Convergence Accelerator Approach to the Numerical Evaluation of Hilbert Transforms Based on Expansions in Hermite Functions

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Abstract—**The evaluation of the Hilbert transform of a function is a very important problem that appears widely in science and engineering, specially in signal analysis. In this work, the numerical evaluation of a class of Hilbert transforms using an expansion in terms of Hermite functions, eigenfunctions of the Fourier transform, is performed. Judicious selection of convergence accelerators allows for the efficient evaluation of the resulting series. The approach is particularly accurate for functions having a Gaussian-like asymptotic behavior. For more slowly decreasing functions, the accuracy of the evaluation decreases.**

Index Terms—**Hilbert-transform, Hermite functions, convergence accelerators.**

I. Introduction

HILBERT transforms occur widely in a variety of problems in science and engineering [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], including applications in the treatment of nonlinear waves, dispersion relations in optical data analysis, scattering problems, and electrocardiography.

Gaussian-type functions play a special role in signal analysis, since they are localized in both the time and frequency domains. The construction of the Hilbert transform of many signals such as Gaussian-type pulses is performed in order to cancel negative frequency components, and the transform is most often carried out numerically [2]. An interesting application of the Hilbert transform of Gaussian functions occurs in the analysis of the heat equation [11]. As a consequence in part, the numerical evaluation of Hilbert transforms has been studied intensely using a number of different approaches [3], [12], [13], [14], [15], [16], [17], [18], [19]. A condensed version of this work was presented at the World Congress of Engineering [20].

The Hilbert transform of a function f, denoted Hf, is defined by

$$(Hf)(x) = \frac{1}{\pi}\, P\int_{-\infty}^{\infty} \frac{f(s)}{x-s}\, ds, \tag{1}$$

where P designates the Cauchy principal value, which can be expressed as

$$(Hf)(x) = \frac{1}{\pi}\lim_{\epsilon\to 0^+}\left[\int_{-\infty}^{x-\epsilon} \frac{f(s)}{x-s}\, ds + \int_{x+\epsilon}^{\infty} \frac{f(s)}{x-s}\, ds\right]. \tag{2}$$

The Hilbert transform is also defined using the opposite sign convention to that given in Eq. (1). H is a linear operator from $L^p(\mathbb{R}) \to L^p(\mathbb{R})$, for $1 < p < \infty$ [3], [21], [22], where $L^p(\mathbb{R})$ denotes the Banach space of Lebesgue integrable functions:

$$L^p(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C}\,,\, \int_{-\infty}^{\infty} |f(x)|^p\, dx < \infty \right\}. \tag{3}$$

The most important case occurs for the Hilbert space $L^2(\mathbb{R})$. If $f(x)$ is an even function, $f(-x) = f(x)$, then Eq. (1) can be expressed as

$$(Hf)(x) = \frac{2x}{\pi}\, P\int_{-\infty}^{\infty} \frac{f(s)}{x^2-s^2}\, ds, \tag{4}$$

and if $f(x)$ is an odd function, $f(-x) = -f(x)$, then

$$(Hf)(x) = \frac{2}{\pi}\, P\int_{-\infty}^{\infty} \frac{sf(s)}{x^2-s^2}\, ds. \tag{5}$$

Equations (4,5) are often referred to as the Kramers-Kronig transforms of even and odd functions, respectively.

A comment on notation is appropriate: $(Hf(s))(x)$ indicates the Hilbert transform of f evaluated at the point x, and s is the dummy integration variable. This is written more concisely as $(Hf)(x)$, when there is no need to specify the integration variable. If there is no risk of confusion, we will write $H[f(x)]$ for the Hilbert transform when the functional form is specified.

There is continuing interest in the development of accurate numerical methods for the evaluation of Hilbert transforms and other related singular integrals [13], [14], [15], [16], [17], [18], [23], [24], [25], [26], [27], [28], [29]. There is also an extensive body of work devoted to the numerical determination of the Kramers-Kronig transforms [25].

Weideman [27] studied the numerical evaluation of the Hilbert transform of a function $f \in L^2(\mathbb{R})$ using an expansion technique in terms of the eigenfunctions of the Hilbert transform operator, which can be written as:

$$\varphi_n(x) = \frac{(1+ix)^n}{(1-ix)^{n+1}}\,, \quad \text{for } n \in \mathbb{Z}. \tag{6}$$

Expanding $f(x)$ in terms of this orthonormal basis set, the following result was found:

$$(Hf)(x) = -ib_0\varphi_0(x) - i\sum_{n=-\infty}^{\infty}{}'\; b_n\, \mathrm{sgn}(n)\varphi(n), \tag{7}$$

Manuscript received August 16, 2011. Support from the National Science Foundation, the Donors of the Petroleum Research Fund, administered by the American Chemistry Society, and the Office of University Research, University of Wisconsin-Eau Claire, are greatly appreciated. We also thank the Spanish Ministerio de Educación, which provided financial assistance for the visit of I. Porras (PR2007-0456), who also acknowledges funding from project FPA2009-14091-C02-02.

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where the prime in the summation signifies that the term with $n = 0$ is omitted, and the coefficients b_n are given by:

$$b_n = \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi_n^*(x) f(x)\, dx. \tag{8}$$

Weideman examined several examples, for which he obtained the coefficients b_n. For the case $f(x) = 1/(1 + x^2)$, the results are exact, since the function can be expanded in a compact closed form in terms of the eigenfunctions in Eq. (6). For functions that decay in a significantly different manner from this example, the eigenfunction decomposition approach of Weideman yielded more slowly convergent series.

II. Theory

In this work an alternative to the Weideman scheme is investigated. Our focus is on more rapidly decaying functions, which arise in several important areas [1], [2], [3]. Two similar methods are presented here for the numerical evaluation of the Hilbert transform in terms of a numerical series.

In our first approach, the Hilbert transform is expressed in terms of an expansion of Hermite functions, which are eigenfunctions of the Fourier transform operator. We start by expanding our function of interest (assumed of $L^2(\mathbb{R})$) as

$$f(x) = \sum_{n=0}^{\infty} \alpha_n u_n(x), \tag{9}$$

where $u_n(x)$ are the orthonormal Hermite functions, defined in terms of the standard Hermite polynomials $H_n(x)$ by:

$$u_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-x^2/2}, \tag{10}$$

and the coefficients α_n are determined from:

$$\alpha_n = \int_{-\infty}^{\infty} f(x) u_n(x)\, dx. \tag{11}$$

An analysis of convergence issues associated with the expansion in Eq. (9) can be found in the work of Boyd [30].

We define the operator

$$\mathcal{T} = \operatorname{sgn} x\, \mathcal{F}, \tag{12}$$

where $\operatorname{sgn} x$ denotes the signum function:

$$\operatorname{sgn} x = \begin{cases} 1 & , \quad x > 0 \\ 0 & , \quad x = 0 \\ -1 & , \quad x < 0 \end{cases}, \tag{13}$$

and $\mathcal{F}$ stands for the Fourier transform operator, a unitary operator on $L^2(\mathbb{R})$. It is obvious that $\mathcal{T}$ is a linear isometric (bounded) operator on $L^2(\mathbb{R})$, therefore it is continuous. The action of $\mathcal{T}$ on the function $f(x)$ can be also expanded in the same basis set:

$$\mathcal{T} f(x) = \sum_{m=0}^{\infty} \mu_m u_m(x), \tag{14}$$

where the coefficients μ_m can be expressed as:

$$\mu_m = \int_{-\infty}^{\infty} \operatorname{sgn} x\, (\mathcal{F} f)(x) u_m(x)\, dx. \tag{15}$$

Applying $\mathcal{T}$ to Eq. (9), it follows that:

$$\operatorname{sgn} x\, (\mathcal{F} f)(x) = \sum_{n=0}^{\infty} \alpha_n \operatorname{sgn} x\, (\mathcal{F} u_n)(x), \tag{16}$$

where we have used the fact that $\mathcal{T}$ is linear and continuous. The basis functions $u_n(x)$ satisfy [31]:

$$(\mathcal{F} u_n)(x) = (-i)^n u_n(x) \tag{17}$$

and

$$(\mathcal{F}^{-1} u_n)(x) = i^n u_n(x). \tag{18}$$

Employing Eq. (17) allows Eq. (16) to be simplified to

$$\operatorname{sgn} x\, (\mathcal{F} f)(x) = \operatorname{sgn} x \sum_{n=0}^{\infty} \alpha_n (-i)^n u_n(x). \tag{19}$$

On substituting Eq. (19) into Eq. (15) leads to the following result for the coefficients μ_m:

$$\mu_m = \sum_{n=0}^{\infty} \alpha_n (-i)^n \int_{-\infty}^{\infty} \operatorname{sgn} x\, u_m(x) u_n(x)\, dx, \tag{20}$$

which can be rearranged, using the property $u_n(-x) = (-1)^n u_n(x)$, to yield

$$\mu_m = \sum_{n=0}^{\infty} \alpha_n (-i)^n \left[1 - (-1)^{n+m}\right] \int_0^{\infty} u_m(x) u_n(x)\, dx. \tag{21}$$

The non-vanishing terms in the previous sum are those for which $n + m$ is odd. Let $m = 2k$ and $n = 2j + 1$, for any k, j non-negative integers, then

$$\mu_{2k} = -i \sum_{j=0}^{\infty} \alpha_{2j+1} (-1)^j I_{k,j} \tag{22}$$

and if $m = 2k + 1$ and $n = 2j$, then

$$\mu_{2k+1} = \sum_{j=0}^{\infty} \alpha_{2j} (-i)^j I_{j,k}, \tag{23}$$

where $I_{k,j}$ is defined by

$$I_{k,j} = 2 \int_0^{\infty} u_{2k}(x) u_{2j+1}(x)\, dx. \tag{24}$$

Using the key connection between the Fourier and Hilbert transforms [3], [21], [22],

$$(\mathcal{F} H f)(x) = -i \operatorname{sgn} x\, (\mathcal{F} f)(x), \tag{25}$$

it follows from Eq. (14) and employing Eq. (18), that

$$\begin{aligned} (Hf)(x) &= -i \mathcal{F}^{-1} \left(\sum_{m=0}^{\infty} \mu_m u_m(y) \right)(x) \\ &= -i \sum_{m=0}^{\infty} \mu_m i^m u_m(x). \end{aligned} \tag{26}$$

Rewriting Eq. (26) to account for the even and odd contributions of m and substituting Eqs. (22) and (23), leads to the final result:

$$\begin{aligned} (Hf)(x) &= -\sum_{k=0}^{\infty} (-1)^k u_{2k}(x) \sum_{j=0}^{\infty} (-1)^j \alpha_{2j+1} I_{k,j} \\ &+ \sum_{k=0}^{\infty} (-1)^k u_{2k+1}(x) \sum_{j=0}^{\infty} (-1)^j \alpha_{2j} I_{j,k}. \end{aligned} \tag{27}$$

The $I_{k,j}$ integral can be evaluated from Eq. (24) by using a relationship between Hermite polynomials and the associated Laguerre polynomials $L_j^{(\alpha)}(x)$, and employing the change of variable $t = x^2$, to obtain

$$I_{k,j} = \frac{(-1)^{k+j} 2^{k+j+1/2} k! j!}{\sqrt{(2k)!(2j+1)!\,\pi}} \int_0^\infty L_k^{(-1/2)}(t) L_j^{(1/2)}(t) e^{-t} dt. \tag{28}$$

The integral in Eq. (28) can be solved in terms of the gamma function $\Gamma(z)$, by writing both polynomials as a series in $L_m^{(0)}$ and then employing the orthogonality relation for Laguerre polynomials, to yield

$$\int_0^\infty L_k^{(-1/2)}(t) L_j^{(1/2)}(t)\, e^{-t}\, dt = \frac{2\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(j+\frac{3}{2}\right)}{\pi(2j-2k+1)\Gamma(k+1)\Gamma(j+1)}, \tag{29}$$

so that

$$I_{k,j} = \frac{(-1)^{k+j} 2^{k+j+3/2}\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(j+\frac{3}{2}\right)}{\pi^{3/2}(2j-2k+1)\sqrt{(2k)!(2j+1)!}}. \tag{30}$$

The preceeding formula can be expressed in terms of double factorials as

$$I_{k,j} = \frac{(-1)^{k+j}\sqrt{2}(2k-1)!!(2j+1)!!}{(2j-2k+1)\sqrt{\pi(2k)!(2j+1)!}}. \tag{31}$$

Equation (27) reflects the fact that the Hilbert transform of an even function is an odd function and *vice versa*. When the function $f(x)$ is even or odd, the right-hand side reduces to only one double sum because every α_{2k+1} or α_{2k}, respectively, is zero. Furthermore, for functions for which all α_m are zero for m greater than a finite value, the second summation of both terms is a finite sum.

The second numerical approach investigated involves applying the Hilbert transform directly to Eq. (9), to obtain

$$(Hf)(x) = \sum_{n=0}^{\infty} \alpha_n H\left[u_n(x)\right]. \tag{32}$$

where we have made use of the fact that H is a continuous operator on $L^2(\mathbb{R})$. Therefore, the right-hand side converges in the norm sense. If we expand explicitely the Hermite polynomials, we can write:

$$\begin{aligned}(Hf)(x) &= \frac{1}{\pi^{1/4}}\sum_{n=0}^{\infty} 2^n\sqrt{(2n)!}\,\alpha_{2n}\chi_n^{(1)} \\ &+ \frac{\sqrt{2}}{\pi^{1/4}}\sum_{n=0}^{\infty} 2^n\sqrt{(2n+1)!}\,\alpha_{2n+1}\chi_n^{(2)}\end{aligned} \tag{33}$$

where

$$\chi_n^{(1)} = \sum_{m=0}^{n}\frac{(-1)^m H\left[x^{2(n-m)}e^{-x^2/2}\right]}{4^m m!(2n-2m)!} \tag{34}$$

and

$$\chi_n^{(2)} = \sum_{m=0}^{n}\frac{(-1)^m H\left[x^{2(n-m)+1}e^{-x^2/2}\right]}{4^m m!(2n-2m+1)!}. \tag{35}$$

The Hilbert transforms $H[x^{2(n-m)}e^{-x^2/2}]$ and $H[x^{2(n-m)+1}e^{-x^2/2}]$, using the recursive relation ([4], App. 1)

$$H\left[t^n f(t)\right](x) = xH\left[t^{n-1}f(t)\right](x) - \frac{1}{\pi}\int_{-\infty}^{\infty} t^{n-1}f(t)dt, \tag{36}$$

TABLE I

EXACT HILBERT TRANSFORMS FOR THE FUNCTIONS INVESTIGATED NUMERICALLY.

Case	$f(x)$	$(Hf)(x)$
1	e^{-ax^2} $(a>0)$	$G(a,x)$
2	xe^{-ax^2} $(a>0)$	$xG(a,x) - \frac{1}{\sqrt{a\pi}}$
3	$x^2e^{-ax^2}$ $(a>0)$	$x^2G(a,x) - \frac{1}{x\sqrt{a\pi}}$
4	$\cos(bx)e^{-ax^2}$ $(a>0, b\geq 0)$	$e^{-ax^2}\mathrm{Im}\left\{e^{ibx}\mathrm{erf}\left[\sqrt{a}\left(\frac{b}{2a}+ix\right)\right]\right\}$
5	$e^{-a\lvert x\rvert}$ $(a>0)$	$\frac{\mathrm{sgn}\,x}{\pi}\left[e^{a\lvert x\rvert}E_1(a\lvert x\rvert) + e^{-a\lvert x\rvert}\mathrm{Ei}(a\lvert x\rvert)\right]$
6	$\frac{1}{1+x^2}$	$\frac{x}{1+x^2}$

can be reduced in closed form to the function $H[e^{-x^2/2}]$, and the latter can be evaluated [3], [4] as:

$$H[e^{-x^2/2}] = -ie^{-x^2/2}\mathrm{erf}(ix/\sqrt{2}), \tag{37}$$

where $\mathrm{erf}(z)$ denotes the error function, defined by ([32], p. 297)

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\, dt. \tag{38}$$

One of the strengths of this approach occurs when the function of interest can be expanded in a finite series using Eq. (9), then the Hilbert transform can be computed accurately. Only one infinite series is encountered for a general function of interest, not taking into account the particular evaluation strategy for $H\left[x^k e^{-x^2/2}\right]$, whereas in the Fourier transform approach, two infinite series will be encountered in many cases.

III. COMPUTATIONAL APPROACH

For the numerical evaluation of the Hilbert transform we have selected the test functions displayed in Table I. These functions show a range of different asymptotic decays and their Hilbert transforms can be evaluated analytically, and therefore provide useful test cases for numerical comparison.

For notational compactness $G(a,x)$ is used to denote the Hilbert transform $H[e^{-ax^2/2}]$, for $a > 0$, and can be expressed as

$$G(a,x) = -ie^{-ax^2}\mathrm{erf}(i\sqrt{a}x). \tag{39}$$

The special functions appearing in Table I are the exponential integral functions, defined by (see for example [32], p. 228)

$$\mathrm{Ei}(z) = -P\int_{-z}^{\infty}\frac{e^{-t}}{t}\, dt, \tag{40}$$

and

$$E_1(z) = \int_1^\infty \frac{e^{-zt}}{t}\, dt, \tag{41}$$

and Im denotes the imaginary part.

For the test functions employed, the corresponding expansion coefficients α_n obtained from Eq. (11) are tabulated in Table II. In these cases, they can be evaluated in closed form, which is an obvious advantage for the application of our method, and we will use them in the computations reported. However, this approach does not require that the expansion coefficients be known explicitely, as they can be determined numerically by Gauss-Hermite quadrature or other numerical integration methods in a straightforward manner. In this table, $\Gamma(a,z)$ denotes the incomplete gamma function and ${}_1F_1(a;b;z)$ designates the Kummer confluent hypergeometric function.

In Table II, we have included some particular cases of special relevance for the functions of Table I, as well as the expansion coefficients for more general expressions in other cases. It is obvious that if the function has a well defined parity, only non-zero coefficients with either an even or odd index arise.

We note parenthetically that it is easy to find examples where the α_m expansion coefficients can be determined analytically, but the Hilbert transform cannot be evaluated in a simple closed form, for example:

$$f(x) = \log|x| e^{-ax^2}, \qquad \text{for } a > 0. \tag{42}$$

The present approach is demonstrated by first considering the simple example

$$f(x) = x^{2m} e^{-x^2/2}. \tag{43}$$

Since this function is even, it can be expanded using Eq. (9) as

$$f(x) = \sum_{n=0}^{\infty} \alpha_{2n} u_{2n}(x). \tag{44}$$

Thus, Eq. (27) simplifies to:

$$(Hf)(x) = \sum_{k=0}^{\infty} (-1)^k u_{2k+1}(x) \sum_{j=0}^{m} (-1)^j \alpha_{2j} I_{j,k}. \tag{45}$$

Let us consider the particular test case $m = 1$, then the expansion coefficients from Table II are $\alpha_0 = \pi^{1/4}/2$, $\alpha_2 = \pi^{1/4}/2^{1/2}$, $\alpha_{2n} = 0$ for $n > 1$, and hence

$$(Hf)(x) = \sqrt{\frac{\sqrt{\pi}}{2}} \sum_{k=0}^{\infty} (-1)^k \left(\sqrt{2} I_{0,k} - I_{1,k}\right) u_{2k+1}(x). \tag{46}$$

As a second example, consider

$$f(x) = x^{2m+1} e^{-x^2/2}. \tag{47}$$

In an analogous fashion, the expansion coefficients can be found from Table II, and Eq. (27) simplifies to

$$(Hf)(x) = -\sum_{k=0}^{\infty} (-1)^k u_{2k}(x) \sum_{j=0}^{m} (-1)^j \alpha_{2j+1} I_{k,j}. \tag{48}$$

In the two preceeding examples, the functions can be written as a finite series using Eq. (9), and the resulting Hilbert transform contains a single infinite series.

A slightly more generalized example is the following function where $a > 0$, and $a \neq 1/2$,

$$f(x) = x^{2m} e^{-ax^2}, \tag{49}$$

and the expansion coefficients are displayed in Table II. The resulting Hilbert transform now involves two infinite series. However, it should be noted that a rescaling of the expansion variable leads to the inner series expansion being a finite sum, which yields a more efficient numerical approach.

That is, the Hilbert transform can be evaluated using

$$(Hf)(x) = \left(\frac{1}{2a}\right)^m H\left[t^{2m} e^{-t^2/2}\right]\left(\sqrt{2a}x\right), \tag{50}$$

and the expansion in Eq. (45).

For the preceding examples in Eqs. (43), (47) and (49), the Hilbert transform can be found in terms of a single infinite series, which is most effectively evaluated by applying convergence acceleration techniques, when direct summation is too inaccurate. We now focus on the general case where a finite series for Eq. (9) cannot be obtained, resulting in a double infinite series. To apply a convergence accelerator to the general case we make use of an interchange of order of summation [33] for Eq. (27), transforming the double infinite series to a combination of finite and infinite series of the form:

$$\begin{aligned}(Hf)(x) &= -\sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} \alpha_{2j+1} I_{k-j,j}\, u_{2(k-j)}(x) \\ &+ \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} \alpha_{2j} I_{j,k-j}\, u_{2(k-j)+1}(x).\end{aligned} \tag{51}$$

Five different convergence accelerators were tested: these were the Levin-u [34] and Levin-t' [35], [36] transformations, the Weniger-1 and Weniger-2 [36], [37], [38], [39] transformations, and the Wynn ϵ-algorithm [40], [41]. These convergence accelerators are given by the following formulas. Each provides an estimate depending on the index k that converges to the value of the sum with increasing k. The Levin-u formula is:

$$u_k = \frac{\displaystyle\sum_{j=0}^{k} (-1)^k \frac{(j+1)^{k-2}}{j!(k-j)!} \frac{S_{j+1}}{A_{j+1}}}{\displaystyle\sum_{j=0}^{k} (-1)^k \frac{(j+1)^{k-2}}{j!(k-j)!} \frac{1}{A_{j+1}}}, \tag{52}$$

the Levin-u formula is:

$$t'_k = \frac{\displaystyle\sum_{j=0}^{k} (-1)^k \frac{(j+1)^{k-1}}{j!(k-j)!} \frac{S_{j+1}}{A_{j+2}}}{\displaystyle\sum_{j=0}^{k} (-1)^k \frac{(j+1)^{k-1}}{j!(k-j)!} \frac{1}{A_{j+2}}}, \tag{53}$$

the Weniger-1 formula is:

$$w_k^{(1)} = \frac{\displaystyle\sum_{j=0}^{k} (-1)^k \frac{(-j-1)_{k-1}}{j!(k-j)!} \frac{S_{j+1}}{A_{j+1}}}{\displaystyle\sum_{j=0}^{k} (-1)^k \frac{(-j-1)_{k-1}}{j!(k-j)!} \frac{1}{A_{j+1}}}, \tag{54}$$

and the Weniger-2 formula is:

$$w_k^{(2)} = \frac{\displaystyle\sum_{j=0}^{k} (-1)^k \frac{(j+1)_{k-1}}{j!(k-j)!} \frac{S_{j+1}}{A_{j+1}}}{\displaystyle\sum_{j=0}^{k} (-1)^k \frac{(j+1)_{k-1}}{j!(k-j)!} \frac{1}{A_{j+1}}}. \tag{55}$$

TABLE II
EXPANSION COEFFICIENTS OF DIFFERENT TEST FUNCTIONS.

$f(x)$	Expansion coefficients α_{2n} or α_{2n+1}
$x^{2m}e^{-x^2/2}$	$\alpha_{2n} = \begin{cases} \dfrac{(2m)!\pi^{1/4}}{2^{2m-n}(m-n)!\sqrt{(2n)!}} & ,n \le m \\ 0 & ,n > m \end{cases}$
$x^{2m+1}e^{-x^2/2}$	$\alpha_{2n+1} = \begin{cases} \dfrac{(2m+1)!\pi^{1/4}}{2^{2m-n+\frac{1}{2}}(m-n)!\sqrt{(2n+1)!}} & ,n \le m \\ 0 & ,n > m \end{cases}$
$x^{2m}e^{-ax^2}$ $(a > 0)$	$\alpha_{2n} = \dfrac{\sqrt{(2n)!\sqrt{\pi}}}{2^{2m+n}\left(a+\frac{1}{2}\right)^{n+m+\frac{1}{2}}} \sum_{j=0}^{n} \dfrac{(-1)^j(2m+2n-2j)!\left(a+\frac{1}{2}\right)^j}{j!(m+n-j)!(2n-2j)!}$
$x^{2m+1}e^{-ax^2}$ $(a > 0)$	$\alpha_{2n+1} = \dfrac{\sqrt{(2n+1)!\sqrt{\pi}}}{2^{2m+n+\frac{3}{2}}\left(a+\frac{1}{2}\right)^{n+m+\frac{3}{2}}} \sum_{j=0}^{n} \dfrac{(-1)^j(2m+2n+2-2j)!\left(a+\frac{1}{2}\right)^j}{j!(m+n+1-j)!(2n+1-2j)!}$
$\cos(bx)e^{-ax^2}$ $(a > 0, b \ge 0)$	$\alpha_{2n} = \dfrac{\sqrt{(2n)!\sqrt{\pi}}}{2^{n}\left(a+\frac{1}{2}\right)^{n+\frac{1}{2}}} \sum_{j=0}^{n} \dfrac{(-1)^j\left(a+\frac{1}{2}\right)^j \; {}_1F_1\left(\frac{1}{2}-j+n;\frac{1}{2};\frac{-b^2}{2(1+2a)}\right)}{j!(n-j)!}$
$e^{-a\|x\|}$ $(a > 0)$	$\alpha_{2n} = \dfrac{2^{2n+\frac{1}{2}}\sqrt{(2n)!}}{\pi^{1/4}} \sum_{j=0}^{n} \dfrac{(-1)^j}{8^j j!(2n-2j)!}\left[\Gamma\left(n-j+\dfrac{1}{2}\right){}_1F_1\left(n-j+\dfrac{1}{2};\dfrac{1}{2};\dfrac{a^2}{2}\right) - \sqrt{2}a\Gamma(n-j+1){}_1F_1\left(n-j+1;\dfrac{3}{2};\dfrac{a^2}{2}\right)\right]$
$\dfrac{1}{1+x^2}$	$\alpha_{2n} = \dfrac{\sqrt{(2n)!}\; e\sqrt{\pi}}{2^n} \sum_{j=0}^{n} \dfrac{(-1)^j\Gamma\left(j-n+\frac{1}{2},\frac{1}{2}\right)}{j!(n-j)!}$

In these expressions A_i is the i-th term of the original series, S_j denotes the partial sum $S_j = \sum_{i=0}^{j} A_i$ and $(n)_k$ denotes a Pochhammer symbol. Finally, the Wynn ϵ-algorithm yields a succesive approximation $\epsilon_k^{(n)}$ to the value of the sum, given by:

$$\epsilon_k^{(n)} = \epsilon_{k-2}^{(n+1)} + \frac{1}{\epsilon_{k-1}^{(n+1)} - \epsilon_{k-1}^{(n)}}, \tag{56}$$

with $\epsilon_{-1}^{(n)} = 0$ and $\epsilon_0^{(n)} = S_n$. This algorithm was computed using the following property [40]:

$$\epsilon_{2k}^{(n)} = \frac{\mathrm{H}_n^{(k+1)}(S)}{\mathrm{H}_n^{(k)}(\Delta^2 S)}, \tag{57}$$

where $\mathrm{H}_n^{(k)}(S)$ is a Hankel determinant depending on the sequence of partial sums $S = \{S_n\}$ given by:

$$\mathrm{H}_n^{(k)}(S) = \begin{vmatrix} S_n & \cdots & S_{n+k-1} \\ \vdots & \cdots & \vdots \\ S_{n+k-1} & \cdots & S_{n+2k-2} \end{vmatrix}, \quad n \ge 0, \ k \ge 1, \tag{58}$$

and Δ is the forward difference operator.

All the numerical calculations were carried out in Mathematica using a computational precision consistent with the particular approach employed, which was 30 digits or less for the entries in Table IV, while the operating precision was extended to 50 digits arithmetic for the entries reported in Table III.

IV. RESULTS AND DISCUSSION

Using the exact formulas from Table I we were able to test the numerical evaluations for the two different approaches. A comparison of the numerical results obtained using Eq. (33), with the exact evaluations of the Hilbert transform for the test functions is shown in Table III.

The test functions 1-4 gave the best results, with the number of significant digits decreasing as the value of x increased. These four test functions exhibit a Gaussian behaviour similar to the Hermite functions $u_n(x)$ and hence the expansion coefficients obtained from Eq. (11) are expected to converge rapidly. The test functions 5 and 6 gave results with a reduced accuracy. Neither of these two test functions have an asymptotic decay that closely matches a Gaussian function, and as a results the expansion coefficients converge much more slowly. By examining either Eq. (33) or Eq. (51), we see that the only difference between the numerical Hilbert transform of two different functions is the expansion coefficients associated with that function. This indicates a direct correlation between the behaviour of the expansion coefficients and the rate of convergence of the Hilbert transform.

For functions with a Gaussian character, the expansion of the present work will improve on the accuracy obtained using the expansion of Weideman [27]. This is expected because of the choice of basis functions for the expansion. As we move to functions exhibiting slower rates of asymptotic decay, the expansion of Weideman will lead to higher accuracy. For example, for cases like $f(x) = x^2 e^{-ax^2}$, $a > 0$, the present approach gives a very compact closed form expansion, which can be evaluated with high efficiency. Here the present approach is significantly better in both computational speed and accuracy compared with the Weideman approach, whilst for a case like $f(x) = (1+x^2)^{-1}$ we encounter the opposite situation, where the latter technique leads to a closed form solution with only two terms, and the approaches described in this work would require much more computational effort. For cases including some oscillatory behaviour, for example, working with $f(x) = (\cos x + \sin x)e^{-x^2}$ or $f(x) = \frac{\cos x}{2+\sin x}e^{-x^2}$, we have found that the computational

TABLE III

A COMPARISON OF NUMERICAL VALUES FOR THE HILBERT TRANSFORM VERSUS THE EXACT EVALUATION FOR THE TEST FUNCTIONS GIVEN IN TABLE I. THE VALUES $a = 12/11$ AND $b = 11/13$ HAVE BEEN EMPLOYED.

Case	x_0	$(Hf)(x_0)$ (from the exact result)	$(Hf)(x_0)$ (numerical result)
1	1/4	0.29200513386717174646471636	0.29200513386717174646471651
	1	0.595985918897246073259532	0.595985918897246073259513
	7	0.07479733191561997571	0.07479733191561997524
2	1/4	-0.4459776510413501694804727	-0.4459776510413501694804716
	1	0.0770069843891029671629	0.0770069843891029671619
	7	0.0046023889011967239	0.0046023889011967220
3	1/4	-0.111494412760337542370118	-0.111494412760337542370149
	1	0.07700698438910296716288	0.07700698438910296716301
	7	0.032216722308377067	0.032216722308377080
4	1/4	0.3370871404860270460661	0.3370871404860270460659
	1	0.559773149180786017822	0.559773149180786017818
	7	0.062416237155745896	0.062416237155745869
5	1/4	0.3157	0.3184
	1	0.4027	0.4036
	7	0.0798005012	0.0798005051
6	1/4	0.2352941	0.2352957
	1	0.500000	0.500008
	7	0.14000	0.14013

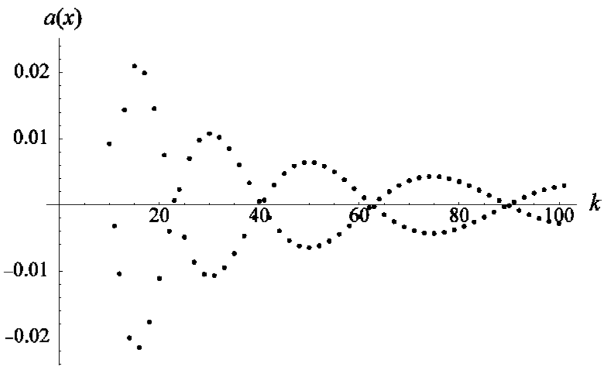

Fig. 1. Plot of the individual terms $a_k(x)$ for the Hilbert transform calculated from Eq. (59) for case 4 with $x = 1$.

speed of both approaches is similar, but the accuracy is better with the approach of the present work.

With respect to the performance of the convergence accelerators employed in this work, the best results were obtained using the Wynn ϵ-algorithm. On writing the general result, given by Eq. (51), in the form:

$$(Hf)(x) = \sum_{k=0}^{\infty} a_k(x), \tag{59}$$

and analyzing the individual terms $a_k(x)$ for case 4 with $x = 1$, the sequence behaved in a non-monotonic manner with irregular signs, as shown in Fig. 1. This would be somewhat similar to the terms generated from the function $\sin(kx)/k$. For this situation, Wynn's algorithm is expected to be the most suitable choice for convergence acceleration [42].

A comment on what constitutes the best convergence accelerator is appropriate. One would normally think about the accuracy of the result, and judge on that basis, however, computational efficiency should also be an issue. In practical applications, two extremes arise. One situation occurs when the individual terms of the series are extremely expensive to evaluate. The other extreme, is where the terms can be evaluated relatively quickly. In the former situation, the best convergence accelerator might arguably be the one that produces a satisfactory accuracy with a modest number of evaluations, even though a competing method can achieve a somewhat improved accuracy, but requiring additional evaluations. In the case where the evaluations can be done relatively quickly, the best convergence accelerator will probably be taken as the one producing the more accurate result, even if it employs slightly additional computational labor. The examples of the present work are of the latter type. In this work, we are concerned with the optimal results (in the accuracy sense) that can be obtained using each convergence accelerator, using *approximately* the same amount of input data. This means that different n and k values will be used to obtain the optimal accuracy, when the different convergence accelerators are employed.

The Levin and Weniger convergence accelerators offered satisfactory accuracy, but less compared to the Wynn ϵ-algorithm. Based on the form of the series, the performance of this method is not surprising [42]. Table IV shows a comparison of the convergence accelerators employed using the test function $f(x) = \cos(bx)e^{-ax^2}$ and employing Eq. (51). The other test functions exhibited similar trends in the relative precision for the different convergence accelerators investigated, so the results in Table IV are representative of the observed trends.

From Table III we can see that the numerical method works well for functions exhibiting a Gaussian asymptotic behavior. By examining the expansion coefficients in Fig. 2 for the test functions employed, we indeed see a trend in their convergence rates as mentioned previously. To keep Fig. 2 uncluttered, only the expansion coefficients for cases 4-6 are shown. The convergence rate for case 4 is slower than for

TABLE IV
A COMPARISON OF THE DIFFERENT CONVERGENCE ACCELERATOR METHODS USING THE FUNCTION $f(x) = \cos(bx)e^{-ax^2}$. THE VALUES $a = 12/11$ AND $b = 11/13$ HAVE BEEN EMPLOYED.

Method	$(Hf)(1)$	$(Hf)(1/4)$
Levin-u	0.5597737	0.3370875
Levin-t'	0.5597723	0.3370865
Weniger-1	0.5597735	0.3370865
Weniger-2	0.5597720	0.33708709
Wynn ϵ	0.55977314921	0.3370871404860286
exact	0.55977314918	0.3370871404860270

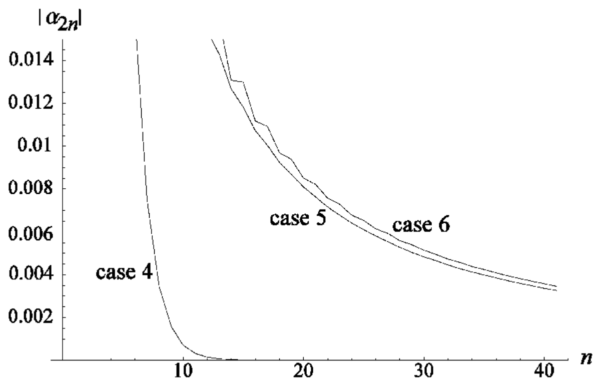

Fig. 2. Comparison of the expansion coefficients for cases 4, 5, and 6.

cases 1-3, but we see from Fig. 2 that the convergence rate of case 4 is much faster than either case 5 or case 6. The results in Fig. 2 provide support for the rate of convergence of the expansion coefficients in either Eq. (33) or Eq. (51) being an important factor, as expected, in determining accurate numerical values for the Hilbert transform. The other important factor in determining accurate results is the optimal selection of convergence accelerator to apply. The numerical method proposed by Weideman also showed that a slower convergence rate for the expansion coefficients resulted in a slower converging Hilbert transform.

The Fourier transform method offered a few more digits of precision over the direct series method for $x = 1/4$. However, for the Fourier method given in Eq. (51), the results were obtained by setting $n = 30$ and $k = 15$ for all calculations using Eq. (57), while for the direct method given in Eq. (33), the results were obtained by utilizing $n = 12$ and $k = 12$ for all calculations employing Eq. (57). For $x = 1$ both methods performed equally well and for $x = 7$ the direct method offered a few more digits of precision over the Fourier method.

Attempts at improving the precision for cases 5 and 6 involved direct summation of the first several terms of the Hilbert transform series that were erratic, and then convergence accelerators were applied to the remaining series, and the results combined. This was tried on both cases 5 and 6 with each of the convergence accelerators, but did not produce any increase in precision over convergence acceleration of the original series. Attempts were also made to split the series in terms of the even and odd contributions of k in Eq. (51). From Fig. 1 we see that splitting the series that way would produce two sequences that are slowly convergent and non-alternating with an irregular sign pattern. The convergence accelerators were applied to both of the resulting series and the results were combined. This also offered no increase in precision for cases 5 and 6. It is usually much more difficult to apply convergence accelerator techniques to series producing irregular input. For a recent study involving this issue, see [43].

Another method that could offer better precision would be to modify the basis functions in Eq. (9). By changing the orthogonal function $u_n(x)$ to a different orthogonal function, whose behavior is similar to the function of interest, then the resulting expansion coefficients would be expected to converge faster, leading to the Hilbert transform series converging faster. This alternative scheme would only be comparable with the direct series approach, because the Fourier series method relies on the function $u_n(x)$ being an eigenfunction of the Fourier transform operator.

One approach to the numerical evaluation of the Hilbert transform of a function f that cannot be evaluated in closed form, is to adopt the following strategy. Select a new function g that mimics the essential features of f, and for which Hg can be evaluated in closed form. Numerical experiments carried out with g, using the exact Hg as a guide, will provide useful information on the best expansion approach, and on the optimal convergence accelerator scheme to employ, when working with the function f. The optimal choice of the expansion basis needs to meet two requirements. The Hilbert transform of the basis functions must be easily determined, or if the Fourier approach is employed, then the Fourier transform of the basis functions and the inverse Fourier transform of $\mathrm{sgn}\, x\,(\mathcal{F}f)(x)$ should be determinable in closed form.

The second requirement is that either the resulting series should be rapidly convergent, or alternatively, accurately summable by convergence acceleration techniques.

V. Conclusion

Two methods have been presented for the numerical evaluation of the Hilbert transform, both of which are computationally efficient, and both yield accurate results for a number of test functions. The approaches are particularly effective for functions exhibiting a Gaussian-like asymptotic behavior. Five different convergence accelerator techniques were employed to sum the series that arise, with the Wynn ϵ-algorithm providing the best accuracy for this particular application.

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