Proof of Constructive Version of the Fan-Glicksberg Fixed Point Theorem Directly by Sperner's Lemma and Approximate Nash Equilibrium with Continuous Strategies: A Constructive Analysis

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***Abstract*—It is often demonstrated that Brouwer's fixed point theorem can not be constructively proved. Therefore, Kakutani's fixed point theorem, the Fan-Glicksberg fixed point theorem and the existence of a pure strategy Nash equilibrium in a strategic game with continuous (infinite) strategies and quasi-concave payoff functions also can not be constructively proved. On the other hand, however, Sperner's lemma which is used to prove Brouwer's fixed point theorem can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's theorem using Sperner's lemma. Thus, Brouwer's fixed point theorem can be constructively proved in its constructive version. It seems that constructive versions of Kakutani's fixed point theorem and the Fan-Glicksberg fixed point theorem can be constructively proved using that of Brouwer's theorem. Then, can we prove a constructive version of the Fan-Glicksberg fixed point theorem directly by Sperner's lemma? We present such a proof, and we will show the existence of an approximate pure strategy Nash equilibrium in a strategic game with continuous strategies and quasi-concave payoff functions. We follow the Bishop style constructive mathematics.**

***Index Terms*—constructive version of the Fan-Glicksberg fixed point theorem, approximate pure strategy Nash equilibrium, continuous strategy, quasi-concave payoff function, Sperner's lemma.**

I. INTRODUCTION

IT is often demonstrated that Brouwer's fixed point theorem can not be constructively proved. Therefore, Kakutani's fixed point theorem, the Fan-Glicksberg fixed point theorem and the existence of a pure strategy Nash equilibrium in a strategic game with continuous (infinite) strategies and quasi-concave payoff functions also can not be constructively proved. On the other hand, however, Sperner's lemma which is used to prove Brouwer's fixed point theorem can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's theorem using Sperner's lemma. Thus, Brouwer's fixed point theorem can be constructively proved in its constructive version. See [4] and [9]. It seems that constructive versions of Kakutani's fixed point theorem and the Fan-Glicksberg fixed point theorem can be constructively proved using that of Brouwer's theorem.

Then, can we prove a constructive version of the Fan-Glicksberg fixed point theorem directly by Sperner's lemma?

We present such a proof, and we will show the existence of an approximate pure strategy Nash equilibrium in a strategic game with continuous strategies and quasi-concave payoff functions.

Let $(p_i)_{i\in I}$ be a family of semi-norms on a locally convex space where I is an index set, for example, the set of positive integers, and F be a finitely enumerable subset of I. A constructive version of the Fan-Glicksberg fixed point theorem states that for any compact and convex valued multi-function (multi-valued function or correspondence) with closed graph from a compact and convex set in a locally convex space to the set of its inhabited (nonempty) subsets, there exists an approximate fixed point. We consider a uniform version of the property of closed graph for multi-functions, and call such a multi-function *a multi-function with uniformly closed graph*, or say that a multi-function uniformly has a closed graph. An approximate fixed point x of a multi-function Φ for each $\varepsilon > 0$ is a point such that $\sum_{i\in F} p_i(x-\Phi(x)) < \varepsilon$ is satisfied for each $F \subset I$, where $p_i(x-\Phi(x)) = \inf_{y\in\Phi(x)} p_i(x-y)$. An approximate pure strategy Nash equilibrium is a state where strategies chosen by all players are best responses each other within the range of ε. We call such strategies *approximate best responses*.

In the next section we prove Sperner's lemma. Our proof is a standard constructive proof. In Section 3 we prove a constructive version of the Fan-Glicksberg fixed point theorem by Sperner's lemma. In Section 4 we will show the existence of an approximate pure strategy Nash equilibrium in a strategic game with continuous strategies and quasi-concave payoff functions. We follow the Bishop style constructive mathematics according to [1], [2] and [3].

II. SPERNER'S LEMMA

To prove Sperner's lemma we use the following simple result in graph theory, Handshaking lemma[1]. A *graph* refers to a collection of vertices and a collection of edges that connect pairs of vertices. Each graph may be undirected or directed. Fig. 1 is an example of an undirected graph. The degree of a vertex of a graph is defined to be the number of

[1]For another constructive proof of Sperner's lemma, see [7].

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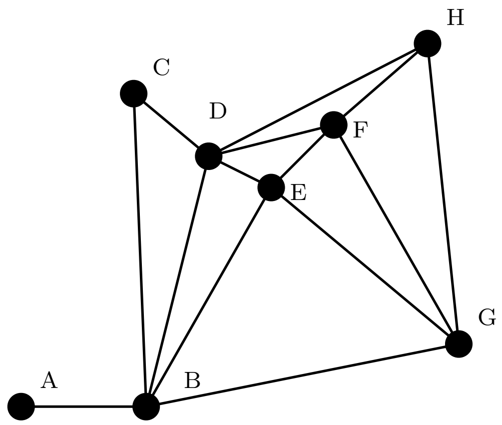

Fig. 1. Example of graph

edges incident to the vertex, with loops counted twice. Each vertex has odd degree or even degree. Let v denote a vertex and V denote the set of all vertices.

Lemma 1 (Handshaking lemma): Every undirected graph contains an even number of vertices of odd degree. That is, the number of vertices that have an odd number of incident edges must be even.

It is a simple lemma. But for completeness of arguments we provide a proof.

Proof: Prove this lemma by double counting. Let $d(v)$ be the degree of vertex v. The number of vertex-edge incidences in the graph may be counted in two different ways: by summing the degrees of the vertices, or by counting two incidences for every edge. Therefore,

$$\sum_{v \in V} d(v) = 2e,$$

where e is the number of edges in the graph. The sum of the degrees of the vertices is therefore an even number. It could happen if and only if an even number of the vertices had odd degree. ∎

Let Δ denote an n-dimensional simplex. n is a positive integer at least 2. For example, a 2-dimensional simplex is a triangle. Let partition or triangulate the simplex. Fig. 2 is an example of partition (triangulation) of a 2-dimensional simplex. In a 2-dimensional case we divide each side of Δ in m equal segments, and draw the lines parallel to the sides of Δ. Then, the 2-dimensional simplex is partitioned into m^2 triangles. We consider partition of Δ inductively for cases of higher dimension. In a 3 dimensional case each face of Δ is an 2-dimensional simplex, and so it is partitioned into m^2 triangles in the way above mentioned, and draw the planes parallel to the faces of Δ. Then, the 3-dimensional simplex is partitioned into m^3 trigonal pyramids. And so on for cases of higher dimension.

Let K denote the set of small n-dimensional simplices of Δ constructed by partition. The vertices of these small simplices of K are labeled with the numbers 0, 1, 2, ..., n subject to the following rule.

1) The vertices of Δ are respectively labeled with 0 to n. We label a point $(1, 0, \dots, 0)$ with 0, a point $(0, 1, 0, \dots, 0)$ with 1, a point $(0, 0, 1 \dots, 0)$ with 2, ..., a point $(0, \dots, 0, 1)$ with n. That is, a vertex whose k-th coordinate ($k = 0, 1, \dots, n$) is 1 and all other coordinates are 0 is labeled with k.
2) If a vertex of simplices of K is contained in an $n-1$-dimensional face of Δ, then that vertex is labeled with some number which is the same as the number of a vertex of that face.
3) If a vertex of simplices of K is contained in an $n-2$-dimensional face of Δ, then that vertex is labeled with some number which is the same as the number of a vertex of that face. And so on for cases of lower dimension.
4) A vertex contained in inside of Δ is labeled with arbitrary number among 0, 1, ..., n.

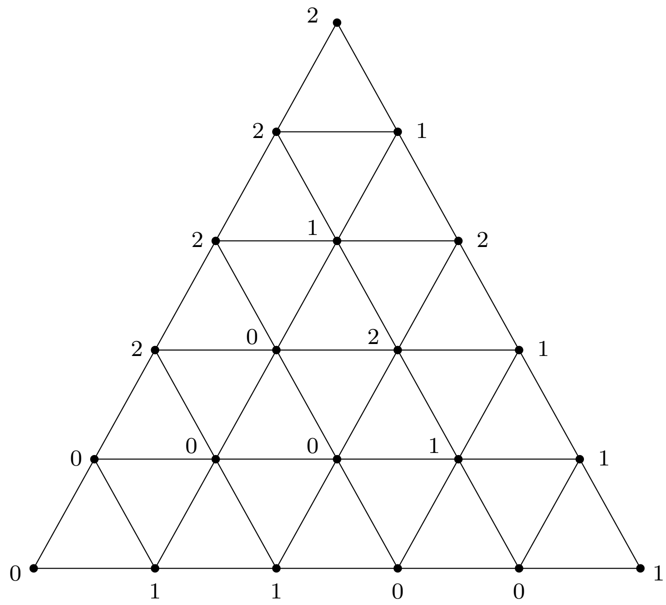

Fig. 2. Partition and labeling of 2-dimensional simplex

A small simplex of K which is labeled with the numbers 0, 1, ..., n is called a *fully labeled simplex*. Now let us prove Sperner's lemma.

Lemma 2 (Sperner's lemma): If we label the vertices of K following above rules 1) $\sim$ 4), then there are an odd number of fully labeled simplices. Thus, there exists at least one fully labeled simplex.

Proof: Our proof is standard. But again for completeness of arguments we provide a proof in Appendix A. ∎

Since n and partition of Δ are finite, the number of small simplices constructed by partition is also finite. Thus, we can constructively find a fully labeled n-dimensional simplex of K through finite steps.

III. Constructive version of the Fan-Glicksberg fixed point theorem

In this section we will prove a constructive version of the Fan-Glicksberg fixed point theorem using Sperner's lemma. The classical Fan-Glicksberg theorem ([5] and [6]) is stated as follows.

Theorem 1 (The Fan-Glicksberg fixed point theorem): Let X be a compact and convex subset of a locally convex space E, and Φ be a compact and convex valued multi-function with closed graph from X to the set of its inhabited subsets. Then, Φ has a fixed point.

A locally convex space consists of a vector space E and a family $(p_i)_{i\in I}$ of semi-norms on X, where I is an index set, for example, a set of positive integers. For each finitely enumerable subset F of I we define a basic neighborhood of a set S as follows[2];

$$V(S,F,\varepsilon)=\{y\in X|\sum_{i\in F}p_i(y-z)<\varepsilon \text{ for some } z\in S\}.$$

The closure of $V(S,F,\varepsilon)$ is denoted by $\overline{V}(S,F,\varepsilon)$, and it is represented as follows;

$$\overline{V}(S,F,\varepsilon)=\{y\in X|\sum_{i\in F}p_i(y-z)\leq\varepsilon \text{ for some } z\in S\}.$$

We call it a closed basic neighborhood of S. Compactness of a set in a locally convex space in constructive mathematics means total boundedness with completeness. According to [3] we define total boundedness of a set in a locally convex space as follows.

Definition 1: (Total boundedness of a set in a locally convex space)

Let X be a subset of E, F be a finitely enumerable subset of I, and $\varepsilon>0$. By an ε-approximation to X relative to F we mean a subset T of X such that for each $x\in X$ there exists $y\in T$ with $\sum_{i\in F}p_i(x-y)<\varepsilon$.

X is totally bounded relative to F if for each $\varepsilon>0$ there exists a finitely enumerable ε-approximation to X relative to F. It is totally bounded if it is totally bounded relative to each finitely enumerable subset of I.

An approximate fixed point of a multi-function is defined as follows;

Definition 2: (Approximate fixed point of a multi-function in a locally convex space) For each $\varepsilon>0$ x^* is an approximate fixed point of a multi-function Φ from X to the set of its inhabited subsets if

$$\sum_{i\in F}p_i(x^*-\Phi(x^*))<\varepsilon,$$

for each finitely enumerable $F\subset I$, where $p_i(x^*-\Phi(x^*))=\inf_{y\in\Phi(x^*)}p_i(x^*-y)$.

A graph of a multi-function Φ from X to the set of its inhabited subsets is

$$G(\Phi)=\cup_{x\in X}\{x\}\times\Phi(x).$$

If $G(\Phi)$ is a closed set, we say that Φ has a closed graph. It implies the following fact.

> Consider sequences $(x(n))_{n\geq 1}$ and $(y(n))_{n\geq 1}$ such that $y(n)\in\Phi(x(n))$. If $x(n)\longrightarrow x$ and $y(n)\longrightarrow y$, then $y\in\Phi(x)$.

According to [3] this means

> If for each basic neighborhood $U(x,F,\varepsilon)$ of x there exists n_0 such that $x(n)\in U(x,F,\varepsilon)$ when $n\geq n_0$, then for the union of basic neighborhoods $\cup_{y\in\Phi(x)}V(y,G,\varepsilon)$ of points in $\Phi(x)$ there exists n_0' such that $y(n)\in\cup_{y\in\Phi(x)}V(y,G,\varepsilon)$ when $n\geq n_0'$.

If X is a metric space, the semi-norm in the definition of a basic neighborhood is replaced by the metric. n_0 and n_0' depend on x and ε. Further we consider a uniform version of this property for multi-functions, and call such a multi-function *a multi-function with uniformly closed graph*, or say that a multi-function uniformly has a closed graph. It means that n_0 and n_0' depend on only ε not on x.

If X is totally bounded relative to each finitely enumerable subset of I, there exists a finitely enumerable τ-approximation $\{x^0,x^1,\dots,x^m\}$ to X relative to each finitely enumerable $F\subset I$, that is, for each $x\in X$ we have $\sum_{i\in F}p_i(x-x^i)<\tau$ for at least one $x^i,\ i=0,1,\dots,m$ for each F. Let

$$\Phi_\tau(x)=\overline{V}(\Phi(x),F,\tau),$$

where $\overline{V}(\Phi(x),F,\tau)$ is a closed basic neighborhood of $\Phi(x)$. If Φ uniformly has a closed graph, Φ_τ also uniformly has a closed graph. Now let

$$X_V=\left\{\sum_{i=0}^n\alpha_ix_i|x_i\in X,\ \sum_{i=0}^n\alpha_i=1,\ \alpha_i\geq 0\right\}. \quad (1)$$

This is the convex-hull of $\{x^0,x^1,\dots,x^m\}$. If X is convex and compact, for $x\in X$ we have

$$\Phi(x)\subset X\subset\overline{V}(X_V,F,\tau).$$

Thus,

$$\Phi(x)\cap\overline{V}(X_V,F,\tau)\neq\emptyset,$$

and so

$$\Phi_\tau(x)\cap X_V\neq\emptyset.$$

Let

$$\Phi_{X_V}(x)=\Phi_\tau(x)\cap X_V \text{ for } x\in X_V.$$

Then, it is a compact and convex valued multi-function with uniformly closed graph from X_V to the set of its inhabited subsets. If the dimension of X_V is n, X_V is homeomorphic to an n-dimensional simplex $\Delta=\{(\alpha^0,\alpha^1,\dots,\alpha^n)|\ \sum_{i=0}^n\alpha_i=1\}$. A multi-function with uniformly closed graph from Δ to the set of its inhabited subsets corresponds one to one to a multi-function with uniformly closed graph from X_V to the set of its inhabited subsets.

The contents of our constructive version of the Fan-Glicksberg fixed point theorem are described in the following theorem.

Theorem 2: (Constructive version of the Fan-Glicksberg fixed point theorem) Let X be a compact (totally bounded and complete) and convex subset of a locally convex space E, and Φ be a compact and convex valued multi-function with uniformly closed graph from X to the set of its inhabited subsets. Then, Φ has an approximate fixed point.

Proof:

1) Let consider a compact and convex valued multi-function Γ with uniformly closed graph from an n-dimensional simplex Δ to the set of its inhabited subsets. We show that we can partition Δ so that the conditions of Sperner's lemma are satisfied. We partition Δ according to the method in the proof of Sperner's lemma, and label the vertices of simplices constructed by partition of Δ. Consider a sequence $(\Delta(m))_{m\geq 1}$ of partitions of Δ. The larger m, the finer the partition and the smaller the diameter of simplices constructed by partition of Δ. Let $x(m)^0,x(m)^1,\dots$ and $x(m)^n$

[2] A set F is finitely enumerable if there exists a natural number N and a mapping of the set $\{1,2,\dots,N\}$ onto F.

be the vertices of a simplex in $\Delta(m)$. The values of Γ at theses vertices are $\Gamma(x(m)^0), \Gamma(x(m)^1), \dots$ and $\Gamma(x(m)^n)$. We can consider a sequence of vertices including the vertices $x(m)^0, x(m)^1, \dots$ and $x(m)^n$ of the same simplex. Denote the sequence by $(x(N))_{N\geq 1}$. And consider a sequence of the values of Γ at these vertices, and denote it by $(\Gamma(x(N)))_{N\geq 1}$. By the uniform version of the closed graph property of Γ,

> Suppose $x(N) \longrightarrow x$, and let $y(N) \in \Gamma(x(N))$. If there exists N_0 such that $|x(N) - x| < \varepsilon$ when $N \geq N_0$, then there exists N_0' such that $|y(N) - \Gamma(x)| < \varepsilon$ (it means $|y(N) - y| < \varepsilon$ for some $y \in \Gamma(x)$) when $N \geq N_0'$.

Consider a simplex in a sufficiently fine partition of Δ. Let x^0 be a vertex of a small n-dimensional simplex constructed by partition of Δ which is labeled, for example, with 0 by the labelling method which will be explained below. We take a point $\varphi(x) \in \Gamma(x)$ for all other vertices of this simplex so that $|\varphi(x^0) - \varphi(x)| < \varepsilon$ is satisfied[3]. It is important how to label the vertices contained in the faces of Δ. We label a vertex x according to the following rule,

$$\text{If } x_k > \varphi_k \text{ or } x_k + \tau > \varphi_k, \text{ we label } x \text{ with } k,$$

where τ is a positive number, and x_k denotes the k-th coordinate of x. If there are multiple k's which satisfy this condition, we label x conveniently for the conditions for Sperner's lemma to be satisfied. We do not randomly label the vertices. For example, let x be a point contained in an $n-1$-dimensional face of Δ such that $x_i = 0$ for one i among $0, 1, 2, \dots, n$ (its i-th coordinate is 0). With $\tau > 0$, we have $\varphi_i > 0$ or $\varphi_i < \tau$[4]. When $\varphi_i > 0$, from $\sum_{j=0}^n x_j = 1$, $\sum_{j=0}^n \varphi_j = 1$ and $x_i = 0$,

$$\sum_{j=1, j\neq i}^n x_j > \sum_{j=1, j\neq i}^n \varphi_j.$$

Then, for at least one j (denote it by k) we have $x_k > \varphi_k$, and we label x with k, where k is one of the numbers which satisfy $x_k > \varphi_k$. Since $\varphi_i > x_i$, i does not satisfy this condition. Assume $\varphi_i < \tau$. $x_i = 0$ implies $\sum_{j=0, j\neq i}^n x_j = 1$. Since $\sum_{j=0, j\neq i}^n \varphi_j \leq 1$, we obtain

$$\sum_{j=0, j\neq i}^n x_j \geq \sum_{j=0, j\neq i}^n \varphi_j.$$

Then, for a positive number τ we have

$$\sum_{j=0, j\neq i}^n (x_j + \tau) > \sum_{j=0, j\neq i}^n \varphi_j.$$

There is at least one $j (\neq i)$ which satisfies $x_j + \tau > \varphi_j$. Denote it by k, and we label x with k. k is one of the numbers other than i such that $x_k + \tau > \varphi_k$ is satisfied. i itself satisfies this condition ($x_i + \tau > \varphi_i$). But, since there is a number other than i which satisfies this condition, we can select a number other than i. We have proved that we can label the vertices contained in an $n-1$-dimensional face of Δ such that $x_i = 1$ for one i among $0, 1, 2, \dots, n$ with the numbers other than i. By similar procedures we can show that we can label the vertices contained in an $n-2$-dimensional face of Δ such that $x_i = 0$ for two i's among $0, 1, 2, \dots, n$ with the numbers other than those i's, and so on.

Consider the case where $x_i = x_{i+1} = 0$. We see that when $\varphi_i > 0$ or $\varphi_{i+1} > 0$,

$$\sum_{j=0, j\neq i, i+1}^n x_j > \sum_{j=0, j\neq i, i+1}^n \varphi_j,$$

and so for at least one j(denote it by k) we have $x_k > \varphi_k$, and we label x with k. On the other hand, when $\varphi_i < \tau$ and $\varphi_{i+1} < \tau$, we have

$$\sum_{j=0, j\neq i, i+1}^n x_j \geq \sum_{j=0, j\neq i, i+1}^n \varphi_j.$$

Then, for a positive number τ we have

$$\sum_{j=0, j\neq i, i+1}^n (x_j + \tau) > \sum_{j=0, j\neq i, i+1}^n \varphi_j.$$

Thus, there is at least one $j (\neq i, i+1)$ which satisfies $x_j + \tau > \varphi_j$. Denote it by k, and we label x with k.

Next consider the case where $x_i = 0$ for all i other than n. If for some i $\varphi_i > 0$, then we have $x_n > \varphi_n$, and label x with n. On the other hand, if $\varphi_j < \tau$ for all $j \neq n$, then we obtain $x_n \geq \varphi_n$. It implies $x_n + \tau > \varphi_n$. Thus, we can label x with n.

Therefore, the conditions for Sperner's lemma are satisfied, and there exists an odd number of fully labeled simplices in K.

2) Consider a fully labeled simplex constructed by a sufficiently fine partition of Δ. Denote vertices of the fully labeled simplex by x^0, x^1, …, x^n. We name these vertices so that $x^0, x^1, \dots, x^n$ are labeled, respectively, with 0, 1, …, n. Then, from 1) of this proof $|x^j - x^0| < \varepsilon$ and $|\varphi(x^j) - \varphi(x^0)| < \varepsilon$ for each $j \neq 0$. The i-th components of x^0 and $\varphi(x^0)$ are denoted by x_i^0 and $\varphi(x^0)_i$, and so on. Suppose $\tau > 0$. About x^0, from the labeling rules we have $x_0^0 + \tau > \varphi(x^0)_0$. About x^1, also from the labeling rules we have $x_1^1 + \tau > \varphi(x^1)_1$ which implies $x_1^1 > \varphi(x^1)_1 - \tau$. $|\varphi(x^0) - \varphi(x^1)| < \varepsilon$ means $\varphi(x^1)_1 > \varphi(x^0)_1 - \varepsilon$. On the other hand, $|x^0 - x^1| < \varepsilon$ means $x_1^0 > x_1^1 - \varepsilon$. Thus, from

$$x_1^0 > x_1^1 - \varepsilon,\ x_1^1 > \varphi(x^1)_1 - \tau,\ \varphi(x^1)_1 > \varphi(x^0)_1 - \varepsilon$$

we obtain

$$x_1^0 > \varphi(x^0)_1 - 2\varepsilon - \tau > \varphi(x^0)_1 - 2\varepsilon - \tau$$

By similar arguments, for each i other than 0,

$$x_i^0 > \varphi(x^0)_i - 2\varepsilon - \tau. \tag{2}$$

[3]There may exist a case such that we can not take a point $\varphi(x)$ for some vertex x so that $|\varphi(x^0) - \varphi(x)| < \varepsilon$ is satisfied. See 3) of this proof about such a case.

[4]In constructive mathematics for any real number x we can not prove that $x \geq 0$ or $x < 0$, that $x > 0$ or $x = 0$ or $x < 0$. But for any distinct real numbers x, y and z such that $x > z$ we can prove that $x > y$ or $y > z$.

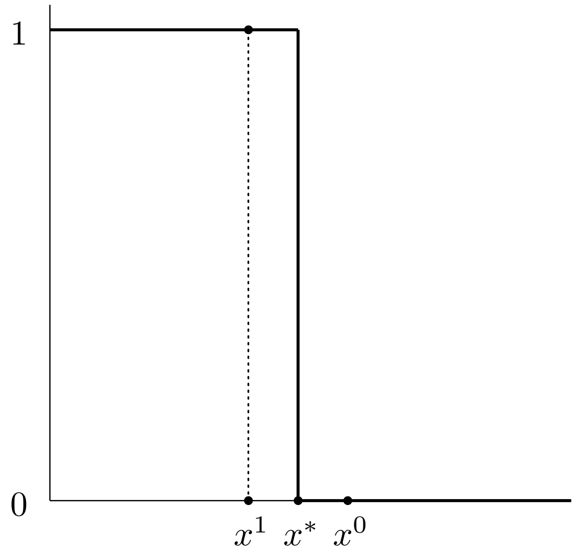

Fig. 3. A multi-function in 1-dimensional case

For $i = 0$ we have $x_0^0 + \tau > \varphi(x^0)_0$. Then,

$$x_0^0 > \varphi(x^0)_0 - \tau \quad (3)$$

Adding (2) and (3) side by side except for some i (denote it by k) other than 0,

$$\sum_{j=0, j\neq k}^{n} x_j^0 > \sum_{j=0,j\neq k}^{n} \varphi(x^0)_j - 2(n-1)\varepsilon - n\tau.$$

From $\sum_{j=0}^{n} x_j^0 = 1$, $\sum_{j=0}^{n} \varphi(x^0)_j = 1$ we have $1 - x_k^0 > 1 - \varphi(x^0)_k - 2(n-1)\varepsilon - n\tau$, which is rewritten as

$$x_k^0 < \varphi(x^0)_k + 2(n-1)\varepsilon + n\tau.$$

Since (2) implies $x_k^0 > \varphi(x^0)_k - 2\varepsilon - \tau$, we have

$$\varphi(x^0)_k - 2\varepsilon - \tau < x_k^0 < \varphi(x^0)_k + 2(n-1)\varepsilon + n\tau.$$

Thus,

$$|x_k^0 - \varphi(x^0)_k| < 2(n-1)\varepsilon + n\tau \quad (4)$$

is derived. On the other hand, adding (2) from 1 to n yields

$$\sum_{j=1}^{n} x_j^0 > \sum_{j=1}^{n} \varphi(x^0)_j - 2n\varepsilon - n\tau.$$

From $\sum_{j=0}^{n} x_j^0 = 1$, $\sum_{j=0}^{n} \varphi(x^0)_j = 1$ we have

$$1 - x_0^0 > 1 - \varphi(x^0)_0 - 2n\varepsilon - n\tau. \quad (5)$$

Then, from (3) and (5) we get

$$|x_0^0 - \varphi(x^0)_0| < 2n\varepsilon + n\tau. \quad (6)$$

Since n is finite, redefining $2n\varepsilon + n\tau$ as ε, (4) and (6) yield

$$|x_i^0 - \varphi(x^0)_i| < \varepsilon \text{ for all } i. \quad (7)$$

Note that $\varphi(x^0) \in \Gamma(x^0)$. Let Δ be X_V and Γ be Φ_{X_V}. Then,

$$x^0 \in V(\Phi(x^0), F, \varepsilon + \tau).$$

Thus, x^0 is an approximate fixed point of Φ.

3) There may exist a case such that we can not take a point $\varphi(x)$ for some vertex x so that $|\varphi(x^0) - \varphi(x)| < \varepsilon$ is satisfied. An example in a 1-dimensional case is a multi-function from $[0, 1]$ to $[0, 1]$ depicted in Fig. 3. The coordinates of the points 0 and 1 are, respectively, $(0, 1)$ and $(1, 0)$. And coordinates of other points in $[0, 1]$ are similar. Even if $|x^0 - x^1| < \varepsilon$ for any $\varepsilon < 0$, $|\varphi(x^0) - \varphi(x^1)| > 0$. x^0 and x^1 are, respectively, numbered with 0 and 1. We consider finer and finer partition of Δ, and sequences $(x^0(n))_{n\geq 1}$ and $(x^1(n))_{n\geq 1}$ such that $x^0(n)$ is labeled with 0, $x^1(n)$ is labeled with 1, $x^0(n) \longrightarrow x^*$, $x^1(n) \longrightarrow x^*$ and $|\varphi(x^0(n)) - \varphi(x^1(n))| > 0$. By the uniform version of the closed graph property of Γ there are points $\varphi^0(x^*)$ and $\varphi^1(x^*)$ such that $\varphi^0(x^*) \in \Gamma(x^*)$, $\varphi^1(x^*) \in \Gamma(x^*)$ and

$$x_0^* > \varphi^0(x^*)_0 - \tau \text{ and } x_1^* > \varphi^1(x^*)_1 - \tau.$$

Since $x_0^* + x_1^* = 1$ and $\varphi^1(x^*)_0 + \varphi^1(x^*)_1 = 1$, the latter implies

$$x_0^* < \varphi^1(x^*)_0 + \tau.$$

Thus,

$$\varphi^0(x^*)_0 - \tau < x_0^* < \varphi^1(x^*)_0 + \tau.$$

Define a point in $\Gamma(x^*)$ by

$$\varphi^*(x^*) = \alpha\varphi^0(x^*) + (1-\alpha)\varphi^1(x^*),\ 0 \leq \alpha \leq 1.$$

By the convexity of $\Gamma(x^*)$, $\varphi^*(x^*) \in \Gamma(x^*)$.
Let

$$\begin{aligned}\alpha =& \frac{\varphi^1(x^*)_0 + \tau - x_0^*}{[\varphi^1(x^*)_0 + \tau - x_0^*] + [x_0^* - \varphi^0(x^*)_0 + \tau]} \\ =& \frac{\varphi^1(x^*)_0 + \tau - x_0^*}{\varphi^1(x^*)_0 - \varphi^0(x^*)_0 + 2\tau},\end{aligned}$$

and

$$1 - \alpha = \frac{x_0^* - \varphi^0(x^*)_0 + \tau}{\varphi^1(x^*)_0 - \varphi^0(x^*)_0 + 2\tau}.$$

Then,

$$\varphi^*(x^*)_0 = \frac{\varphi^0(x^*)_0(\tau - x_0^*) + \varphi^1(x^*)_0(\tau + x_0^*)}{\varphi^1(x^*)_0 - \varphi^0(x^*)_0 + 2\tau}.$$

And so we have

$$x_0^* - \varphi^*(x^*)_0 = \frac{\tau[2x_0^* - \varphi^0(x^*)_0 - \varphi^1(x^*)_0]}{\varphi^1(x^*)_0 - \varphi^0(x^*)_0 + 2\tau}.$$

Since τ is arbitrary, for any $\varepsilon > 0$ we obtain

$$|x_0^* - \varphi^*(x^*)_0| < \varepsilon.$$

Similarly

$$|x_1^* - \varphi^*(x^*)_1| < \varepsilon$$

is derived. A case of higher dimension is similar.

We have completed the proof. ■

IV. Approximate pure strategy Nash equilibrium of a strategic game with continuous strategies and quasi-concave payoff functions

In this section, using the constructive version of the Fan-Glicksberg fixed point theorem proved in the previous section, we will prove the existence of an approximate pure strategy Nash equilibrium in a strategic game with continuous (infinite) strategies and quasi-concave payoff functions. Consider a strategic game such that there are m players with an infinite number of pure strategies for each player. The set of pure strategies of player i is denoted by $S_i, i = 1, 2, \dots, m$. S_i is a compact and convex subset of a locally convex space. Let $\mathbf{S} = \Pi_{i=1}^m S_i$ be the set of profiles of strategies of all players. Denote a pure strategy of player i by s_i, a profile of strategies of all players by $\mathbf{s} = (s_1, s_2, \dots, s_m)$, and a profile of strategies of players other than i by $\mathbf{s}_{-i}$. There exists a family $(p_j)_{j\in I}$ of semi-norms on S_i for each i, and we define the semi-norm on $\mathbf{S}$ by the sum of the semi-norms on S_i over all i.

The payoff function of player i is denoted by $\pi_i(s_i, \mathbf{s}_{-i})$. $\pi_i(s_i, \mathbf{s}_{-i})$ is uniformly continuous and quasi-concave. We define, constructively, quasi-concavity of payoff functions with respect to s_i as follows;

Definition 3 (Quasi-concavity): $\pi_i(s_i, \mathbf{s}_{-i})$ is quasi-concave if for any $s_i, s'_i \in S_i$, $\delta > 0$ we have

$$\pi_i(\lambda s_i + (1-\lambda)s'_i, \mathbf{s}_{-i}) > \min(\pi_i(s_i, \mathbf{s}_{-i}), \pi_i(s'_i, \mathbf{s}_{-i})) - \delta.$$

Each player chooses one of strategies s_i satisfying the following condition.

$$\pi_i(s_i, \mathbf{s}_{-i}) > \pi_i(s'_i, \mathbf{s}_{-i}) - \varepsilon \text{ for all } s'_i \in S_i.$$

ε is a positive number. We call such a strategy s_i an *approximate best response* of player i to $\mathbf{s}_{-i}$, and denote the set of approximate best responses of player i by $ABR_i(\mathbf{s}_{-i})$.

Since $\pi_i(s_i, \mathbf{s}_{-i})$ is uniformly continuous and S_i is totally bounded, there exists $\sup \pi_i(s_i, \mathbf{s}_{-i})$. Thus, for some $s^*_i \in S_i$ and $\frac{\varepsilon}{2}$, we have $\pi_i(s^*_i, \mathbf{s}_{-i}) > \sup \pi_i(s_i, \mathbf{s}_{-i}) - \frac{\varepsilon}{2}$. From total boundedness of S_i, for any δ and each $F \subset I$ there exists a finitely enumerable δ-approximation $\{t_1, t_2, \dots, t_m\}$ to S_i such that for any $t \in S_i$ $\sum_{j\in F} p_j(t_i - t) < \delta$ for at least one t_i. Uniform continuity of $\pi_i(s_i, \mathbf{s}_{-i})$ implies that there exists some $\delta > 0$ such that when $\sum_{j\in F} p_j(t_i - s^*_i) < \delta$ we have $|\pi_i(t_i, \mathbf{s}_{-i}) - \pi_i(s^*_i, \mathbf{s}_{-i})| < \frac{\varepsilon}{2}$. Therefore, we can constructively find at least one t_i such that $\pi_i(t_i, \mathbf{s}_{-i}) > \sup \pi_i(s_i, \mathbf{s}_{-i}) - \varepsilon$ for each $F \subset I$.

A set of approximate best responses of all players at a profile $\mathbf{s}$ is a multi-function from $\mathbf{S} = (S_1, S_2, \dots . S_m)$ to the set of its inhabited subsets, and it is denoted by

$$\mathbf{ABR}(\mathbf{s}) = (ABR_1(\mathbf{s}_{-1}), ABR_2(\mathbf{s}_{-2}), \dots, ABR_m(\mathbf{s}_{-m})).$$

An approximate Nash equilibrium is a state where all players choose their approximate best responses each other, that is, an approximate fixed point of $\mathbf{ABR}(\mathbf{s})$ is an approximate Nash equilibrium. Now we show that $\mathbf{ABR}(\mathbf{s})$ satisfies the conditions for the constructive version of the Fan-Glicksberg fixed point theorem.

1) $\mathbf{ABR}(\mathbf{s})$ is convex.

Let $\mathbf{s}, \mathbf{s}' \in \mathbf{ABR}(\mathbf{s})$. Denote $\mathbf{s} = (s_1, s_2, \dots, s_n)$ and $\mathbf{s}' = (s'_1, s'_2, \dots, s'_n)$. By the quasi-concavity of payoff functions we have, for each player i

$$\pi_i(\lambda s_i + (1-\lambda)s'_i, \mathbf{s}_{-i}) > \pi_i(s_i, \mathbf{s}_{-i}) - \delta,$$

or

$$\pi_i(\lambda s_i + (1-\lambda)s'_i, \mathbf{s}_{-i}) > \pi_i(s'_i, \mathbf{s}_{-i}) - \delta.$$

Since $s_i,\ s'_i \in ABR_i(\mathbf{s}_{-i})$, we have

$$\pi_i(\lambda s_i + (1-\lambda)s'_i, \mathbf{s}_{-i}) > \pi_i(s''_i, \mathbf{s}_{-i}) - \delta - \varepsilon \quad \text{for all } s''_i \in S_i.$$

Thus, $\lambda s_i + (1-\lambda)s'_i$ is an approximate best response of player i to $\mathbf{s}_{-i}$, and $\mathbf{ABR}(\mathbf{s})$ is a convex set.

2) $\mathbf{ABR}(\mathbf{s})$ uniformly has a closed graph. Consider a sequence of profiles of strategies of players other than i $(\mathbf{s}_{-i}(m))_{m\geq 1}$ and a sequence of strategies of player i $(s_i(m))_{m\geq 1}$ such that $s_i(m) \in ABR_i(\mathbf{s}_{-i}(m))$. Let $\mathbf{s}_{-i}(m) \longrightarrow \mathbf{s}_{-i}$ and $s_i(m) \longrightarrow s_i$. By the uniform continuity of $\pi_i(s_i, \mathbf{s}_{-i})$, for $m \geq M$ with some natural number M we have $\pi_i(s_i, \mathbf{s}_{-i}(m)) > \pi_i(s_i(m), \mathbf{s}_{-i}(m)) - \varepsilon$ and $\pi_i(s_i, \mathbf{s}_{-i}) > \pi_i(s_i, \mathbf{s}_{-i}(m)) - \varepsilon$. Thus,

$$\pi_i(s_i, \mathbf{s}_{-i}) > \pi_i(s_i(m), \mathbf{s}_{-i}(m)) - 2\varepsilon.$$

Since $s_i(m) \in ABR_i(\mathbf{s}_{-i}(m))$, $\pi_i(s_i(m), \mathbf{s}_{-i}(m)) > \pi_i(s'_i, \mathbf{s}_{-i}(m)) - \varepsilon$ for all $s'_i \in S_i$, and again by the uniform continuity of $\pi_i(s_i, \mathbf{s}_{-i})$, $\pi_i(s'_i, \mathbf{s}_{-i}(m)) > \pi_i(s'_i, \mathbf{s}_{-i}) - \varepsilon$. Then

$$\pi_i(s_i, \mathbf{s}_{-i}) > \pi_i(s'_i, \mathbf{s}_{-i}) - 4\varepsilon \text{ for all } s'_i \in S_i.$$

Therefore, $s_i \in ABR_i(\mathbf{s}_{-i})$. This holds for all i, and so $\mathbf{ABR}(\mathbf{s})$ uniformly has a closed graph. The conditions for the constructive version of the Fan-Glicksberg fixed point theorem are satisfied.

There exists a point $\mathbf{s}^*$ such that for any δ and each $F \in I$,

$$\sum_{j\in F} p_j(\mathbf{s} - \mathbf{s}^*) < \delta \text{ for some } \mathbf{s} \in \mathbf{ABR}(\mathbf{s}^*).$$

This means

$$\sum_{j\in F} p_j(s_i - s^*_i) < \delta \text{ for some } s_i \in ABR_i(\mathbf{s}^*_{-i}). \tag{8}$$

Since $\pi_i(s_i, \mathbf{s}_{-i})$ is uniformly continuous with respect to s_i, (8) implies

$$\pi_i(s^*_i, \mathbf{s}^*_{-i}) > \pi_i(s'_i, \mathbf{s}^*_{-i}) - \tau - \varepsilon \text{ for all } s'_i \in S_i,$$

for $\tau > 0$ and $\varepsilon > 0$. The value of τ is determined corresponding to the value of δ. Then, s^*_i is an approximate best response of player i to $\mathbf{s}^*_{-i}$, and we have completed the proof of the existence of an approximate pure strategy Nash equilibrium.

V. Concluding Remarks

In this paper we have presented a proof of a constructive version of the Fan-Glicksberg fixed point theorem for multi-functions in a locally convex space using Sperner's lemma, and applied it to a proof of the existence of a Nash equilibrium in a strategic game with continuous strategies and quasi-concave payoff functions. We are studying some related

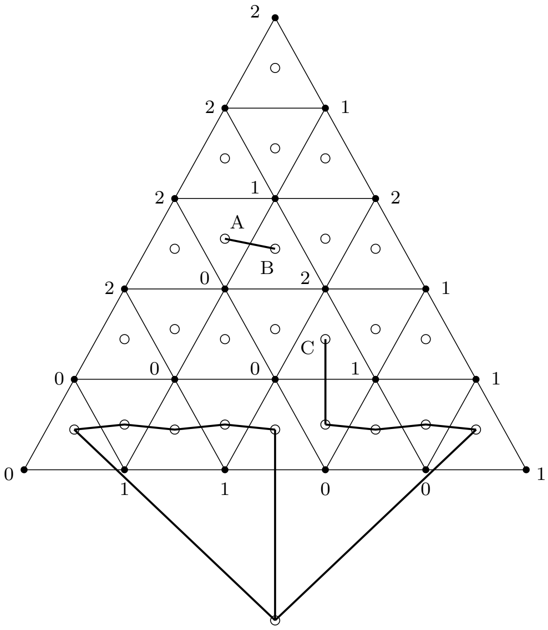

Fig. 4. Sperner's lemma

problems such as the existence of an approximate equilibrium of a competitive economy with multi-valued demand and supply functions, and the existence of an approximate core in a NTU (non-transferable utility) game. A related paper is [8] in which a constructive version of Tychonoff's fixed point theorem for single-valued functions in a locally convex space has been proved.

Appendix A
Proof of Sperner's lemma

We prove this lemma by induction about the dimension of Δ. When $n = 0$, we have only one point with the number 0. It is the unique 0-dimensional simplex. Therefore the lemma is trivial. When $n = 1$, a partitioned 1-dimensional simplex is a segmented line. The endpoints of the line are labeled distinctly, with 0 and 1. Hence in moving from endpoint 0 to endpoint 1 the labeling must switch an odd number of times, that is, an odd number of edges labeled with 0 an 1 may be located in this way.

Next consider the case of 2 dimension. Assume that we have partitioned a 2-dimensional simplex (triangle) Δ as explained above. Consider the face of Δ labeled with 0 and 1[5]. It is the base of the triangle in Fig. 4. Now we introduce a dual graph that has its nodes in each small triangle of K plus one extra node outside the face of Δ labeled with 0 and 1 (putting a dot in each small triangle, and one dot outside Δ). We define edges of the graph that connect two nodes if they share a side labeled with 0 and 1. See Fig. 4. White circles are nodes of the graph, and thick lines are its edges. Since from the result of 1-dimensional case there are an odd number of faces of K labeled with 0 and 1 contained in the face of Δ labeled with 0 and 1, there are an odd number of edges which connect the outside node and inside nodes. Thus, the outside node has odd degree. Since by the Handshaking lemma there are an even number of nodes which have odd degree, we have at least one node inside the triangle which has odd degree. Each node of our graph except for the outside node is contained in one of small triangles of K. Therefore, if a small triangle of K has one face labeled with 0 and 1, the degree of the node in that triangle is 1; if a small triangle of K has two such faces, the degree of the node in that triangle is 2, and if a small triangle of K has no such face, the degree of the node in that triangle is 0. Thus, if the degree of a node is odd, it must be 1, and then the small triangle which contains this node is labeled with 0, 1 and 2 (fully labeled). In Fig. 4 triangles which contain one of the nodes A, B, C are fully labeled triangles. Now assume that the theorem holds for dimensions up to $n-1$. Assume that we have partitioned an n-dimensional simplex Δ. Consider the fully labeled face of Δ which is a fully labeled $n-1$-dimensional simplex. Again we introduce a dual graph that has its nodes in small n-dimensional simplices of K plus one extra node outside the fully labeled face of Δ (putting a dot in each small n-dimensional simplex, and one dot outside Δ). We define the edges of the graph that connect two nodes if they share a face labeled with $0, 1, \dots, n-1$. Since from the result of $n-1$-dimensional case there are an odd number of fully labeled faces of small simplices of K contained in the $n-1$-dimensional fully labeled face of Δ, there are an odd number of edges which connect the outside node and inside nodes. Thus, the outside node has odd degree. Since, by the Handshaking lemma there are an even number of nodes which have odd degree, we have at least one node inside the simplex which has odd degree. Each node of our graph except for the outside node are contained in one of small n-dimensional simplices of K. Therefore, if a small simplex of K has one fully labeled face, the degree of the node in that simplex is 1; if a small simplex of K has two such faces, the degree of the node in that simplex is 2, and if a small simplex of K has no such face, the degree of the node in that simplex is 0. Thus, if the degree of a node is odd, it must be 1, and then the small simplex which contains this node is fully labeled.

> If the number (label) of a vertex other than vertices labeled with $0, 1, \dots, n-1$ of an n-dimensional simplex which contains a fully labeled $n-1$-dimensional face is n, then this n-dimensional simplex has one such face, and this simplex is a fully labeled n-dimensional simplex. On the other hand, if the number of that vertex is other than n, then the n-dimensional simplex has two such faces.

We have completed the proof of Sperner's lemma.

[5]We call edges of triangle Δ *faces* to distinguish between them and edges of a dual graph which we will consider later.

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