A Study on Rough, Fuzzy and Rough Fuzzy Bi-ideals of Ternary Semigroups

Sompob Saelee and Ronnason Chinram

***Abstract*—A ternary semigroup is a nonempty set together with a ternary multiplication which is associative. Any semigroup can be reduced to a ternary semigroup but a ternary semigroup does not necessarily reduce to a semigroup. The notion of fuzzy sets was introduced by Zadeh in 1965 and that of rough sets by Pawlak in 1982. Applications of the fuzzy set theory and rough set theory have been found in various fields. The theory of fuzzy sets and rough sets were studied in various kinds of algebraic systems. In this paper, we study rough, fuzzy and rough fuzzy bi-ideals of ternary semigroups.**

***Index Terms*—bi ideals, rough bi-ideals, fuzzy bi-ideals, rough fuzzy bi-ideals, ternary semigroups.**

I. Introduction

THE formal definition of a ternary algebraic structure was given by Lehmer [13] in 1932, but earlier such structures were studied by Kasner [9] and Prüfer [18]. Lehmer investigated certain triple systems called triplexes which turn out to be commutative ternary group. The notion of ternary semigroups was known to Banach (cf. [14]) who is credited with an example of a ternary semigroup which does not reduce to a semigroup. A nonempty set T is called a *ternary semigroup* if there exists a ternary operation $T \times T \times T \to T$, written as $(x_1, x_2, x_3) \mapsto x_1x_2x_3$ satisfying the following identity for any $x_1, x_2, x_3, x_4, x_5 \in T$,

$$(x_1x_2x_3)x_4x_5 = x_1(x_2x_3x_4)x_5 = x_1x_2(x_3x_4x_5).$$

Any semigroup can be reduced to a ternary semigroup but a ternary semigroup does not necessarily reduce to a semigroup, for example, $\mathbb{Z}^-$ is a ternary semigroup while $\mathbb{Z}^-$ is not a semigroup under the multiplication over integers. In [14], Los proved that every ternary semigroup can be embedded in a semigroup.

Let T be a ternary semigroup. For nonempty subsets A, B and C of T, let $ABC := \{abc \mid a \in A, b \in B \text{ and } c \in C\}$. A nonempty subset S of T is called a *ternary subsemigroup* if $SSS \subseteq S$. In [20], Sioson studied ideal theory in ternary semigroups. A nonempty subset A of a ternary semigroup T is called a *left ideal* of T if $TTA \subseteq A$, a *right ideal* of T if $ATT \subseteq A$ and a *lateral ideal* of T if $TAT \subseteq A$. If A is a left, right and lateral ideal of T, A is called an *ideal* of T. A ternary subsemigroup B of T is called a *bi-ideal* of T if $BTBTB \subseteq B$. A ternary subsemigroup Q of T is called a *quasi-ideal* of T if $QTT \cap TQT \cap TTQ \subseteq Q$ and $QTT \cap TTQTT \cap TTQ \subseteq Q$. The intersection of a left ideal, a right ideal and a lateral ideal is a quasi-ideal and every quasi-ideal contains in this way [20]. Later, ideal theory for ternary semigroups has studied by Dixit and Dewan. They studied quasi-ideals and bi-ideals in ternary semigroups [5] and minimal quasi-ideals in ternary semigroups [4]. In [5], Dixit and Dewan proved that every quasi-ideal of T is a bi-ideal of T but the converse is not true in general by giving example. Later, ternary semigroups were studied by some author, for example, Shabir and Bano [21] studied prime bi-ideals in ternary semigroups and Iampan [7] studied filters in ternary semigroups, etc. Recently, Santiago and Bala [19] studied regular ternary semigroups and cover semigroups of ternary semigroups.

The notion of fuzzy sets was introduced by Zadeh [23]. Fuzzy set theory is a generalization of set theory. Applications of the fuzzy set theory have been found in various fields. The notion of rough sets was introduced by Pawlak [16]. Rough set is described by a pair of ordinary sets called the upper and lower approximations. The notion of a rough set has often been compared to that of a fuzzy set, sometimes with a view to prove that one is more general, or, more useful than the other. Several researchs were conducted on the generalizations of the notion of fuzzy sets and rough sets. The fuzzy sets and rough sets were studied in various kinds of algebraic systems. Moreover, they were studied in various kinds of ternary algebraic systems. For example, Kavikumar and Khamis [10] studied fuzzy ideals and fuzzy quasi-ideals in ternary semirings, Kavikumar, Khamis and Jun [11] studied fuzzy bi-ideals in ternary semirings, Chinram and Malee studied L-fuzzy ideals [1] and k-fuzzy ideals [15] in ternary semirings, Davvaz [3] studied fuzzy hyperideals in ternary semihyperrings and Chinram and Saelee [2] studied fuzzy ideals and fuzzy filters of ordered ternary semigroups, etc. Rough ideals in semigroups were studied by Kuroki [12]. In [22], Xiao and Zhang studied rough prime ideals and rough fuzzy prime ideals. Later, Petchkhaew and Chinram [17] studied rough ideals and fuzzy rough ideals of ternary semigroups analogous to that of semigroups considered by

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Kuroki [12] and Xiao and Zhang [22]. Moreover, they studied fuzzy ideals of ternary semigroups analogous to that of semigroups.

In this paper, we study rough, fuzzy and rough fuzzy bi-ideals of ternary semigroups.

II. Rough bi-ideals of ternary semigroups

Kar and Maity [8] studied congruences on ternary semigroups. Congruence is a special type of equivalence relation which plays a vital role in the quotient structure of algebraic structures. Let T be a ternary semigroup. A *congruence* ρ on T is an equivalence relation on T such that for all $a, b, x, y \in T$,

$$(a, b) \in \rho \text{ implies } (xya, xyb), (xay, xby), (axy, bxy) \in \rho.$$

For $a \in T$, the ρ-congruence class containing a denoted by $[a]_\rho$. A congruence ρ of T is called *complete* if $[a]_\rho[b]_\rho[c]_\rho = [abc]_\rho$ for all $a, b, c \in T$. Let ρ be a congruence on T and A a nonempty subset of T. The sets

$$\rho_-(A) = \{x \in T \mid [x]_\rho \subseteq A\} \text{ and}$$

$$\rho^-(A) = \{x \in T \mid [x]_\rho \cap A \neq \emptyset\}$$

are called the *ρ-lower* and *ρ-upper approximations* of A, respectively.

Example II.1. Define a relation ρ on a ternary semigroup $\mathbb{Z}^-$ under the usual multiplication by

$$x\rho y \leftrightarrow 3 \mid x - y \text{ for all } x, y \in \mathbb{Z}^-.$$

Then ρ is a congruence on $\mathbb{Z}^-$ and $\mathbb{Z}^-/\rho = \{[-1]_\rho, [-2]_\rho, [-3]_\rho\}$. Let $A = \{-3, -6\}$. We have that $\rho_-(A) = \emptyset$ and $\rho^-(A) = [-3]_\rho$.

This proposition is similar to the proof of Theorem 2.1 in Kuroki [12].

Proposition II.1. ([*17*]) *Let ρ and λ be congruences on a ternary semigroup T and A and B nonempty subsets of T. The following statements are true.*

(1) $\rho_-(A) \subseteq A \subseteq \rho^-(A)$.
(2) $\rho^-(A \cup B) = \rho^-(A) \cup \rho^-(B)$.
(3) $\rho_-(A \cap B) = \rho_-(A) \cap \rho_-(B)$.
(4) $A \subseteq B$ *implies* $\rho_-(A) \subseteq \rho_-(B)$.
(5) $A \subseteq B$ *implies* $\rho^-(A) \subseteq \rho^-(B)$.
(6) $\rho_-(A) \cup \rho_-(B) \subseteq \rho_-(A \cup B)$.
(7) $\rho^-(A \cap B) \subseteq \rho^-(A) \cap \rho^-(B)$.
(8) $\rho \subseteq \lambda$ *implies* $\lambda_-(A) \subseteq \rho_-(A)$.
(9) $\rho \subseteq \lambda$ *implies* $\rho^-(A) \subseteq \lambda^-(A)$.

Proposition II.2. ([*17*]) *Let ρ be a complete congruence on a ternary semigroup T and A, B and C nonempty subsets of T. Then (1) $\rho^-(A)\rho^-(B)\rho^-(C) \subseteq \rho^-(ABC)$ and (2) $\rho_-(A)\rho_-(B)\rho_-(C) \subseteq \rho_-(ABC)$.*

A nonempty subset A of a ternary semigroup T is called a *ρ-upper rough bi-ideal* of T if $\rho^-(A)$ is a bi-ideal of T and A is called a *ρ-lower rough bi-ideal* of T if $\rho_-(A)$ is a bi-ideal of T.

Lemma II.3. ([*17*]) *Let ρ be a congruence on a ternary semigroup T and A a nonempty subset of T. If A is a ternary subsemigroup of T, then A is a ρ-upper rough ternary subsemigroup of T.*

Theorem II.4. *Let ρ be a congruence on a ternary semigroup T and A a nonempty subset of T. If A is a bi-ideal of T, then A is a ρ-upper rough bi-ideal of T.*

Proof. Assume A is a bi-ideal of T. Then $\rho^-(A) \neq \emptyset$. By Lemma II.3, we have $\rho^-(A)\rho^-(A)\rho^-(A) \subseteq \rho^-(A)$. By Theorem II.2 and Proposition II.1(5), we have $\rho^-(A)T\rho^-(A)T\rho^-(A) = \rho^-(T)\rho^-(A)\rho^-(T)\rho^-(A)\rho^-(A) \subseteq \rho^-(ATATA) \subseteq \rho^-(A)$. □

However, the converse of this theorem is not true in general. For example, we can see in Example II.1, $\rho^-(A)$ is a bi-ideal of $\mathbb{Z}^-$ but A is not.

Theorem II.5. *Let ρ be a complete congruence on a ternary semigroup T and A a nonempty subset of T such that $\rho_-(A) \neq \emptyset$. If A is a bi-ideal of T, then A is a ρ-lower rough bi-ideal of T.*

Proof. The proof of this theorem is similar to the proof of Theorem II.4 □

Proposition II.6. *Let T be a ternary semigroup and ρ a complete congruence on T. Then $T/\rho = \{[a]_\rho \mid a \in T\}$ is a ternary semigroup under the ternary operation by $[a]_\rho[b]_\rho[c]_\rho = [abc]_\rho$ for all $a, b, c \in T$.*

Let ρ be a complete congruence on a ternary semigroup T. The ternary semigroup T/ρ is called a *quotient ternary semigroup of T by a congruence ρ*.

Let ρ be a congruence on a ternary semigroup T. The ρ-lower and ρ-upper approximations can be presented in an equivalent form as shown below:

$$\rho_-(A)/\rho = \{[x]_\rho \in T/\rho \mid [x]_\rho \subseteq A\} \text{ and}$$

$$\rho^-(A)/\rho = \{[x]_\rho \in T/\rho \mid [x]_\rho \cap A \neq \emptyset\},$$

respectively. Now we discuss these sets as subsets of a quotient ternary semigroup T/ρ.

Lemma II.7. ([*17*]) *Let ρ be a complete congruence on a ternary semigroup T. If A is a ternary subsemigroup of T, then $\rho^-(A)/\rho$ is a subsemigroup of T/ρ.*

Theorem II.8. *Let ρ be a complete congruence on a ternary semigroup T. If A is a bi-ideal of T, then $\rho^-(A)/\rho$ is a bi-ideal of T/ρ.*

Proof. Assume A is a bi-ideal of T. By Lemma II.7, we have $\rho^-(A)/\rho$ is a ternary subsemigroup of T/ρ. Let

$[x]_\rho, [y]_\rho, [z]_\rho \in \rho^-(A)/\rho$ and $[a]_\rho, [b]_\rho \in T/\rho$. Then there exist $x' \in [x]_\rho \cap A, y' \in [y]_\rho \cap A, z' \in [z]_\rho \cap A$ and $a' \in [a]_\rho, b' \in [b]_\rho$. Since ρ is complete, $x'a'y'b'z' \in [x]_\rho[a]_\rho[y]_\rho[b]_\rho[x]_\rho = [xaybz]_\rho$. Since A is a bi-ideal of T, $x'a'y'b'z' \in A$. Then $[xaybz]_\rho \cap A \neq \emptyset$. Hence $[x]_\rho[a]_\rho[y]_\rho[b]_\rho[z]_\rho \in \rho^-(A)/\rho$. $\square$

Lemma II.9. ([*17*]) *Let ρ be a complete congruence on a ternary semigroup T. If A is a ternary subsemigroup of T, then $\rho_-(A)/\rho$ is a ternary subsemigroup of T/ρ.*

Theorem II.10. *Let ρ be a complete congruence on a ternary semigroup T and A a nonempty subset of T such that $\rho_-(A)/\rho \neq \emptyset$. If A is a bi-ideal of T, then $\rho_-(A)/\rho$ is a bi-ideal of T/ρ.*

Proof. Assume A is a bi-ideal of T. By Lemma II.9, we have $\rho_-(A)/\rho$ is a ternary subsemigroup of T/ρ. Let $[x]_\rho, [y]_\rho, [z]_\rho \in \rho_-(A)/\rho$ and $[a]_\rho, [b]_\rho \in T/\rho$. Then $[x]_\rho \subseteq A, [y]_\rho \subseteq A$ and $[z]_\rho \subseteq A$. Since A is a bi-ideal of T, $[x]_\rho[a]_\rho[y]_\rho[b]_\rho[z]_\rho \subseteq A$. Therefore $[x]_\rho[a]_\rho[y]_\rho[b]_\rho[z]_\rho \in \rho_-(A)/\rho$. $\square$

III. Fuzzy bi-ideals of ternary semigroups

First, we recall the definition of fuzzy subsets. Let T be a ternary semigroup. A function f from T to the unit interval $[0,1]$ is called a *fuzzy subset* of T. The ternary semigroup T itself is a fuzzy subset of T such that $T(x) = 1$ for all $x \in T$, denoted also by T. If $A \subseteq T$, the *characteristic function* f_A of A is a fuzzy subset of T defined as follows:

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Now we study fuzzy bi-ideals of ternary semigroups. Let T be a ternary semigroup. A fuzzy subset f of T is called a *fuzzy bi-ideal* of T if $f(abc) \geq \min\{f(a), f(b), f(c)\}$ and $f(abcde) \geq \min\{f(a), f(c), f(e)\}$ for all $a, b, c, d, e \in T$.

Theorem III.1. *Let T be a ternary semigroup and A a nonempty subset of T. Then A is a bi-ideal of T if and only if f_A is a fuzzy bi-ideal of T.*

Proof. Assume A is a bi-ideal of T. Let $a, b, x, y, z \in T$.
Case 1 : $x, y, z \in A$. Since A is a bi-ideal of T, $xyz, xaybz \in A$. Therefore $f_A(xyz) = 1 \geq \min\{f_A(x), f_A(y), f_A(z)\}$ and $f_A(xaybz) = 1 \geq \min\{f_A(x), f_A(y), f_A(z)\}$.
Case 2 : $x \notin A$ or $y \notin A$ or $z \notin A$. Thus $f_A(x) = 0$ or $f_A(y) = 0$ or $f_A(z) = 0$. Hence $\min\{f_A(x), f_A(y), f_A(z)\} = 0 \leq f_A(xyz)$ and $\min\{f_A(x), f_A(y), f_A(z)\} = 0 \leq f_A(xaybz)$. $\square$

Let T be a ternary semigroup. A nonempty subset S of T is called a *prime subset* of T if for all $x, y, z \in T, xyz \in S$ implies $x \in S$ or $y \in S$ or $z \in S$. A bi-ideal S of T is called a *prime bi-ideal* of T if S is a prime subset of T. A fuzzy subset f of T is called a *prime fuzzy subset* of T if $f(xyz) \leq \max\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$. A fuzzy bi-ideal f of T is called a *prime fuzzy bi-ideal* of T if f is a prime fuzzy subset of T.

Lemma III.2. ([*17*]) *Let T be a ternary semigroup and A a nonempty subset of T. Then A is a prime subset of T if and only if f_A is a prime fuzzy subset of T.*

Theorem III.3. *Let T be a ternary semigroup and A a nonempty subset of T. Then A is a prime bi-ideal of T if and only if f_A is a prime fuzzy bi-ideal of T.*

Proof. It follows from Lemma III.2 and Theorem III.1. $\square$

Let f be a fuzzy subset of a set (a ternary semigroup) T. For any $t \in [0,1]$, the set

$$f_t = \{x \in T \mid f(x) \geq t\} \text{ and } f_t^s = \{x \in T \mid f(x) > t\}$$

are called a *t-levelset* and a *t-strong levelset* of f, respectively [22].

Theorem III.4. *Let f be a fuzzy subset of a ternary semigroup T. Then f is a fuzzy bi-ideal of T if and only if for all $t \in [0,1]$, if $f_t \neq \emptyset$, then f_t is a bi-ideal of T.*

Proof. Assume f is a fuzzy bi-ideal of T. Let $t \in [0,1]$ such that $f_t \neq \emptyset$. Let $x, y, z \in f_t$. Then $f(x) \geq t$, $f(y) \geq t$ and $f(z) \geq t$. Since f is a fuzzy bi-ideal of T, $f(xyz) \geq \min\{f(x), f(y), f(z)\} \geq t$ and $f(xaybz) \geq \min\{f(x), f(y), f(z)\} \geq t$ for all $a, b \in T$. Therefore $xyz, xaybz \in f_t$. Hence f_t is a bi-ideal of T. Conversely, assume for all $t \in [0,1]$, if $f_t \neq \emptyset$, then f_t is a bi-ideal of T. Let $a, b, x, y, z \in T$. Choose $t = \min\{f(x), f(y), f(z)\}$. Then $x, y, z \in f_t$. This implies that $f_t \neq \emptyset$. By assumption, we have f_t is a bi-ideal of T. So $xyz, xaybx \in f_t$. Therefore $f(xyz) \geq t$ and $f(xaybz) \geq t$. Hence $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ and $f(xaybz) \geq \min\{f(x), f(y), f(z)\}$. $\square$

Lemma III.5. ([*17*]) *Let f be a fuzzy subset of a ternary semigroup T. Then f is a prime fuzzy subset of T if and only if for all $t \in [0,1]$, if $f_t \neq \emptyset$, then f_t is a prime subset of T.*

Theorem III.6. *Let f be a fuzzy subset of a ternary semigroup T. Then f is a prime fuzzy bi-ideal of T if and only if for all $t \in [0,1]$, if $f_t \neq \emptyset$, then f_t is a prime bi-ideal of T.*

Proof. It follows from Lemma III.5 and Theorem III.4. $\square$

Theorem III.7. *Let f be a fuzzy subset of a ternary semigroup T. Then f is a fuzzy bi-ideal of T if and only if for all $t \in [0,1]$, if $f_t^s \neq \emptyset$, then f_t^s is a bi-ideal of T.*

Proof. The proof of this theorem is similar to the proof of Theorem III.4 $\square$

Theorem III.8. *Let f be a fuzzy subset of a ternary semigroup T. Then f is a prime bi-ideal of T if and only if for all $t \in [0,1]$, if $f_t^s \neq \emptyset$, then f_t^s is a prime bi-deal of T.*

Proof. It follows from Lemma III.5 and Theorem III.7. □

IV. Rough fuzzy bi-ideals of ternary semigroups

Let f be a fuzzy subset of a ternary semigroup T and ρ be a congruence on T. Then the fuzzy sets $\rho^-(f)$ and $\rho_-(f)$ are defined by

$$\rho^-(f)(x) = \sup_{a\in[x]_\rho} f(a) \text{ and } \rho_-(f)(x) = \inf_{a\in[x]_\rho} f(a)$$

are called the ρ-*upper* and ρ-*lower approximations* of a fuzzy set f, respectively [6].

Example IV.1. Define a relation ρ on a ternary semigroup $\mathbb{Z}^-$ under the usual multiplication by

$$x\rho y \leftrightarrow 2 \mid x-y \text{ for all } a,b \in \mathbb{Z}^-.$$

It is easy to see that ρ is a congruence on $\mathbb{Z}^-$. Let $f(x) = \dfrac{1}{-x}$ for all $x \in \mathbb{Z}^-$. Then

$$\rho_-(f)(x) = 0 \text{ for all } x \in \mathbb{Z}^-$$

and

$$\rho^-(f)(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}^- \text{ is odd,} \\ \dfrac{1}{2} & \text{if } x \in \mathbb{Z}^- \text{ is even.} \end{cases}$$

Lemma IV.1. ([*17*]) *Let ρ be a congruence on a ternary semigroup T, f a fuzzy subset of T and $t \in [0,1]$, then (1) $(\rho_-(f))_t = \rho_-(f_t)$ and (2) $(\rho^-(f))^s_t = \rho^-(f^s_t)$.*

Theorem IV.2. *Let ρ be a complete congruence on a ternary semigroup T. If f is a fuzzy bi-ideal of T, then $\rho^-(f)$ and $\rho_-(f)$ are fuzzy bi-ideals of T.*

Proof. It follows from Theorem III.4, Theorem III.7, Theorem II.4, Theorem II.5 and Lemma IV.1. □

Note that if $\rho^-(f)$ and $\rho_-(f)$ are fuzzy bi-ideals of a ternary semigroup T, in general f need not be a fuzzy bi-ideal of T. For example, we can see in Example IV.1, $\rho_-(f)$ is a fuzzy bi-ideal of T but f is not.

V. Problems of homomorphisms

Let T_1 and T_2 be ternary semigroups. A mapping φ from T_1 to T_2 is called a *homomorphism* from T_1 to T_2 if $\varphi(abc) = \varphi(a)\varphi(b)\varphi(c)$ for all $a,b,c \in T_1$. Then the set $\kappa = \{(a,b) \in T_1 \times T_1 \mid \varphi(a) = \varphi(b)\}$ is called the *kernel* of φ. We have that κ is a congruence on T_1.

Lemma V.1. ([*17*]) *Let φ be an onto homomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x),\varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. The following statements are true.*

(1) *ρ_1 is a congruence on T_1.*
(2) *If ρ_2 is complete and φ is 1-1, then ρ_1 is complete.*
(3) *$\varphi(\rho_1^-(A)) = \rho_2^-(\varphi(A))$.*
(4) *$\varphi(\rho_{1-}(A)) \subseteq \rho_{2-}(\varphi(A))$.*
(5) *If φ is 1-1, then $\varphi(\rho_{1-}(A)) = \rho_{2-}(\varphi(A))$.*

Lemma V.2. ([*17*]) *Let φ be an onto homomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x),\varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_1^-(A)$ is a ternary subsemigroup of T_1 if and only if $\rho_2^-(\varphi(A))$ is a ternary subsemigroup of T_2.*

Theorem V.3. *Let φ be an onto homomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x),\varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_1^-(A)$ is a bi-ideal of T_1 if and only if $\rho_2^-(\varphi(A))$ is a bi-ideal of T_2.*

Proof. Assume $\rho_1^-(A)$ is a bi-ideal of T_1. By Lemma V.2, we have $\rho_2^-(\varphi(A))$ is a ternary subsemigroup of T_2. $\varphi(\rho_1^-(A))\varphi(T_1)\varphi(\rho_1^-(A))\varphi(T_1)\varphi(\rho_1^-(A)) = \varphi(\rho_1^-(A)T_1\rho_1^-(A)T_1\rho_1^-(A)) \subseteq \varphi(\rho_1^-(A))$. By Lemma V.1(3), $\rho_2^-(\varphi(A))\rho_2^-(T_2)\rho_2^-(\varphi(A))\rho_2^-(T_2)\rho_2^-(\varphi(A)) \subseteq \rho_2^-(\varphi(A))$. So $\rho_2^-(\varphi(A))$ is a bi-ideal of T_2. The proof of the converse is similar. □

Lemma V.4. ([*17*]) *Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x),\varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_1^-(A)$ is a prime subset of T_1 if and only if $\rho_2^-(\varphi(A))$ is a prime subset of T_2.*

Theorem V.5. *Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x),\varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_1^-(A)$ is a prime bi-ideal of T_1 if and only if $\rho_2^-(\varphi(A))$ is a prime bi-ideal of T_2.*

Proof. It follows from Lemma V.4 and Theorem V.3. □

Lemma V.6. ([*17*]) *Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x),\varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_{1-}(A)$ is a ternary subsemigroup of T_1 if and only if $\rho_{2-}(\varphi(A))$ is a ternary subsemigroup of T_2.*

Theorem V.7. *Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x),\varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_{1-}(A)$ is a bi-ideal of T_1 if and only if $\rho_{2-}(\varphi(A))$ is a bi-ideal of T_2.*

Proof. By Lemma V.1(5), we have $\rho_{1-}(A) = \rho_{2-}(\varphi(A))$. The proof of this theorem is similar to the proof of TheoremV.3. □

Lemma V.8. ([*17*]) *Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x),\varphi(y)) \in \rho_2\}$ and A*

a nonempty subset of T_1. Then $\rho_{1-}(A)$ is a prime subset of T_1 if and only if $\rho_{2-}(\varphi(A))$ is a prime subset of T_2.

Theorem V.9. *Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x, y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_{1-}(A)$ is a prime bi-ideal of T_1 if and only if $\rho_{2-}(\varphi(A))$ is a prime bi-ideal of T_2.*

Proof. It follows from LemmaV.8 and TheoremV.7. □

We can obtain the following conclusion easily in a quotient ternary semigroup.

Corollary V.10. *Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a complete congruence on T_2, $\rho_1 = \{(x, y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. The following statements are true.*

(1) *$\rho_{1-}(A)/\rho_1$ is a bi-ideal of T_1/ρ_1 if and only if $\rho_{2-}(\varphi(A))/\rho_2$ is a bi-ideal of T_2/ρ_2.*
(2) *$\rho_1^-(A)/\rho_1$ is a bi-ideal of T_1/ρ_1 if and only if $\rho_2^-(\varphi(A))/\rho_2$ is a bi-ideal of T_2/ρ_2.*
(3) *$\rho_{1-}(A)/\rho_1$ is a prime bi-ideal of T_1/ρ_1 if and only if $\rho_{2-}(\varphi(A))/\rho_2$ is a prime bi-ideal of T_2/ρ_2.*
(4) *$\rho_1^-(A)/\rho_1$ is a prime bi-ideal of T_1/ρ_1 if and only if $\rho_2^-(\varphi(A))/\rho_2$ is a prime bi-ideal of T_2/ρ_2.*

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