Exact and Explicit Solution Algorithm for Linear Programming Problem with a Second-Order Cone

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Abstract—**This paper proposes a exact solution algorithm to explicitly obtain the exact optimal solution of a second-order cone programming problem with box constraints of decision variables. The proposed solution algorithm is based on a parametric solution approach to determine the optimal strict region of parameters, and the main procedures are to perform deterministic equivalent transformations for the main problem and to solve the KKT condition of auxiliary problem without the loss of optimality.**

Index Terms—**Second-order cone programming problem, KKT condition, Exact optimal solution, Parametric approach.**

I. Introduction

Second-order cone programming (SOCP) problem is one of the most important optimization problems with linear and second-order cone constraints. A wide range of problems can be formulated as SOCP problem; linear programming (LP) problems, convex quadratic programming problems, some stochastic and robust programming problems, various practical problems of engineering, control and management science ([1], [2], [3], [4]). SOCP problem itself is a subclass of semi-definite programming (SDP) problems, and so SOCP problems can be solved as SDP problems in theory. Then, some interior-point methods have been developed to solve SOCP problems ([5], [6], [7]). As one of other approaches, Cai and Toh [8] proposed the reduced augmented equation approach for SOCP problems. However, these approaches sometimes have failed to deliver solutions with satisfactory accuracy and been far more efficient computationally to solve SOCP problems directly. Therefore, many researchers have developed more efficient solution methods for SOCP and SDP problems (e.g., [9], [10]).

On the other hand, as a traditional solution approach for linear programming, Danzig [11] proposed the Simplex method, and it has been the centre of solution methods for linear programming up to the present since it is often more efficient to solve standard problems using the Simplex method than the interior-point method. In recently, Muramatsu [12] developed the efficient solution method for a SOCP problem using the pivoting method based on the Simplex method.

Thus, many researchers have considered various types of theoretical and practical approaches for SOCP problems using interior-point methods and linear programming. However, the explicit optimal solution is not obtained by these solution approaches. It is obvious that the strict and efficient solution method to obtain the explicit optimal solution has substantial advantages in theory. Most recently, Hasuike [13] developed the exact solution algorithm to obtain the explicit optimal solution for a mathematical programming problem with a single second-order cone. However, this study does not consider box constraints for decision variables, particularly upper limited value of decision variables. Therefore, in this paper, we extend the mathematical model of Hasuike [13] and propose a new solution method based on linear programming to explicitly obtain the strict optimal solution of a SOCP problem with box constraints of decision variables. In order to solve the main SOCP problem, we perform the deterministic equivalent transformations, and show that the main problem is equivalent to a parametric quadratic programming problem. Furthermore, we show some theorems to obtain the strict optimal solution explicitly, and develop the efficient and strict solution method based on linear programming.

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II. Formulation of a SOCP Problem

In this paper, we deal with the following problem with a second-order cone and rectangle constraints of decision variables:

$$\begin{array}{ll} \text{Minimize} & c_0 x_0 + \boldsymbol{c}^t \boldsymbol{x} \\ \text{subject to} & \mathbf{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{p} \leq \boldsymbol{x} \leq \boldsymbol{q}, \\ & \begin{pmatrix} x_0 \\ \boldsymbol{x} \end{pmatrix} \in \mathcal{K}_{n+1} \end{array} \tag{1}$$

where x_0 and $\boldsymbol{x} \in R^n$ are decision variables, and $\boldsymbol{c} \in R^n$, $c_0 > 0$, $\mathbf{A} \in R^{m\times(n+1)}, (m < n)$, $\boldsymbol{b} \in R^m$, $\boldsymbol{p}, \boldsymbol{q} \in R^n$. Then, second-order cone $\mathcal{K}_{n+1}$ is defined by $\mathcal{K}_{n+1} = \left\{ (x_0, \boldsymbol{x}) \in R^{n+1} \,\middle|\, x_0 \geq \sqrt{\sum_{j=1}^{n} x_j^2} \right\}$.

In general, robust programming problems and safety first models in stochastic programming are equivalently transformed into problem (1), and so problem (1) is one of standard SOCP problems. With respect to SOCP problem (1), the efficient solution approach based on pivoting method of linear programming has been proposed (for instance, [12]), but the global convergence has not been completely proved. Furthermore, the optimal solution of the SOCP problem has not been explicitly obtained until now. Therefore, we develop the new solution approach to overcome these disadvantages.

III. Development of the Exact Solution Method for the Proposed SOCP Problem

First, since parameter x_0 is in only objective function and minimizing x_0 is equivalent to minimizing $\sqrt{\sum_{j=1}^{n} x_j^2}$, we transform the main problem (1) into the following problem without the loss of optimality:

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$$\begin{array}{ll} \text{Minimize} & \boldsymbol{c}^t\boldsymbol{x} + c_0\sqrt{\sum\limits_{j=1}^{n} x_j^2} \\ \text{subject to} & \mathbf{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{p} \leq \boldsymbol{x} \leq \boldsymbol{q} \end{array} \tag{2}$$

Problem (2) is a convex programming problem, and so we may find a global optimal solution by using the nonlinear or convex programming approaches. However, in general, nonlinear programming approaches are not efficient than linear programming, and so it is not appropriate to solve the large-scale problems. Furthermore, it is also difficult to represent the exact optimal solution analytically. Therefore, in order to solve this problem analytically and explicitly, we consider the following auxiliary problem P^S introducing a parameter S:

(Problem P^S)

$$\begin{array}{ll} \text{Minimize} & S\boldsymbol{c}^t\boldsymbol{x} + \frac{c_0}{2}\left(\sum\limits_{j=1}^{n} x_j^2\right) \\ \text{subject to} & \mathbf{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{p} \leq \boldsymbol{x} \leq \boldsymbol{q} \end{array} \tag{3}$$

This problem is a quadratic programming problem if parameter S is fixed, and so it is more efficiently solved than the main SOCP problems. Subsequently, with respect to the relation between problem (2) and its auxiliary problem (3), the following theorem holds.

Theorem 1

Let the optimal solution of problem (3) be $(x_0(S), \boldsymbol{x}(S))$. Then, if $S = \sqrt{\sum\limits_{j=1}^{n} x_j(S)^2}$ is satisfied, $(x_0(S), \boldsymbol{x}(S))$ is also the optimal solution of main problem (2).

Proof

We compare Karush-Kuhn-Tucker (KKT) conditions of problem (2) with that of problem (3). KKT condition of problem (2) is as follows:

$$\begin{aligned} L &= \boldsymbol{c}^t\boldsymbol{x} + c_0\sqrt{\sum_{j=1}^{n} x_j^2} + \sum_{i=1}^{m}\lambda_i\left(\sum_{j=1}^{n} a_{ij}x_j - b_i\right) \\ &\quad + \sum_{j=1}^{n} u_j(x_j - q_j) + \sum_{j=1}^{n} v_j(p_j - x_j) \\ \frac{\partial L}{\partial x_j} &= c_j + c_0\frac{x_j}{\sqrt{\sum\limits_{j=1}^{n} x_j^2}} + \sum_{i=1}^{m}\lambda_i a_{ij} + u_j - v_j = 0 \end{aligned} \tag{4}$$

KKT condition of auxiliary problem (3) is also obtained as follows:

$$\begin{aligned} L' &= S\boldsymbol{c}^t\boldsymbol{x} + \frac{c_0}{2}\left(\sum_{j=1}^{n} x_j^2\right) + \sum_{i=1}^{m}\lambda'_i\left(\sum_{j=1}^{n} a_{ij}x_j - b_i\right) \\ &\quad + \sum_{j=1}^{n} u'_j(x_j - q_j) + \sum_{j=1}^{n} v'_j(p_j - x_j) \\ \frac{\partial L'}{\partial x_j} &= Sc_j + c_0x_j + \sum_{i=1}^{m}\lambda'_i a_{ij} + u'_j - v'_j = 0 \end{aligned} \tag{5}$$

With respect to these KKT conditions, we set $\lambda_i = \frac{\lambda'_i}{S}, u_i = \frac{u'_i}{S}, v_i = \frac{v'_i}{S}$, KKT condition of problem (3) is equal to that of problem (2). Therefore, this theorem holds.

This theorem means that we can obtain the optimal solution of main SOCP problem (1) by solving the auxiliary quadratic programming problem (3) setting the appropriate parameter S. Furthermore, let $g(S) = S - \sqrt{\sum\limits_{j=1}^{n} x_j(S)^2}$. Then, the following theorem to determine the appropriate value of S is derived.

Theorem 2

Let the optimal solution to main problem be $(x_0^*, \boldsymbol{x}^*)$ and optimal value $S^* = \sqrt{\sum\limits_{j=1}^{n} x_j^{*2}}$. Then the following relationship holds:

$$\begin{array}{lll} S^* > S & \Leftrightarrow & g(S) > 0 \\ S^* = S & \Leftrightarrow & g(S) = 0 \\ S^* < S & \Leftrightarrow & g(S) < 0 \end{array} \tag{6}$$

Proof

First, we introduce the following two lemmas to prove this theorem.

Lemma1

With respect to S, $\boldsymbol{c}^t\boldsymbol{x}$ is a decreasing function.

Proof

Set $0 < \bar{S} < S$, and optimal solutions $\boldsymbol{x}(S)$ and $\boldsymbol{x}(\bar{S})$ for auxiliary problems P^S and $P^{\bar{S}}$, respectively. Then, the following relations holds based on the optimality of auxiliary problems P^S and $P^{\bar{S}}$:

$$\begin{cases} S\boldsymbol{c}^t\boldsymbol{x}(S) < S\boldsymbol{c}^t\boldsymbol{x}(\bar{S}) \\ \bar{S}\boldsymbol{c}^t\boldsymbol{x}(\bar{S}) < \bar{S}\boldsymbol{c}^t\boldsymbol{x}(S) \end{cases}$$

From the difference of two inequalities,

$$\begin{aligned} &(S - \bar{S})\,\boldsymbol{c}^t\boldsymbol{x}(S) < (S - \bar{S})\,\boldsymbol{c}^t\boldsymbol{x}(\bar{S}) \\ &\Leftrightarrow \quad (S - \bar{S})\left(\boldsymbol{c}^t\boldsymbol{x}(S) - \boldsymbol{c}^t\boldsymbol{x}(\bar{S})\right) < 0 \end{aligned}$$

holds. Since $S - \bar{S} > 0$ holds derived from the assumption, the following relation is derived:

$$\boldsymbol{c}^t\boldsymbol{x}(S) - \boldsymbol{c}^t\boldsymbol{x}(\bar{S}) < 0 \Leftrightarrow \boldsymbol{c}^t\boldsymbol{x}(S) < \boldsymbol{c}^t\boldsymbol{x}(\bar{S}) \tag{7}$$

Therefore, this lemma holds.

Lemma 2

$\sum\limits_{j=1}^{n} x_j^2$ is an increasing function of S.

Proof

In a way similar to Lemma 1, we set $0 < \bar{S} < S$, and optimal solutions $\boldsymbol{x}(S)$ and $\boldsymbol{x}(\bar{S})$ for auxiliary problems P^S and $P^{\bar{S}}$, respectively. Then, with respect to $P^{\bar{S}}$,

$$\begin{aligned} &\bar{S}\boldsymbol{c}^t\boldsymbol{x}(\bar{S}) + \frac{1}{2}\left(\sum_{j=1}^{n}(x_j(\bar{S}))^2\right) \\ &< \bar{S}\boldsymbol{c}^t\boldsymbol{x}(S) + \frac{1}{2}\left(\sum_{j=1}^{n}(x_j(S))^2\right) \end{aligned}$$

Then, due to formula (7) in Lemma 1, the following relation is derived:

$$\begin{aligned}
&0 < \bar{S}\boldsymbol{c}^t\boldsymbol{x}\left(\bar{S}\right) - \bar{S}\boldsymbol{c}^t\boldsymbol{x}\left(S\right) \\
&< \frac{1}{2}\left(\sum_{j=1}^{n} x_j\left(S\right)^2\right) - \frac{1}{2}\left(\sum_{j=1}^{n} x_j\left(\bar{S}\right)^2\right) \\
&\Leftrightarrow \quad 0 < \frac{1}{2}\left(\sum_{j=1}^{n} x_j\left(S\right)^2\right) - \frac{1}{2}\left(\sum_{j=1}^{n} x_j\left(\bar{S}\right)^2\right) \\
&\Leftrightarrow \quad \sum_{j=1}^{n} x_j\left(\bar{S}\right)^2 < \sum_{j=1}^{n} x_j\left(S\right)^2
\end{aligned} \tag{8}$$

Therefore, from assumption $0 < \bar{S} < S$, this lemma holds.

Furthermore, since the feasible region of problem (3) is same as that of problem (2) which is a bounded region and optimal solutions $\boldsymbol{x}\left(S\right)$ are continuous to S, it holds $-\boldsymbol{c}^t\boldsymbol{x}\left(S\right) > -\infty$, $\sum_{j=1}^{n} x_j\left(S\right)^2 < \infty$. Therefore, $\sum_{j=1}^{n} x_j^2$ is also a continuous function to S, and $g\left(0\right) < g\left(S^*\right) < g\left(\infty\right)$ holds. Consequently, from Lemmas 1 and 2, the Mean value theorem and uniqueness of $\boldsymbol{x}$, Theorem 2 is derived.

This theorem means that we obtain the strict optimal solution $x_0^*, \boldsymbol{x}^*$, and the strict value of S by updating S according to the value of $g\left(S\right)$. Subsequently, with respect to each value of S, we consider the following simultaneous equations derived from KKT conditions of auxiliary problem P^S:

$$\begin{array}{l}
Sc_j + c_0x_j + \sum_{i=1}^{m} \lambda_i' a_{ij} + u_j' - v_j' = 0,\ (j = 1, 2, ..., n) \\
\sum_{j=1}^{n} a_{ij}x_j = b_i, (i = 1, 2, ..., m) \\
u_j'\left(x_j - q_j\right) = 0,\ (j = 1, 2, ..., n) \\
v_j'\left(p_j - x_j\right) = 0,\ (j = 1, 2, ..., n)
\end{array} \tag{9}$$

We solve the simultaneous equations and find the optimal solution to each S using Wolfe's method [14], which is based on linear programming and pivoting method. Therefore, updating S using the iterative solution method such as the bi-section algorithm and considering Theorem 2, we obtain the optimal solution of main problem (1).

Furthermore, in order to obtain the explicit optimal solution by using finite iteration with respect to updating of S, we show the following theorem.

Theorem 3

With respect to auxiliary problem P^S, the range $[S_L, S_U]$ with same active constraints of P^S including S is unique.

Proof

With respect to auxiliary problem P^S, we consider the following case where active constraints of P^S are known:

$$\begin{array}{ll}
\text{Minimize} & S\boldsymbol{c}^t\boldsymbol{x} + \frac{c_0}{2}\left(\sum_{j=1}^{n} x_j^2\right) \\
\text{subject to} & \mathbf{A}\boldsymbol{x} = \boldsymbol{b}, \\
& x_i = p_i,\ (i \in J_p) \\
& x_i = q_i,\ (i \in J_q) \\
& p_i < x_i < q_i,\ (i \notin J_p, J_q)
\end{array} \tag{10}$$

where J_p and J_q are the index sets of active constraints $x_i = p_i$ and $x_i = q_i$, respectively. Then, this KKT conditions are represented as follows:

$$\begin{array}{l}
\frac{\partial L'}{\partial x_j} = Sc_j + c_0x_j + \sum_{i=1}^{m} \lambda_i' a_{ij} - u_j' = 0 \\
\lambda_i' = 0,\ \sum_{j=1}^{n} a_{ij}x_j = b_i, (i = 1, 2, ..., m) \\
x_i = p_i, u_i' \geq 0,\ (i \in J_p) \\
x_i = q_i, v_i' \geq 0,\ (i \in J_q) \\
u_i' = 0, v_i' = 0,\ (i \notin J_p, J_q)
\end{array} \tag{11}$$

Furthermore, focusing on decision variables $p_i < x_i < q_i$, $(i \notin J_p, J_q)$, we consider the following revised KKT conditions:

$$\begin{array}{l}
\frac{\partial L'}{\partial x_j} = Sc_j + c_0x_j + \sum_{i=1}^{m} \lambda_i' a_{ij} = 0,\ (j \notin J_p, J_q) \\
\sum_{j \notin J_p, J_q} a_{ij}x_j = b_i', (i = 1, 2, ..., m) \\
b_i' = b_i - \sum_{j \in J_p} a_{ij}p_j - \sum_{j \in J_q} a_{ij}q_j
\end{array} \tag{12}$$

Since the total number of decision variables x_i and λ_i' in the KKT except for parameter S is equal to that of equations and all equations are linear, the solutions x_i and λ_i' of the KKT conditions are represented as the following linear equations with respect to S:

$$\begin{aligned}
&\left\{\begin{array}{l} Sc_j + c_0x_j + \sum_{i=1}^{m} \lambda_i' a_{ij} = 0,\ (j \notin J_p, J_q) \\ \sum_{j \notin J_p, J_q} a_{ij}x_j = b_i', (i = 1, 2, ..., m) \end{array}\right. \\
&\Leftrightarrow \quad \mathbf{D}\begin{pmatrix} \boldsymbol{\lambda}' \\ \boldsymbol{x}' \end{pmatrix} = \begin{pmatrix} -S\boldsymbol{c} \\ \boldsymbol{b}' \end{pmatrix}, \\
&\qquad \mathbf{D} = \begin{pmatrix} \mathbf{A}^t & \mathbf{E}_{c_0} \\ \mathbf{0} & \mathbf{A} \end{pmatrix}, \\
&\qquad \mathbf{E}_{c_0} = \begin{pmatrix} c_0 & 0 & \cdots & 0 \\ 0 & c_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_0 \end{pmatrix} \\
&\Leftrightarrow \quad \begin{pmatrix} \boldsymbol{\lambda}' \\ \boldsymbol{x}' \end{pmatrix} = \mathbf{D}^{-1}\begin{pmatrix} -S\boldsymbol{c} \\ \boldsymbol{b}' \end{pmatrix} \\
&\qquad = \mathbf{D}^{-1}\begin{pmatrix} \mathbf{0} \\ \boldsymbol{b}' \end{pmatrix} - S\mathbf{D}^{-1}\begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{b}' \end{pmatrix} \\
&\qquad = \boldsymbol{\alpha} - S\boldsymbol{\beta} \\
&\Leftrightarrow \quad \left\{\begin{array}{l} \lambda_i = \alpha_k - S\beta_k \\ x_j = \alpha_j - S\beta_j,\ (j \notin J_0) \end{array}\right.
\end{aligned} \tag{13}$$

Furthermore, if there exists another S' including another range $[S_L', S_U']$ with same active constraints in problem (10), the optimal solution is obtained as $\hat{x}_j' = \alpha_j - S'\beta_j$, $(j \notin J_p, J_q)$ from the same KKT condition in a way similar to (13). However, it contradicts with uniqueness of optimal solution since problem (10) is a convex programming problem and the solution derived from the KKT condition is unique. Consequently, it is clear to obtain the unique range $[S_L, S_U]$ of parameter S.

Finally, considering the following relation to the optimality of parameter S in Theorem 2;

$$S^2 - \left(\sum_{j \notin J_0} x_j^2 \right) = 0$$

we obtain the following solution substituting the solutions in (13):

$$\begin{aligned}
&S^2 - \sum_{j \notin J_p, J_q} (\alpha_j - S\beta_j)^2 = 0, \\
&\Leftrightarrow \quad S^2 - \left(\sum_{j \notin J_p, J_q} \alpha_j^2 \right) + \left(2 \sum_{j \notin J_p, J_q} \alpha_j \beta_j \right) S \\
&\qquad - \left(\sum_{j \notin J_p, J_q} \beta_j^2 \right) S^2 = 0 \\
&\Leftrightarrow \quad S^* = \begin{cases} \dfrac{\sum_{j \notin J_p, J_q} \alpha_j^2}{2 \sum_{j \notin J_p, J_q} \alpha_j \beta_j}, \left(\sum_{j \notin J_p, J_q} \beta_j^2 = 1 \right) \\ \dfrac{\sum_{j \notin J_p, J_q} \alpha_j \beta_j \pm \sqrt{D(\alpha, \beta)}}{\left(\left(\sum_{j \notin J_p, J_q} \beta_j^2 \right) - 1 \right)}, \\ \left(\sum_{j \notin J_p, J_q} \beta_j^2 \neq 1 \right) \end{cases} \\
&\qquad D(\alpha, \beta) = \left(\sum_{j \notin J_p, J_q} \alpha_j \beta_j \right)^2 \\
&\qquad\qquad - \left(\left(\sum_{j \notin J_p, J_q} \beta_j^2 \right) - 1 \right) \left(\sum_{j \notin J_p, J_q} \alpha_j^2 \right)
\end{aligned} \tag{14}$$

If $\left(\sum_{j \notin J_p, J_q} \beta_j^2 \right) - 1 < 0$, the optimal value of S is automatically solved as follows due to $S > 0$:

$$S = \frac{\sum_{j \notin J_p, J_q} \alpha_j \beta_j - \sqrt{D(\alpha, \beta)}}{\left(\left(\sum_{j \notin J_p, J_q} \beta_j^2 \right) - 1 \right)}$$

If $\left(\sum_{j \notin J_p, J_q} \beta_j^2 \right) - 1 > 0$, considering $[S_L, S_U]$ including optimal value S^*, we can determine the plus or minus sign in the optimal solution. Therefore, from the determination of this optimal value S^*, we obtain the strict optimal solution explicitly. As a result of the discussion, we develop the following strict solution method.

Solution algorithm

STEP1: Set the initial value of parameter $S_L \leftarrow 0$, $S_U \leftarrow M$ where M is a sufficient large value and two families of index sets to active constraint for auxiliary problem P^S as $FIS_L \leftarrow \phi$ and $FIS_U \leftarrow \phi$.

STEP2: Set $S \leftarrow \dfrac{S_L + S_U}{2}$.

STEP3: Solve auxiliary problem P^S using the Wolfe method and obtain the optimal solutions $\boldsymbol{x}(S)$.

STEP4: Calculate $g(S) = S - \sqrt{\sum_{j=1}^{n} (x_j(S))^2}$. If $g(S) = 0$, $\boldsymbol{x}(S)$ are optimal solutions of the main problem and terminate this algorithm. If $g(S) > 0$, $S_L \leftarrow S$, update FIS_L to the present index sets of active constrains, and go to STEP5. If $g(S) < 0$, $S_U \leftarrow S$, update FIS_U to the present index sets of active constrains, and go to STEP5.

STEP5: If $FIS_L = FIS_U$, go to STEP 6. If are not, return to STEP 2.

STEP6: Solve the following simultaneous linear equations derived from the KKT condition;

$$\begin{aligned}
&Sc_j + c_0 x_j + \sum_{i=1}^{m} \lambda'_i a_{ij} = 0, \; (j \notin J_p, J_q) \\
&\sum_{j \notin J_p, J_q} a_{ij} x_j = b'_i, \; (i = 1, 2, ..., m) \\
&\left(b'_i = b_i - \sum_{j \in J_p} a_{ij} p_j - \sum_{j \in J_q} a_{ij} q_j \right)
\end{aligned}$$

and obtain solutions $x_j = \alpha_j S + \beta_j$.

STEP7: Solve the following quadratic equations of S:

$$S^2 - \left(\sum_{j \notin J_p, J_q} x_j^2 \right) = 0$$

and obtain the optimal value S^*. Then, substitute S^* into x_j in STEP6 and obtain the optimal solution x_j^*.

Our solution method converges within finite iteration since both constraints in the main problem and ranges $[S_L, S_U]$ with same active constraints for P^S are finite. Then, it is also efficient since it bases on linear programming and one quadratic equation. Furthermore, some types of the main SOCP problem (1) can be explicitly solved by our proposed approach. Therefore, our solution approach has the advantage comparing other solution methods based on linear programming and interior-point algorithm. Then, the solution method may be applied to more general SOCP problems by extending some procedures. Furthermore, in (13), since elements of matrix $\mathbf{D}$ are sparse, it may be possible to develop more efficient solution method using this sparseness.

IV. Conclusion

In this paper, we have considered a SOCP problem with box constrains of decision variables based on stochastic or robust programming problem, and developed the strict solution algorithm based on KKT conditions and equivalently transformations. Using the proposed solution algorithm, we have obtained the exact optimal solution explicitly. However, the main problem is still a restricted SOCP problem, and so it may be difficult to apply this solution algorithm to general SOCP problems directly. Therefore, as a future work, we will develop more general solution algorithm to find the exact optimal solution explicitly.

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