Optimal Fee to Recover the Setup Cost for Resource Extraction

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***Abstract*—Setup cost for extracting a publically owned nonrenewable resource may be recovered by charging a fee for each unit of the resource extracted. This paper derives the exact analytical expression of the optimal recovery fee.**

***Index Terms*—Nonrenewable resource, optimal control, optimal recovery fee, setup cost**

I. INTRODUCTION

ECONOMIC literature on the setup cost in the context of nonrenewable resources is scant and limited. For examples, Kemp and Long (1986) say that setup cost may or may not depend on the proposed path of extraction, and Hartwick et al. (1986) mention that the setup cost can be payable recurrently instead of in one lump sum, but both papers are short of addressing how the setup cost can be optimally recovered. In this paper, we intend to fill the void by analytically deriving an optimal unit extraction fee to recover the setup cost within a reasonable framework.

There are several papers that are related to the above topic in a broad domain. The Herfindahl (1967) finds that for the optimal order of mineral extraction, the least cost site will be exploited first if the quality difference can be completely incorporated into unit extraction costs. Lewis (1982) shows that in a partial equilibrium context, the Herfindahl principle even can be extended to the case where quality difference cannot be completely incorporated into unit extraction costs.

Lewis's remark was challenged by Chakravorty and Krulce (1994). These two authors formulate a model in partial equilibrium context, specifically for oil and coal, which have quality difference that cannot be characterized independently of demand. Coal and oil have no quality difference when used to produce electricity but there is a significant quality difference if used for transportation. They show that there is a certain period when both resources are extracted simultaneously. During this phase, higher cost coal is extracted to generate electricity even though cheaper oil is available because oil has a comparative advantage in transportation. Hence, it is optimal to conserve oil for future transportation demand and generate electricity with coal, although coal is more expensive than oil.

Im, Charkravorty and Roumasset (2006) show that discontinuous extraction of a non-renewable resource is still possible even without setup costs. The model is formulated in a partial equilibrium context. Assuming different production costs for different resources and multiple demands, they find that there are periods in which the resources are simultaneously extracted. This is not a challenge to the Herfindahl principle, as the model assumes that quality difference cannot always be incorporated into unit extraction costs. When the quality difference can be characterized by unit extraction costs, the model confirms that the Herfindahl principle holds but leaves the setup cost problem unaddressed.

Chakravorty, Krulce, and Roumasset (2006) address a different issue arising in nonrenewable resources. They develop a model with multiple demands and find that taxes on a resource or on any economy's sector may have effects that are quite different than in the classic models. Their results imply that Ricardian absolute advantage results in dynamic specialization while Ricardian comparative advantage leads to intersectoral specialization.

Im and Vu (2010) theoretically reconsider the validity of Herfindahl rule, the least-cost-first extraction, in the general equilibrium framework of neoclassical growth theory. They find that Herfindal rule is valid if and only if population growth rate is nonnegative.

In sum, most papers do not discuss the optimal recovery of setup costs, and none of the papers that discussed this issue derive an exact analytical expression of the optimal recovery fee. Our paper makes an attempt to fill this gap. Our theoretical model is derived using optimal control theory discussed in Chiang (1992), Akulenko (1994), and Cannarsa and Sinestrari (2004).

II. THE MODEL

In this paper we assumes a publically owned nonrenewable resource stock (Q_0), say an oil field or a landfill, is planned to be exhausted over a given time period $[0,T]$. Further assumed is that there is a setup cost incurred (of $t=0$), and thereafter the extraction cost per unit of resource (c) is constant over the entire planned extraction

Manuscript received July 10, 2010; revised December 26, 2010.
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period. Under these assumptions, the optimal control problem would be to maximize the present value of a steam of social benefits from consuming the resource extracted over the time period:

$$\underset{q_t}{Max}\ W = \int_0^T e^{rt}\left(U(q_t) - cq_t\right)dt - F \tag{1}$$

subject to

$$\dot{Q}(t) \equiv dQ(t)/dt = -q_t; \tag{1.1}$$

$$Q(0) = Q_0; \quad Q(T) = 0; \tag{1.2}$$

$$q_t \geq 0 \quad \forall t \in (0,T);$$

$$[\ F, Q_0, r, T \in (0,\infty) \text{ are all given}], \tag{1.3}$$

where r is social time preference rate, q_t the resource extraction rate at time t, $U(q_t)$ the social utility function characterized by the usual assumptions in economic theory:

$$\partial U(q_t)/\partial q_t > 0;\ \partial^2 U(q_t)/\partial {q_t}^2 < 0;$$

$$\lim_{q_t \to 0} \partial U(q_t)/\partial q_t = \infty$$

$$\text{for all } t \in [0,T]; \tag{2}$$

and $Q(t)$, the resource stock at time t, is

$$Q(t) = Q_0 - \int_0^t q_t dt \qquad t \in [0,T]. \tag{3}$$

Defining a constant σ such that

$$F = \int_0^T e^{-rt}\sigma dt$$

or equivalently as

$$\sigma = rF/(e^{-rT} - 1), \tag{4}$$

the objective functional W in (1) can be re-expressed as

$$W = \int_0^T e^{-rt}\left(U(q_t) - cq_t - \sigma\right)dt. \tag{5}$$

Hence, the current-value Hamiltonian for the optimal control problem (1) will be

$$\begin{aligned} H_c &= \left(U(q_t) - cq_t - \sigma\right) + \lambda(t)\dot{Q} \\ &= U(q_t) - (c + \lambda(t))q_t - \sigma. \end{aligned} \tag{6}$$

The maximum principle conditions are:

$$\underset{\tau,\, q_t}{Max}\ H_c, \text{ subject to} \tag{7.1}$$

$$\dot{Q}(t) = -q_t; \tag{7.2}$$

$$Q(0) = Q_0; \quad Q(T) = 0; \tag{7.3}$$

$$q_t \geq 0 \quad \forall t \in [0,T];$$

$$[\ F, Q_0, r, T \in (0,\infty) \text{ are all given }]. \tag{7.4}$$

F in (1) is given, hence so is σ in H_c Therefore, F has no bearing on the optimal paths of extraction rate q_t and the resource stock $Q(t)$.

III. OPTIMAL RECOVERY OF SETUP COSTS

Instead of externally funding the setup as in Section 1, suppose that the setup cost F is recovered by charging a fixed fee, τ, for each unit of resource extracted ($\tau>0$). Then, the setup cost must be equal to the definite integral of the present value of the setup cost recovered at t over the entire extraction period $[0,T]$:

$$F = \int_0^T e^{-rt}(\tau q_t)dt = \tau\int_0^T e^{-rt} q_t dt, \tag{8}$$

where τq_t measures the setup cost recovered at $t \in [0,T]$ and $e^{-rt}\tau q_t$ is its present value. With the equality integral constraint added, the optimal control problem (1) for the social planner is now revised to:

$$\underset{\tau,\, q_t}{Max}\ W = \int_0^T e^{-rt}\left(U(q_t) - cq_t - \sigma\right)dt \tag{9}$$

subject to

$$F = \tau\int_0^T e^{-rt} q_t dt; \tag{9.1}$$

$$\dot{Q}(t) = -q_t; \tag{9.2}$$

$$Q(0) = Q_0; \quad Q(T) = 0; \tag{9.3}$$

$$\tau > 0,\ q_t \geq 0 \quad \forall t \in [0,T];$$

$$(\ F, Q_0, r, T \in (0,\infty) \text{ are all given}), \tag{9.4}$$

where σ is as defined in (4).

Since the optimal control problem (9) is a standard isoperimetric optimization problem, we define an artificial state variable as usual (e.g., Chiang, p. 280):

$$\Gamma(t) = -\tau J(t), \tag{10}$$

where $J(t)$ is defined for notational economy as

$$J(t) = \int_0^t e^{-rt} q_t dt \quad \left(\Rightarrow \dot{J}(t) = e^{-rt} q_t\right). \tag{11}$$

Then, from (10) derive

$$\Gamma(0) = -\tau J(0) = 0; \quad \Gamma(T) = -F\,; \tag{12}$$

Also,

$$\dot{\Gamma}(t) = -\tau \dot{J}(t) = -\tau e^{-rt} q_t\,. \tag{13}$$

Hence, the optimal control problem (9) can be restated as:

$$\underset{\tau,\, q_t}{Max}\ W = \int_0^T e^{-rt} \left(U(q_t) - c\, q_t - \sigma\right) dt \tag{14}$$

subject to

$$\dot{Q}(t) = -q_t\,; \tag{14.1}$$

$$\dot{\Gamma}(t) = -\tau \dot{J}(t) = -e^{-rt} \tau q_t\ ; \tag{14.2}$$

$$Q(0) = Q_0; \quad Q(T) = 0\,; \tag{14.3}$$

$$\Gamma(0) = 0; \quad \Gamma(T) = -F\ ; \tag{14.4}$$

$$\tau > 0,\ q_t \geq 0 \quad \forall\, t \in [0,T];$$

$$[\ F, Q_0, r, T \in (0,\infty) \text{ are all given}]. \tag{14.5}$$

The Hamiltonian for (14) will be

$$H_c = \left(U(q_t) - cq_t + \sigma\right) + \lambda(t)\dot{Q}(t) + \mu(t)\dot{\Gamma}(t)$$

$$= \left(U(q_t) - cq_t - \sigma\right) + \lambda(t)(-q_t) + \mu(t)(-\tau q_t)$$

$$= U(q_t) - \left(c + \mu(t)\tau + \lambda(t)\right) q_t - \sigma$$

(15)

The maximum principle conditions for the optimal control problem (14) follow as:

$$\underset{\tau, q_t}{Max}\ H_c \quad \text{for all } t \in [0,T] \tag{16.1}$$

subject to

$$\dot{Q}(t) = \partial H_c / \partial \lambda = -q\,; \tag{16.2}$$

$$\dot{\lambda}(t) = -(\partial H_c / \partial Q) + r\lambda(t) = r\lambda(t)$$

$$\Rightarrow \quad \lambda(t) = \lambda(0)e^{rt}\,; \tag{16.3}$$

$$\dot{\Gamma}(t) = \partial H_c / \partial \mu = -\tau q_t\,; \tag{16.4}$$

$$\dot{\mu}(t) = -(\partial H_c / \partial \Gamma) + r\mu(t) = r\mu(t)$$

$$\Rightarrow \quad \mu(t) = \mu(0)e^{rt}\,; \tag{16.5}$$

$$\tau > 0;\ q_t \geq 0;\ Q(t) \geq 0 \quad \forall t \in [0,T] \tag{16.6}$$

$$Q(0) = Q_0;\ Q(T) = 0\,; \tag{16.7}$$

$$\Gamma(0) = 0; \quad \Gamma(T) = -F\,. \tag{16.8}$$

The necessary conditions (16.1) to (16.8) are also sufficient for the global maximization of W in light of Mangasarian theorem on the ground that $U(q_t)$ is differentiable and strictly concave in q_t as assumed in (2), and both $\dot{Q}(t)$, and $\dot{\Gamma}(t)$ in (16.2) and (16.4) are linear in q_t.

IV. DERIVATION OF OPTIMAL RECOVERY FEE

We can derive the optimal τ from the first-order conditions for maximizing H_c with respect to the control variables in (161.). However, the transversality condition $\Gamma(T) = -F$ in (16.8) can be alternatively expressed as F $F = -\tau J(T)$ in view of (10), which is an obvious constraint on τ. Therefore, we can incorporate the transversality condition into the Hamiltonian in (15), forming the Lagrangean Hamiltonian for optimization:

$$L(\theta, \tau, q_t) = U(q_t) - \left(c + \mu(t)\tau + \lambda(t)\right) q_t - \sigma$$

$$+\theta\left(F - \tau J(T)\right), \tag{17}$$

where θ denotes Lagrangian multiplier. The first order conditions for maximizing L follow:

$$\partial L / \partial \theta = 0\,;$$

(17.1)

$$\partial L / \partial \tau = 0\,; \tag{17.2}$$

$$\partial L / \partial q_t \leq 0; \quad q_t \geq 0; \quad \partial L / \partial q_t\, q_t = 0$$

(nonnegativity conditions for q_t). (17.3)

However, $\partial L / \partial q_t \leq 0$ in (17.3) cannot hold for $q_t = 0$ at any $t \in [0,T]$ in view of $\lim_{q_t \to 0} \partial U(q_t) / \partial q_t = \infty$ assumed in (2):

$$\lim_{q_t \to 0} \partial L / \partial q_t = \lim_{q_t \to 0} \partial U(q_t) / \partial q_t - (c + \mu(t)\tau$$

$$+\lambda(t)) - \theta\tau T e^{-rT} = \infty \leq 0\,, \tag{18}$$

which is impossible. Therefore, on the optimal path $q_t > 0$ at each $t \in [0,T]$, which implies $\partial L / \partial q_t = 0$ in light of the complementary slackness condition in (17.1). As a result, the first order conditions (17.1) to (17.3) are all equalities:

$$\partial L / \partial \theta = F - \tau J(T) = 0\,; \tag{19.1}$$

$$\partial L / \partial \tau = -\mu(t) q_t - \theta J(T) = 0\,; \tag{19.2}$$

$$\partial L/\partial q_t = \left[\partial U(q_t)/\partial q_t - (c+\mu(t)\tau+\lambda(t))\right]$$
$$-\theta\tau(\partial J(T)/\partial q_t) = 0, \quad (19.3)$$

in which

$$\partial J(T)/\partial q_t = \partial\left(\int_0^T e^{-rt} q_t dt\right)/\partial q_t = \int_0^T e^{-rt} dt$$
$$= (1-e^{-rT})/r,$$

(20)

where $J(T)$ is $J(t)$, defined in (11), for $t = T$.

Substituting $J(T)$ from (19.1) into (19.2), then substituting θ from (19.2) into (19.3), then substituting (20) and $\mu(t) = \mu(0)e^{rt}$ from (16.5) into (19.3), and then dividing through (19.3) by e^{rt}, we obtain a quadratic function of τ as

$$\left[\mu(0)(1-e^{-rT})q_t\right]\tau^2 - \left(\mu(0)rF\right)\tau$$
$$+\left[e^{-rt} rF\left(\partial U(q_t)/\partial q_t - c - \lambda(t)\right)\right] = 0,$$

(21)

where $\mu(0) > 0$ (See Appendix C for proof).

Taking the definite integral of (21) over $[0,T]$ yields

$$\int_0^T \left[\mu(0)(1-e^{-rT})q_t\tau^2\right]dt - \int_0^T \left(\mu(0)rF\tau\right)dt$$
$$+\int_0^T e^{-rt}\left[rF\left(\partial U(q_t)/\partial q_t - c - \lambda(t)\right)\right]dt = 0.$$

(22)

Moving all the factors independent of time variable t outside of the integrals gives

$$\mu(0)(1-e^{-rT})\left(\int_0^T q_t dt\right)\tau^2 - \mu(0)rF\left(\int_0^T dt\right)\tau$$
$$+ rF\left[\int_0^T e^{-rt}\left(\partial U(q_t)/\partial q_t - c - \lambda(t)\right)dt\right] = 0. \quad (23)$$

Since $\int_0^T q_t\, dt = Q_0 - Q(T) = Q_0$ in view of (3) and (9.3), and $\int_0^T dt = T$, (23) can be re-expressed as

$$\left[\mu(0)(1-e^{-rT})Q_0\right]\tau^2 - \left(\mu(0)rFT\right)\tau$$
$$+ rF\int_0^T e^{-rt}\left(\partial U(q_t)/\partial q_t - c - \lambda(t)\right)dt = 0,$$

(24)

or simply

$$\alpha\tau^2 + \beta\tau + \gamma = 0, \quad (25)$$

where

$$\alpha \equiv \mu(0)(1-e^{-rT})Q_0; \quad \beta \equiv -\mu(0)rFT;$$
$$\gamma \equiv rF\int_0^T e^{-rt}\left(\partial U(q_t)/\partial q_t - c - \lambda(t)\right)dt. \quad (26)$$

Hence, the solution of (24) for τ denoted by τ^* is obtained as

$$\tau^* = \left(-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}\right)/(2\alpha). \quad (27)$$

However, the bordered Hessian for the Lagrangian Hamiltonian in (17) satisfies the condition for a unique global maximum as shown in Appendix A, therefore τ^* in (17) should have one solution, which requires $\beta^2 - 4\alpha\gamma = 0$ in (27) so that

$$\tau^* = -\beta/2\alpha = rFT/\left[2Q_0(1-e^{-rT})\right]. \quad (28)$$

The optimal extraction fee τ^* in (28) explicitly shows the determinants of the optimal unit fee and their respective effects on τ^* are: $\partial\tau^*/\partial F > 0$, $\partial\tau^*/\partial Q_o < 0$, $\partial\tau^*/\partial r > 0$, and $\partial\tau^*/\partial T > 0$ as shown in Appendix B.

V. INTERNALIZED SETUP COSTS

Replacing τ in (8) with τ^*, we obtain:

$$F = \tau^* \int_0^T e^{-rt} q_t\, dt.$$

(29) (29)

Substituting F from (29) into the initial objective functional (1), we find that it is reduced to a simple optimal control problem with one control variable q_t and one state variable $Q(t)$:

$$\underset{q_t}{Max}\, W^* = \int_0^T e^{-rt}\left[U(q_t) - (c+\tau^*)q_t\right]dt \quad (30)$$

subject to

$$\dot{Q}(t) = -q_t;$$

(30.1)

$$Q(0) = Q_0, \quad Q(T) = 0 \quad (30.2)$$

$$q_t > 0 \quad \text{for every } t \in [0,T] \quad (30.3)$$

[$Q_0, r, T \in (0,\infty)$ are given] .

Then, the Hamiltonian becomes

$H_c^* = U(q_t) - (c + \tau^* + \lambda(t))q_t$,

and the maximum principle conditions are:

$$\underset{q_t}{Max}\ H_c^* = U(q_t) - \left(c+\tau^*+\lambda(t)\right) q_t$$

$$\text{for all } t \in [0,T] \qquad (31.1)$$

subject to

$$\dot{Q} = \partial H_c / \partial \lambda = -q\,; \qquad (31.2)$$

$$\dot{\lambda}(t) = -\partial H_c / \partial Q + r\lambda(t) = r\lambda(t)\,; \qquad (31.3)$$

$Q(0) = Q_0$; $Q(T) = 0$;

$q_t \geq 0 \quad \forall t \in [0,T]$;

[$F, Q_0, r, T \in (0,\infty)$ are all given].

In the optimal control problem (30), the setup cost has been absorbed or internalized as an additional unit extraction cost.

VI. Conclusion

We have derived the optimal unit extraction fee to recover the setup costs for nonrenewable resource extraction. The analytical results show that the optimal extraction fee is completely free of the social utility function, solely depending on given parameters, which may be counterintuitive. The analytical procedure used in this paper may be useful in determining optimal recovery fees such as optimal toll at tollgates to recover the construction cost of a highway and a fee for each unit of waste dumped into a landfill to recover its setup cost.

Appendixes

Appendix A

The second partial derivatives of $L = L(\theta, \tau, q_t)$ in (17) are:

$$\partial^2 L / \partial \theta^2 = 0\,; \qquad (A.1)$$

$$\partial^2 L / \partial \tau^2 = 0\,; \qquad (A.2)$$

$$\partial^2 L / \partial q_t^2 = \partial^2 U(q_t) / \partial q_t^2\,; \qquad (A.3)$$

$$\partial^2 L / \partial \theta \partial \tau = \partial^2 L / \partial \tau \partial \theta = -J(T)\,; \qquad (A.4)$$

$$\partial^2 L / \partial \theta \partial q_t = \partial^2 L / \partial q_t \partial \theta = -\tau(1 - e^{-rt}) / r\,; \qquad (A.5)$$

$$\partial^2 L / \partial \tau \partial q_t = \partial^2 L / \partial q_t \partial \tau$$

$$= \mu(t)\{[(1 - e^{-rT}) / rJ(T)]\} q_t - 1\,. \qquad (A.6)$$

Therefore, the bordered Hessian for L is positive:

$$\left|\bar{\mathrm{H}}\right| = \begin{vmatrix} \partial^2 L / \partial \theta^2 & \partial^2 L / \partial \theta \partial \tau & \partial^2 L / \partial \theta \partial q_t \\ \partial^2 L / \partial \tau \partial \theta & \partial^2 L / \partial \tau^2 & \partial^2 L / \partial \tau \partial q_t \\ \partial^2 L / \partial q_t \partial \theta & \partial^2 L / \partial q_t \partial \tau & \partial^2 L / \partial q_t^{\,2} \end{vmatrix}$$

$$= -\left(J(T)\right)^2 (\partial^2 U / \partial q_t^2) > 0 \quad \forall\, (\theta, \tau, q_t)\,, \qquad (A.7)$$

note that $\partial^2 U / \partial q_t^2 < 0$ as assumed in (2) .

$\left|\bar{\mathrm{H}}\right| > 0\ \forall\,(\theta, \tau, q_t)$ in (A.7) is the necessary and sufficient condition for H_c subject to $F = -\tau J(t)$ to have the global maximum (See Chiang (1984, p. 385).

Appendix B

From (28)

$$\tau^* = rFT / \left[\, 2Q_0 (1 - e^{-rT})\right].$$

Differentiating τ^* partially with respect to each of the parameters ($F, Q_O, r, T > 0$ as assumed) yields

$$\partial \tau^* / \partial F = rT / \left[2Q_0 (1 - e^{-rT})\right] > 0\,; \qquad (B.1)$$

$$\partial \tau^* / \partial Q_0 = -rFT / \left[2Q_0^{\,2} (1 - e^{-rT})\right] < 0\,; \qquad (B.2)$$

$$\partial \tau^* / \partial r = (FT / 2Q_0)\left[(1 - e^{-rT} - rTe^{-rT}) / \left(1 - e^{-rT}\right)^2\right]$$

$$\equiv y\left[FT / (2Q_0)\right] > 0 \quad (\because\ y > 0)\,; \qquad (B.3)$$

$$\partial \tau^* / \partial T = y\left[rF / (2Q_0)\right] > 0\,.$$

(B.4)

Proof for

$$y = (1 - e^{-rT} - rTe^{-rT}) / (1 - e^{-rT})^2 \equiv a / b > 0:$$

Let $x = rT$. Then, $x > 0$ $(\because r,\ T > 0)$.

Since $a = 1 - e^{-x} - xe^{-x} = 0$ for $x = 0$,

and $da / dx = xe^{-x} > 0$ for all $x > 0$,

$a > 0$ for all $x > 0$.

Also $b = (1 - e^{-x})^2 > 0$ for all $x > 0$.

Therefore, $y > 0$.

Appendix C

To definitize $\mu(0)$, substitute α, β, γ defined in (26) into $\beta^2 - 4\alpha\gamma = 0$:

$$\left(\mu(0)rFT\right)^2 - 4\left[\mu(0)(1-e^{-rT})Q_O\right]$$
$$\cdot\left\{rF\int_0^T e^{-rT}\left[\partial U(q_t)/\partial q_t - c - \lambda(t)\right]dt\right\} = 0\,. \quad \text{(C.1)}$$

From (C.1) follows the nontrivial solution for $\mu(0)$ as

$$\mu(0) = \left[4Q_0(1-e^{-rT})/rFT^2\right]$$
$$\cdot\int_0^T e^{-rT}\left(\partial U(q_t)/\partial q_t - c - \lambda(t)\right)dt\,. \quad \text{(C.2)}$$

In addition, (31.1) implies that for q_t maximizing H_c^*

$$\partial U(q_t)/\partial q_t - \left(c+\tau^*+\lambda(t)\right)q_t = 0\;. \quad \text{(C.3)}$$

Therefore, using (28), (32), and (C.3), we can express $\mu(0)$ in (C.2) as

$$\mu(0) = (2/T)\left[2Q_0(1-e^{-rT})/rFT\right]$$
$$\cdot\int_0^T e^{-rt}\left(\partial U(q_t)/\partial q_t - c - \lambda(t)\right)dt$$
$$= (2/T)(1/\tau^*)\int_0^T e^{-rt}\left(\partial U(q_t)/\partial q_t - c - \lambda(t)\right)dt$$
$$= 2/(T\tau^*)$$
$$\cdot\int_0^T e^{-rt}\left\{\left[\partial U(q_t)/\partial q_t - \left(c+\tau^*+\lambda(t)\right)\right] + \tau^*\right\}dt$$
$$= 2/(T\tau^*)\int_0^T e^{-rt}\tau^*\,dt$$
$$= \tau^*\left[2/(T\tau^*)\right]\int_0^T e^{-rt}dt$$
$$= (2/T)\int_0^T e^{-rt}dt$$
$$= 2(1-e^{-rT})/(rT) > 0 \quad (\because r, T > 0)\,. \quad \text{(C.4)}$$

REFERENCES

[1] Akulenko, L.D., *Problems and Methods of Optimal Control,* Kluwer Academic Publishers, The Netherlands, 1994.

[2] Cannarsa, P. and C. Sinestrari, *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control,* Birkhauser, Boston, 2004.

[3] Chakravorty, U. and Krulce, D. L., "Heterogeneous Demand and Order of Resource Extraction." *Econometrica,* 62(6), pp. 1445-52, November 1994

[4] Chakravorty, U., Krulce L. D., and J. Roumasset, "Specialization and Nonrenewable Resources: Ricardo meets Ricardo," *Journal of Economic Dynamic and Control*, vol. 29, No. 9, pp. 1517-1545, January 2006.

[5] Chiang, A.C., *Fundamental Methods of Mathematical Economics*, 3rd Edition, McGraw-Hill, Inc., New York, 1984.

[6] Chiang, A.C., *Elements of Dynamic Optimization.* McGraw-Hill, Inc., New York, 1992.

[7] Hartwick, J., Kemp, M., and N. Long, "Set-up Costs and Theory of Exhaustible Resources," *Journal of Environmental Economics and Management*, Vol. 13, pp. 212-224, 1986.

[8] Herfindahl, Orris C., "Depletion and Economic Theory." In Mason Gaffiney, ed., *Extractive Resources and Taxation*. WI: University of Wisconsin Press, Madison: pp. 68-90, 1967.

[9] Im, I.E, Chakravorty U., and J. Roumasset,"Discontinuous Extraction of a Nonrenewable Resource," *Economics Letters,* Vol 90, pp. 6-11, 2006.

[10] Im, I. E. and T.B. Vu, "Herfindahl Rule Revisited in the Framework of Neoclassical Growth Theory," unpublished, pp. 1-11, 2010.

[11] Kemp. M.C., and N. Long, "The Optimal Order of Exploitation." In Kemp (ed.) *Essays in the Economics of Exhaustible Resources*. North-Holland Press, Holland: pp. 39-74, 1984.

[12] Lewis, T.R., 1982, "Sufficient Conditions for Extracting Least Cost Resource First," *Econometrica*, Vol. 50, pp. 1081-1083, 1982.

[13] Mangasarian, O.L., 1966, "Sufficient Conditions for the Optimal Control of Nonlinear Systems." *SIAM Journal of Control*, Vol. 4, pp. 139-152, 1966.