zero of the function Φ. In conclusion, we obtain the following unknown parameters: $b = \frac{\beta}{y_{\varepsilon,\beta}} = b_{\varepsilon,\beta}, B = \frac{K}{1+\delta}(\frac{\beta}{y_{\varepsilon,\beta}})^{\delta} = B_0$ and $A = -1 + \frac{y_{\varepsilon,\beta}}{\beta}\frac{K\delta}{1+\delta} = A_0$. Thus, $\hat{P}_{\beta}^{\varepsilon}$ fulfills (23). Taking account of (26), (24) is satisfied by $\hat{P}_{\beta}^{\varepsilon}$. Now we propose to show that $\hat{P}_{\beta}^{\varepsilon}$ verifies (25). It is easy to check, taking account of equation (21) that

$$\forall x \in]b_{\varepsilon,\beta}, \beta[, \quad \hat{P}_{\beta}^{\varepsilon}(x) > \frac{K\delta}{1+\delta}(1-y_{\varepsilon,\beta}) + \beta - x.$$

Now, we will show that for ε small enough, $\frac{K\delta}{1+\delta}(1-y_{\varepsilon,\beta}) + \beta - x > K$. Put

$$g_{\beta}(\varepsilon) \triangleq \frac{K\delta}{1+\delta}(1-y_{\varepsilon,\beta}) + \beta - K = \frac{2rK}{\varepsilon^2+2rK}(1-y_{\varepsilon,\beta}) + \beta - K. \quad (51)$$

Using the second order Taylor expansion (35) (Subsection (IV-C) below) of $y_{\varepsilon,\beta}$, we obtain

$$g_{\beta}(\varepsilon) = (\beta-K)\left(1-\frac{1}{K}\right) + (\beta-K)\frac{\varepsilon^2}{2rK^2} + o(\varepsilon^2).$$

This shows that in a neighborhood of $\varepsilon = 0$, $g_{\beta}(\varepsilon) > 0$, since $K > 1$ and $\beta - K > 0$. Therefore, there exists $\epsilon_0 > 0$ such that if $\varepsilon < \epsilon_0$, then $\hat{P}_{\beta}^{\varepsilon}(x) > K - x$ for all $x \in]b_{\varepsilon,\beta}, \beta[$. Hence $\hat{P}_{\beta}^{\varepsilon}$ verifies (25).

The function $\hat{P}_{\beta}^{\varepsilon}$ is clearly in the space $\mathcal{H}$ and is decreasing on $[0,\beta]$ as $\frac{\partial \hat{P}_{\beta}^{\varepsilon}}{\partial x}(x)$

$$= \left\{\begin{array}{ll} -1 & \text{if} \quad 0 < x < b_{\varepsilon,\beta} \\ -B_0\left[\beta^{-1-\delta} + \delta x^{-1-\delta}\right] & \text{if} \quad b_{\varepsilon,\beta} < x < \beta \\ . & \end{array}\right\} < 0$$

Finally, the function $\hat{P}_{\beta}^{\varepsilon}$ is clearly convex since it is continuous non-increasing, null on $[\beta,\infty[$, affine on $[0, b_{\varepsilon,\beta}]$ and convex on $[b_{\varepsilon,\beta}, \beta]$ as $\frac{\partial^2 \hat{P}_{\beta}^{\varepsilon}}{\partial x^2}(x) = \delta(\delta+1)Bx^{-\delta-2} > 0$ for all $x \in]b_{\varepsilon,\beta}, \beta[$. It is now clear that $(b_{\varepsilon,\beta}, \hat{P}_{\beta}^{\varepsilon})$ solves the variational inequality (23)-(27) for ε small enough.

Appendix F
Proof of proposition (7)

By hypothesis $(\mathbf{H_3})$, it comes that v_{∞}^{ε} is convex, non-increasing on $[0,\infty[$ and $v_{\infty}^{\varepsilon} \geq \psi$. On the other hand, we have for all $T > 0$, $v_{\infty}^{\varepsilon}(x) \geq e^{-rT}\mathbb{E}\left[\psi\left(xe^{\left(r-\frac{\varepsilon^2}{2}\right)T+\varepsilon B_T}\right)\right]$. It follows that $v_{\infty}^{\varepsilon} > 0$ for all $x \geq 0$. Furthermore,

$$\forall x \leq b_{\varepsilon,\infty}, v_{\infty}^{\varepsilon}(x) = \psi(x) \text{ and } \forall x > b_{\varepsilon,\infty}, v_{\infty}^{\varepsilon}(x) > \psi(x). \quad (52)$$

Since the stopping time $\gamma_{\infty}^{\varepsilon,x}$ is optimal for the problem $(\mathcal{P}_{\varepsilon,\infty}^{am})$, we conclude by definition of $\gamma_{\infty}^{\varepsilon,x}$ and by equation (52) that

$$\gamma_{\infty}^{\varepsilon,x} = \tau^{\varepsilon,x}(b_{\varepsilon,\infty}). \quad (53)$$

where for $z > 0$, $\tau^{\varepsilon,x}(z)$ is given by equation (16). Set

$$\varphi(z) \triangleq \mathbb{E}\left[e^{-r\tau^{\varepsilon,x}(z)}\psi\left(S_{\tau^{\varepsilon,x}(z)}^{\varepsilon,x}\right)\right].$$

Since the stopping time $\tau^{\varepsilon,x}(b_{\varepsilon,\infty})$ is optimal for the problem $(\mathcal{P}_{\varepsilon,\infty}^{am})$, the function φ reaches its maximum at the point $z = b_{\varepsilon,\infty}$. Let us compute explicitly $v_{\infty}^{\varepsilon}(x) = \sup_{z>0}\varphi(z)$.

- If $z \geq x$, then $\tau^{\varepsilon,x}(z) = 0$ and $\varphi(z) = \psi(x)$.
- If $0 < z < x$, we obtain using the continuity of the trajectories of $S_t^{\varepsilon,x}$, $\tau^{\varepsilon,x}(z) \triangleq \inf\{t \geq 0 / S_t^{\varepsilon,x} = z\}$ and then

$$\varphi(z) = \psi(z)E\left[e^{-r\tau^{\varepsilon,x}(z)}\right].$$

Since $\tau^{\varepsilon,x}(z)$ is the first hitting time by the drifted brownian motion $B_t + \nu t$, (where ν is given by (6)) of the level $a(z)$ given by equation (7), it comes using Laplace transformation (see e.g. [7]), that for all $\lambda \in \mathbb{R}$:

$$\mathbb{E}\left[e^{-\frac{\lambda^2}{2}\tau^{\varepsilon,x}(z)}\right] = e^{\nu a(z)}\exp[-|a(z)|\sqrt{\nu^2+\lambda^2}].$$
$$= e^{\frac{1}{\varepsilon^2}\log\left(\frac{z}{x}\right)\left[r-\frac{\varepsilon^2}{2}+\sqrt{(r-\frac{\varepsilon^2}{2})^2+\lambda^2\varepsilon^2}\right]},$$

since $\log\left(\frac{z}{x}\right) < 0$. For $\lambda = \sqrt{2r}$ and $0 \leq z < x$ we get $\mathbb{E}\left[e^{-r\tau^{\varepsilon,x}(z)}\right] = \left(\frac{x}{z}\right)^{-\delta}$. Therefore, taking account of $(\mathbf{H_3})$, we get

$$\varphi(z) = \left\{\begin{array}{ll} \psi(x) & \text{if } z \geq x \\ \psi(z)\left(\frac{x}{z}\right)^{-\delta} & \text{if } z \in [0,x] \cap [0,k] \\ 0 & \text{if } z \in [0,x] \cap [k,+\infty[\end{array}\right.$$

The maximum of φ is reached on the interval $]0,x] \cap]0,k[$, hence $b_{\varepsilon,\beta} \in]0,x] \cap]0,k[$. For all $z \in]0,x] \cap]0,k[$, we have

$$\varphi'(z) = x^{-\delta}z^{\delta-1}\left[z\psi'(z) + \delta\psi(z)\right].$$

Consequently, $b_{\varepsilon,\infty}\psi'(b_{\varepsilon,\infty}) + \delta\psi(b_{\varepsilon,\infty}) = 0$ and $b_{\varepsilon,\infty} \in]0,x] \cap]0,k[$. We conclude that

- If $x \leq b_{\varepsilon,\infty}$, $\max_z \varphi(z) = \varphi(x) = \psi(x)$.
- If $x > b_{\varepsilon,\infty}$

$\max_z \varphi(z) = \varphi(b_{\varepsilon,\infty}) = \psi(b_{\varepsilon,\infty})\left(\frac{x}{b_{\varepsilon,\infty}}\right)^{-\delta}$.

This implies the demanded formulas.

Appendix G
Proof of lemma (3)

For $\alpha \in \mathbb{R}$, $R_\alpha(Y)$ is defined by (47). For $z > 0$, $a(z)$ is defined by (7) and X_t is given by equation (46). For $b_{\varepsilon,\infty} < x < \beta$, we have $a(b_{\varepsilon,\infty}) < 0 < a(\beta)$ and $T_\beta^{\varepsilon,x} = R_{a(\beta)}(Y)$ and $\gamma_\infty^{\varepsilon,x} = R_{a(b_{\varepsilon,\infty})}(Y)$. It follows that (see [7]):

$$\begin{aligned} f(\beta) &= E\Big[e^{-r(R_{a(\beta)}(X)\wedge R_{a(b_{\varepsilon,\infty})}(X))} \\ &\times\ 1_{\{R_{a(\beta)}(X)\wedge R_{a(b_{\varepsilon,\infty})}(X)=R_{a(\beta)}(X)\}}\Big] \\ &= e^{\nu a(\beta)}\frac{\sinh[-\kappa_0 a(b_{\varepsilon,\infty})]}{\sinh[\frac{\kappa_0}{\varepsilon}\log\left(\frac{\beta}{b_{\varepsilon,\infty}}\right)]} \\ &= 2C\left(\frac{x}{b_{\varepsilon,\infty}}\right)^{\frac{1}{2}}\frac{\left(\frac{\beta}{b_{\varepsilon,\infty}}\right)^{\frac{\kappa_0}{\varepsilon}-\frac{1}{2}}}{\left(\frac{\beta}{b_{\varepsilon,\infty}}\right)^{\frac{2\kappa_0}{\varepsilon}}-1} \end{aligned}$$

where $C \triangleq \sinh(-\kappa_0 A)$.

Appendix H
Proof of lemma (4)

Set

$$\zeta(\beta,b) = \left(\frac{b}{\beta}\right)^{1+\delta} + \delta - \frac{1+\delta}{K}x. \tag{54}$$

For all $\beta > K$, there exists a unique $b_{\varepsilon,\beta} \in]0,K[$ such that

$$\zeta(\beta, b_{\varepsilon,\beta}) = 0. \tag{55}$$

The function $\zeta \in \mathcal{C}^1(]K,\infty[\times]0,K[)$, with

$$\forall(\beta,b) \in]K,\infty[\times]0,K[,\quad \frac{\partial\zeta}{\partial b}(\beta,b) = \frac{1+\delta}{\beta}\left(\frac{b}{\beta}\right)^{\delta} \neq 0.$$

Let us fix $\varepsilon > 0$ and denote $b_\varepsilon(\beta) \triangleq b_{\varepsilon,\beta}$. The implicit function theorem shows then that $\beta \mapsto b_\varepsilon(\beta)$ is $\mathcal{C}^1(]K,+\infty[)$ and for all $\beta \in]K,\infty[$, $b'_\varepsilon(\beta) = -\frac{\frac{\partial\zeta}{\partial\beta}(\beta,b_\varepsilon(\beta))}{\frac{\partial\zeta}{\partial b}(\beta,b_\varepsilon(\beta))} = \frac{\beta}{b_\varepsilon(\beta)} > 0$. Hence the function $\beta \mapsto b_\varepsilon(\beta)$ is increasing on $]K,\infty[$ and bounded above by K. Consequently $b_\varepsilon(\beta)$ has a limit as $\beta \to \infty$. Let $b_{\varepsilon,\infty} \triangleq \lim_{\beta\to\infty} b_{\varepsilon,\beta}$. Passing to the limit as β goes to infinity in (55), we get $\zeta(\beta, b_{\varepsilon,\infty}) = \delta - \frac{1+\delta}{K}b_{\varepsilon,\beta} = 0$, which implies (40).

Appendix I
Proof of lemma (5)

Since the function $\beta \mapsto b_{\varepsilon,\beta}$ is increasing to $b_{\varepsilon,\infty}$ as β goes to ∞, it comes that for almost all $\omega \in \Omega$ we have,

- If $x \geq b_{\varepsilon,\infty}$, then the map $\beta \mapsto \gamma_\beta^{\varepsilon,x}(\omega)$ is decreasing and above $\gamma_\infty^{\varepsilon,x}(\omega)$.
- If $x < b_{\varepsilon,\infty}$, then for β large enough, we have $x < b_{\varepsilon,\beta}$ and the map $\beta \mapsto \gamma_\beta^{\varepsilon,x}(\omega)$ is increasing and below $\gamma_\infty^{\varepsilon,x}(\omega)$.

Therefore, $\gamma_\beta^{\varepsilon,x}$ converges a.s. as β goes to ∞. On the other hand, using Laplace transformation (54), expressions (53) of $\gamma_\infty^{\varepsilon,x}$ and (33) of $\gamma_\beta^{\varepsilon,x}$, we get

$$\begin{aligned} \mathbb{E}\left[e^{-\frac{\lambda^2}{2}\gamma_\infty^{\varepsilon,x}}\right] &= e^{\nu a(b_{\varepsilon,\infty})-|a(b_{\varepsilon,\infty})|\sqrt{\nu^2+\lambda^2}}, \\ \mathbb{E}\left[e^{-\frac{\lambda^2}{2}\gamma_\beta^{\varepsilon,x}}\right] &= e^{\nu a(b_{\varepsilon,\beta})-|a(b_{\varepsilon,\beta})|\sqrt{\nu^2+\lambda^2}}, \end{aligned}$$

where ν is given by equation (6) and $a(z)$ is defined by equation (7).

This shows the convergence of the Laplace transformation of $\gamma_\beta^{\varepsilon,x}$ to $\gamma_\infty^{\varepsilon,x}$ and therefore, the convergence a.s. of $\gamma_\beta^{\varepsilon,x}$ to $\gamma_\infty^{\varepsilon,x}$, since $\gamma_\beta^{\varepsilon,x}$ converges a.s.

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