Pricing And Hedging of Asian Option Under Jumps

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Abstract—**In this paper we study the pricing and hedging problems of "generalized" Asian options in a jump-diffusion model. We choose the minimal entropy martingale measure (MEMM) as equivalent martingale measure and we derive a partial-integro differential equation for their price. We discuss the minimal variance hedging including the optimal hedging ratio and the optimal initial endowment. When the Asian payoff is bounded, we show that for the exponential utility function, the utility indifference price goes to the Asian option price evaluated under the MEMM as the Arrow-Pratt measure of absolute risk-aversion goes to zero.**

Index Terms—**Geometric Lévy process, Generalized Asian options, Minimal entropy martingale measure, Partial-integro differential equation, Utility indifference pricing, Exponential utility function.**

I. Introduction

Asian options have payoff depending on the average price of the underlying asset during some part of their life. Therefore, their payoff must be expressed as a function of the asset price history. But, in contrast to the European options, they have not a closed form solution for their price, even when the underlying follows a geometric Brownian model.

In the setting of the continuous-time market models, many approaches and numerical approximations are proposed to price Asian options: For instance, [35] provides tight analytic bounds for the Asian option price, In [20] is computed the Laplace transform of the Asian Option price, [27] uses Monte Carlo simulations, [25] follows a method based on binomial trees. [13] introduces change of numeraire technique and [32] uses it to reduce the PDE problem in two variables to obtain a lower bound for the price. In a recent paper [10] is derived a closed-form solution for the price of an average strike as well as an average price geometric Asian option, by making use of the path integral formulation.

But, in practice, continuous time models are unrealistic, since the price process really presents discontinuities. In particular, Lévy processes [6, 33, 2] became the most popular and tractable for the market modeling and finance applications.

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To price Asian options in such setting, [12] presents generalized Laplace transform for continuously sampled Asian option where underlying asset is driven by a Lévy Process, [29] proposes a binomial tree method under a particular jump-diffusion model. For the case of semimartingale models, [36] shows that the pricing function is satisfying a partial-integro differential equation. For the precise numerical computation of the PIDE using difference schemes we refer interested readers to explore with [11]. In [3] is derived the range of price for Asian option when underlying stock price is following geometric Brownian motion and jump (Poisson). In [30] is derived bounds for the price of a discretely monitored arithmetic Asian option when the underlying asset follows an arbitrary Lévy process. The paper [37] develops a simple network approach to American exotic option valuation under Lévy processes using the fast Fourier transform. [23] uses lattices to price fixed-strike European-style Asian options that are discretely monitored. In [28] is studied a certain one-dimensional, degenerate parabolic partial differential equation with a boundary condition which arises in pricing of Asian options. It is proven that the generalized solution of the problem is indeed a classical solution. [24] provides a semi explicit valuation formula for geometric Asian options, with fixed and floating strike under continuous monitoring, when the underlying stock price process exhibits both stochastic volatility and jumps.

To overcome the problem of replicating the option by trading in the underlying asset in such models, the authors of [15] where the first to introduce a mean-variance criterion for hedging contingent claims in incomplete markets. They derive, under a martingale measure setting, a unique strategy minimizing the variance of the total error of replication due to hedging in a given claim. More precisely, they propose to approximate the target payoff H by optimally choosing the initial capital c and a self-financing trading strategy $(\Delta_t)_{t\in[0,T]}$ in the risky asset S in order to minimize the quadratic hedging error:

$$J(c,\Delta) = \mathbb{E}^{\mathbb{Q}}\left[c + \int_0^T \Delta_t dS_t - H\right]^2. \tag{1}$$

The choice of EMM $\mathbb{Q}$ is questionable, since the market in presence of jumps may be incomplete. To choose an EMM many methods are proposed: Minimal martingale measure [16], Esscher martingale measure [21, 7] and Minimal entropy martingale measure introduced (MEMM) in [19]. The last one has the advantage that it conserves the Lévy property of the process under the change of measure. In particular the MEMMs for geometric Lévy processes have been discussed by [18, 19, 31].

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In this paper, we study generalized Asian option pricing and hedging in a financial market where the underlying price is modeled by geometric jump-diffusion Lévy process. The considered market is in fact incomplete. Deeply influenced by [19], we choose the MEMM as equivalent martingale measure under which we show that the price process of a generalized Asian option is a solution of a two-dimensional parabolic-hyperbolic partial-integro differential equation.

The paper is structured as follows: In section (III) we describe the geometric jump-diffusion market model and we provide some of their useful properties. Next, we introduce the concept of minimal entropy martingale measure under which we study the properties of the underlying asset price. In section (IV), we derive a partial-integro differential equation for generalized Asian options and we make some vague study of the numerical computation. Section (V) is devoted to study minimal variance hedging including the optimal initial endowment and hedging strategy minimizing the quadratic risk under the MEMM. In section (VI), we show that the MEMM generalized Asian price is the limit of the corresponding utility indifference price as the risk-aversion parameter tends to zero. Finally, section (VII) gives some concluding remarks.

II. Generalized Asian options

Let $(S_t)_{0\leq t\leq T}$ be the stock price process with jumps and $(Y_t)_{0\leq t\leq T}$ be the arithmetic average value of the underlying over the life time interval $[0,t]$:

$$Y_t \quad = \quad \frac{1}{t}\int_0^t S_u du. \tag{2}$$

Consider an Asian option maturing at time T based on Y_T with an $\mathcal{F}_T$-measurable payoff H:

$$H \quad = \quad \varphi(S_T, Y_T),$$

where φ satisfies the following Lipschitz assumption:

$(\mathbf{A_1})$ $\qquad \exists a>0; \forall (S,S',Y,Y')\in\mathbb{R}_+^4,$

$|\varphi(S,Y)-\varphi(S',Y')|\leq a[|S-S'|+|Y-Y'|]$.

This kind of options will be termed as *generalized Asian options*. As examples we cite the payoff $\varphi(S_T,Y_T) = |S_T - Y_T|$ of a straddle Asian option and the payoff $\varphi(S_T,Y_T) = (\zeta.(Y_T - K_1S_T - K_2))^+$, which is a fixed strike Asian option, for $K_1 = 0$, and a floating strike Asian option for $K_2 = 0$. The constant $\zeta = \pm 1$ determines whether the option is call or put.

The price V_t of a generalized Asian option depends on t, S_t and on the path that the asset price followed up to time t. In particular, we can not invoke the Markov property to claim that V_t is a function of t and S_t.

To overcome this difficulty, we increase the state variable S_t by using the second process Y_t given by Equation (2) which follow the stochastic differential equation:

$$dY_t \quad = \quad \frac{S_t - Y_t}{t}dt. \tag{3}$$

The process (S_t, Y_t) constitutes then a two-dimensional Markov process. Furthermore, the payoff $H = \varphi(S_T, Y_T)$ is a function of T and the final value (S_T, Y_T) of this process. This implies the existence of some function $\phi(t,S,Y)$ such that the generalized Asian option price under any EMM $\mathbb{Q}$ is written as:

$$\phi(t,S_t,Y_t) \quad = \quad \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}\varphi(S_T,Y_T)/\mathcal{F}_t].$$

III. Asian options in a geometric jump-diffusion Lévy market model

In this section, we focus on the market where the underlying price process is modeled by a geometric jump-diffusion Lévy process. And we make an appropriate choice of EMM for such incomplete market.

A. Description of the market model

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a given complete probability space. We consider a market which consists of a bond with price process given by $B_t = e^{rt}$, where r denotes the (constant) interest rate and a non-dividend paying risky asset (the stock or index). The price process $(S_t)_{0\leq t\leq T}$ of the risky asset is modeled by the following geometric jump-diffusion Lévy process:

$$S_t = S_0\exp(rt + L_t),\ 0\leq t\leq T, \tag{4}$$

where L_t is a Lévy process given by:

$$L_t = \mu t + \sigma B_t + \sum_{j=1}^{N_t} U_j, \tag{5}$$

here μ is a drift, σ is the underlying volatility supposed to be positive and $(N_t)_{0\leq t\leq T}$ is a Poisson process with intensity $\lambda > 0$. The jump sizes $(U_j)_{j\geq 1}$ are i.i.d. $\mathbb{P}$-integrable r.v.s with a distribution F and independent of $(N_t)_{0\leq t\leq T}$. We denote by $\hat{S}_t = e^{-rt}S_t = S_0\exp(L_t)$ the discounted underlying price process.

We assume that the σ-fields generated respectively by $(B_t)_{0\leq t\leq T}$, $(N_t)_{0\leq t\leq T}$ and $(U_j)_{j\geq 1}$ are independents and we take $\mathcal{F}_t := \sigma(B_s, N_s, U_j\mathbf{1}_{\{j\leq N_s\}}; s\leq t, j\geq 1), t\geq 0$ as filtration. Let $\tau_1,\cdots,\tau_j,\cdots$ be the dates of jumps of the stock price S_t; i.e. the jump times of $(N_t)_{0\leq t\leq T}$ and $V_1,\cdots,V_j,\cdots$ their proportions of jumps at these times, respectively; i.e.

$$\Delta S_{\tau_j} = S_{\tau_j} - S_{\tau_j^-} = S_{\tau_j^-}V_j \qquad \text{for all } j\geq 1,$$

where $V_j = e^{U_j} - 1$. The r.v.'s $(V_j)_{j\geq 1}$ are i.i.d. with a common distribution G and takes its values in the interval $(-1,+\infty)$. Recall that the expression (5) can be rewritten in integral form as

$$L_t = \mu t + \sigma B_t + \int_{[0,t]}\int_{\mathbb{R}} xJ_L(ds,dx),$$

and L_t is a Lévy process with the characteristic triplet $(\mu + \int_{|x|\leq 1} x\nu_L(dx), \sigma^2, \nu_L)$, where $\nu_L(dx) = \lambda F(dx)$ is the Lévy measure, $J_L(dt,dx) = \sum_{j\geq 1}\delta_{(\tau_j,\Delta L_{\tau_j})}$ is the random Poisson measure of L on $[0,+\infty[\times\mathbb{R}$ with intensity $\nu_L(dt,dx) := \nu_L(dx)dt$.

Hence L has the following Lévy-Itô decomposition:

$$L_t = \left(\mu + \int_{|x|\leq 1} x\nu_L(dx)\right)t + \sigma B_t \tag{6}$$
$$+ \int_{[0,t]}\int_{|x|>1} xJ_L(ds,dx) + \int_{[0,t]}\int_{|x|\leq 1} x\widetilde{J}_L(ds,dx),$$

where $\widetilde{J}_L(dt,dx)$ is the compensated measure of J_L, i.e. $\widetilde{J}_L(dt,dx) := (J_L - \nu_L)(dt,dx)$.
Since $\int_{\mathbb{R}} |x|\,\nu_L(dx) = \lambda\mathbb{E}^{\mathbb{P}}[|U_1|] < \infty$, it comes that the first moment of L_t (for all t) is finite ($\mathbb{E}[|L_t|] < \infty$). Consequently, the expression (6) resumes to this form:

$$L_t = \left(\mu + \int_{\mathbb{R}} x\nu_L(dx)\right)t + \sigma B_t + \int_0^t \int_{\mathbb{R}} (J_L - \nu_L)(ds,dx).$$

Set $X_t := L_t + rt$, then X_t is a Lévy process with characteristic triplets (b, σ^2, ν_L), where

$$b = \mu + \int_{|x|\le 1} x\nu_L(dx) + r. \qquad (7)$$

According to Proposition 8.22 of [8], there exists a Lévy process $\hat{X}$ such that S_t is the Doleans-Dade exponent of $\hat{X}_t$: $S = \mathcal{E}(\hat{X})$. The process $\hat{X}$ is given by

$$\begin{aligned}\hat{X}_t &= X_t + \frac{\sigma^2}{2}t + \sum_{0\le u\le t}(1 + \Delta X_u - e^{\Delta X_u}) \\ &= L_t + \left(r + \frac{\sigma^2}{2}\right)t + \sum_{0\le u\le t}(1 + \Delta L_u - e^{\Delta L_u})\end{aligned}$$

with $\Delta X_t = X_t - X_{t^-} = X_t - \lim_{u\uparrow t} X_u$ and satisfies the stochastic differential equation

$$S_t = S_0 + \int_0^t S_{u-}d\hat{X}_u.$$

or in integral form

$$\begin{aligned}\hat{X}_t &= b_1 t + \sigma B_t + \int_0^t \int_{|x|\le 1}(e^x - 1)\tilde{J}_L(du,dx) \\ &+ \int_0^t \int_{|x|>1}(e^x - 1)J_L(du,dx),\end{aligned}$$

where

$$b_1 = \frac{1}{2}\sigma^2 + b + \int_{|x|\le 1}(e^x - 1 - x)\nu_L(dx). \qquad (8)$$

Note that, $\hat{X}_t$ is a Lévy process under $\mathbb{P}$ with characteristics $(\hat{b}, \sigma^2, \hat{\nu})$, and has the following Lévy-Itô decomposition [19]:

$$\begin{aligned}\hat{X}_t &= \hat{b}t + \sigma B_t + \int_0^t \int_{|x|>1} xJ_{\hat{X}}(du,dx) \qquad (9) \\ &+ \int_0^t \int_{|x|\le 1} x\tilde{J}_{\hat{X}}(du,dx),\end{aligned}$$

where $\hat{\nu}(dy) = \nu o J^{-1}(dy)$ with $J(x) = e^x - 1$, $\hat{b} = b_1 + \int_{\{x<-1\}}(e^x - 1)\nu_L(dx) - \int_{\{log2<x\le 1\}}(e^x - 1)\nu_L(dx)$, $J_{\hat{X}}$ is the random Poisson measure of $\hat{X}$ with intensity $\hat{\nu}(du,dx) = \hat{\nu}(dx)du$ and $\tilde{J}_{\hat{X}}(du,dx) = (J_{\hat{X}} - \hat{\nu})(du,dx)$ its compensator.

Denote by $(\hat{R}_t)$ be the simple return process for $(\hat{S}_t)$ [7]. i.e. $\hat{R}_t = \int_0^t \frac{1}{\hat{S}_{u-}}d\hat{S}_u$. It is easy to see that

$$\hat{R}_t = \hat{X}_t - rt \qquad (10)$$

B. Minimal entropy martingale measure and geometric jump-diffusion Lévy processes

The fundamental asset pricing theorem shows that the choice of an arbitrage-free method is equivalent to the choice of an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$. To do so, influenced by [19], we use minimal entropy martingale method for geometric Lévy processes.

Denote by EMM($\mathbb{P}$) the set of all EMMs of S; i.e. the set of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ such that the process $\{e^{-rt}S_t, 0 \le t \le T\}$ is an $(\mathcal{F}_t, \mathbb{Q})$-martingale and $\mathbb{Q} \sim \mathbb{P}$.

Definition 1: The relative entropy of $\mathbb{Q} \in$ EMM($\mathbb{P}$), with respect to $\mathbb{P}$, is defined by:

$$\mathbb{H}(\mathbb{Q}|\mathbb{P}) := \begin{cases} \int_\Omega \log\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]d\mathbb{Q}, & \text{if } \mathbb{Q} \ll \mathbb{P}; \\ \infty & \text{otherwise.}\end{cases}$$

If an EMM $\mathbb{P}^*$ satisfies the following condition:

$$\mathbb{H}(\mathbb{P}^*|\mathbb{P}) \le \mathbb{H}(\mathbb{Q}|\mathbb{P}), \qquad \text{for all } \mathbb{Q} \in \text{EMM}(\mathbb{P}), \quad (11)$$

then, $\mathbb{P}^*$ is called the *minimal entropy martingale measure* (MEMM) of $(S_t)_{0\le t\le T}$.
A general feature of the MEMM is that its statistical properties resemble the original process so the specification of the prior is quite important. It has also the advantage that the Lévy property is preserved for geometric Lévy processes and can be defined in terms of the Esscher transform and Return process [14, 7]. The following theorem resumes the main properties and results concerning the MEMM.

Theorem 1: [19] Suppose there exists a constant β^* such that the following assumptions hold:
$(\mathbf{A_2})$

$$\int_{x>1} e^x e^{\beta^*(e^x-1)}\nu_L(dx) = \lambda\mathbb{E}^{\mathbb{P}}\left[1_{U_1>1}e^{U_1}e^{\beta^*(e^{U_1}-1)}\right] < \infty$$

$(\mathbf{A_3})$ $\mu + \left(\frac{1}{2} + \beta^*\right)\sigma^2 + \int_{\mathbb{R}}(e^x - 1)e^{\beta^*(e^x-1)}\nu_L(dx) = 0$

then,

1) The Esscher transform define a probability measure $\mathbb{P}^*$ on $\mathcal{F}_T$ by:

$$\left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{F}_t} := \frac{e^{\beta^*\hat{R}_t}}{\mathbb{E}^{\mathbb{P}}[e^{\beta^*\hat{R}_t}]} = e^{\beta^*\hat{X}_t - b_0 t}, \quad \forall t \in [0,T],$$

where $(\hat{R}_t)_{t\in[0,T]}$ is the simple return process given by Equation (10), $(\hat{X}_t)$ is defined by (9), and

$$\begin{aligned}b_0 &= \frac{\beta^*}{2}(1+\beta^*)\sigma^2 + \beta^* b + \int_{\mathbb{R}}(e^{\beta^*(e^x-1)} - 1 \\ &- \beta^* x 1_{\{|x|\le 1\}})\nu_L(dx),\end{aligned}$$

with b defined by Equation (7).

2) The probability measure $\mathbb{P}^*$ is actually in EMM($\mathbb{P}$). In addition, $\mathbb{P}^*$ is the MEMM for $(S_t)_{t\ge 0}$ and the relative entropy of $\mathbb{P}^*$ with respect to $\mathbb{P}$ has the form:

$$\begin{aligned}\mathbb{H}(\mathbb{P}^*|\mathbb{P}) &= -T\left[\beta^*(b-r) + \frac{\beta^*(1+\beta^*)}{2}\sigma^2\right. \\ &+ \left.\int_{\mathbb{R}}(e^{\beta^*(e^x-1)} - 1 - \beta^* x 1_{\{|x|\le 1\}})\nu_L(dx)\right].\end{aligned}$$

where b is the drift (7).

3) The Stochastic process (L_t) remains a Lévy process under $\mathbb{P}^*$ with characteristic triplet $(\mu^*, \sigma^2, \nu_L^*)$ and Lévy-Itô decomposition:

$$\begin{aligned} L_t &= \mu^* t + \sigma B_t^* + \int_0^t \int_{|x|\leq 1} x(J_L - \nu_L^*)(ds, dx) \\ &+ \int_0^t \int_{|x|>1} x J_L(ds, dx), \end{aligned} \tag{12}$$

where $\mu^* = \mu + \beta^*\sigma^2 + \int_{|x|\leq 1} x e^{\beta^*(e^x-1)}\nu_L(dx)$, $\nu_L^*(dx) = e^{\beta^*(e^x-1)}\nu_L(dx)$ and $B_t^* = B_t - \sigma\beta^* t$ is a $\mathbb{P}^*$-Brownian motion.

An easy calculus of the density $\eta_T := \frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_T}$ of the MEMM $\mathbb{P}^*$ leads to the explicit expression:

$$\begin{aligned} \eta_T = \exp\Bigg[&\beta^*\sigma B_T - \frac{1}{2}(\beta^*\sigma)^2 T \\ &+ \int_{[0,T]} \int_{\mathbb{R}} \beta^*(e^x - 1) J_L(ds \times dx) \\ &- T \int_{\mathbb{R}} \left(e^{\beta^*(e^x-1)} - 1\right) \nu_L(dx)\Bigg]. \end{aligned}$$

Remark 1: Assumption $(\mathbf{A_2})$ ensures that the first exponential moment of L_t under $\mathbb{P}^*$ is finite, i.e. $(\hat{S}_t)_{0\leq t\leq T}$ is $\mathbb{P}^*$-integrable. Consequently, the r.v. $V_1 = e^{U_1} + 1$ is also $\mathbb{P}^*$-integrable.

The decomposition (12) of L can be rewritten as:

$$L_t = \mu^* t + \sigma B_t^* + \left(\sum_{j=1}^{N_t} U_j - t\lambda\kappa\right).$$

where $\kappa = \mathbb{E}^{\mathbb{P}}[U_1 e^{\beta^*(e^{U_1}-1)} \mathbf{1}_{\{|U_1|\leq 1\}}]$. The expression of S_t is finally given by:

$$S_t = S_0 \exp(\mu_0 t + \sigma B_t^*) \prod_{j=1}^{N_t} (1 + V_j),$$

where

$$\begin{aligned} \mu_0 &= \mu + \beta^*\sigma^2 + \int_{|x|\leq 1} x e^{\beta^*(e^x-1)} \nu_L(dx) - \lambda\kappa + r \\ &= \mu + \beta^*\sigma^2 + r. \end{aligned}$$

Since L is a Lévy process with characteristic triplet $(\mu^*, \sigma^2, \nu_L^*)$ under $\mathbb{P}^*$, it comes according to Proposition 8.20 of [8] and taking account of assumption $(\mathbf{A_3})$, that L is a $\mathbb{P}^*$-square integrable martingale and has the following expression:

$$\begin{aligned} \hat{S}_t &= S_0 + \sigma \int_0^t \hat{S}_{s-} dB_s^* \\ &+ \int_0^t \int_{-\infty}^{+\infty} \hat{S}_{s-}(e^x - 1)(J_L - \nu_L^*)(ds, dx). \end{aligned} \tag{13}$$

The integral of the predictable random function $\theta(\omega, s, x) = \hat{S}_{s^-}(\omega)(e^x - 1)$, with respect to the Poisson random measure J_L, is given by:

$$\begin{aligned} \int_0^t \int_{-\infty}^{+\infty} \theta(s, x) J_L(ds, dx) &= \sum_{\substack{s\in[0,t] \\ \Delta L_s \neq 0}} \theta(s, \Delta L_s) \\ = \sum_{j=1}^{N_t} \theta(\tau_j, U_j) = \sum_{j=1}^{N_t} (e^{U_j} - 1)\hat{S}_{\tau_j^-} &= \sum_{j=1}^{N_t} V_j \hat{S}_{\tau_j^-}. \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} &\int_0^t \int_{-\infty}^{+\infty} \hat{S}_{s-}(e^x - 1)\nu_L^*(ds, dx) \\ &= \lambda \int_0^t \hat{S}_s ds \int_{\mathbb{R}} (e^x - 1) e^{\beta^*(e^x-1)} F(dx) \\ &= \lambda \mathbb{E}^*[V_1] \left(\int_0^t \hat{S}_s ds\right). \\ &= (\mu + \beta^*\sigma^2) \left(\int_0^t \hat{S}_s ds\right). \end{aligned}$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$. The last equality follow from assumption $(\mathbf{A_1})$. It follows from the Equation (13) that $\hat{S}_t$ satisfies:

$$\begin{aligned} \hat{S}_t &= S_0 + \int_0^t \hat{S}_s \left[\sigma dB_s^* + (\mu + \beta^*\sigma^2) dt\right] \\ &+ \sum_{j=1}^{N_t} V_j \hat{S}_{\tau_j^-}. \end{aligned}$$

IV. Pricing and partial-integro differential equation

Recall that $H = \varphi(S_T, Y_T)$ is the payoff of a generalized Asian option in the exponential jump-diffusion market model, where φ is Borel and fulfills assumption $(\mathbf{A_1})$. Using the stationary and the independence of the increments of L, the value $V_t = e^{-r(T-t)}\mathbb{E}^*[H/\mathcal{F}_t]$ of the option at time t is a function of t, S_t and Y_t:

$$V_t = \phi(t, S_t, Y_t),$$

where ϕ is the following measurable function on $[0, T] \times \mathbb{R}^2$:

$$\begin{aligned} \phi(t, S, Y) &= e^{-r(T-t)} \mathbb{E}^*[H \mid S_t = S, Y_t = Y] \\ = \mathbb{E}^*\Bigg[&e^{-r(T-t)} \varphi\Bigg(S e^{L_{T-t} + r(T-t)}, \frac{t}{T} Y \\ &+ \frac{S}{T} \int_0^{T-t} e^{L_u + ru} du\Bigg)\Bigg]. \end{aligned} \tag{14}$$

We have the following useful lemma

Lemma 1: On each compact set $K_0 = [0, T] \times [0, S_0] \times [0, Y_0]$, $Y_0, S_0 > 0$, the function ϕ is Lipschitz; i.e. there exist a constant $a_0 > 0$ such that for all $(t, S, Y), (t', S', Y') \in K_0$ we have:

$$|\phi(t, S, Y) - \phi(t', S', Y')| \leq a_0[|t - t'| + |S - S'| + |Y - Y'|].$$

Proof: See Appendix (A). ∎

A. Partial-integro differential equation

Proposition 1: Under the assumption $(\mathbf{A_1})$, the function ϕ, given by (14) is continuous on $[0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$. Furthermore, if ϕ is in the class $C^{1,2,1}(]0, T[\times \mathbb{R}_+^* \times \mathbb{R}_+^*)$, then it fulfills the following partial-integro differential equation on $[0, T[\times \mathbb{R}_+^* \times \mathbb{R}_+^*$:

$$\mathcal{L}\phi(t, S, Y) = 0$$

with the following terminal condition

$$\forall S > 0, \forall Y > 0, \phi(T, S, Y) = \varphi(S, Y),$$

where

$$\begin{aligned}
\mathcal{L}\phi &= \frac{\partial\phi}{\partial t} + \left(r+\mu+\beta^*\sigma^2\right) S\frac{\partial\phi}{\partial S} \\
&+ \frac{S-Y}{t}\frac{\partial\phi}{\partial Y} + \frac{\sigma^2}{2}S^2\frac{\partial^2\phi}{\partial S^2} - (r+\lambda)\phi \\
&+ \lambda\int_{-1}^{+\infty} e^{\beta^* v}\phi(t,S(1+v),Y)G(dv) \\
&= \frac{\partial\phi}{\partial t} + \left(r+\mu+\beta^*\sigma^2\right) S\frac{\partial\phi}{\partial S} + \frac{S-Y}{t}\frac{\partial\phi}{\partial Y} \\
&+ \frac{\sigma^2}{2}S^2\frac{\partial^2\phi}{\partial S^2} - (r+\lambda)\phi \\
&+ \lambda\int_{-\infty}^{+\infty} e^{\beta^*(e^u-1)}\phi(t,Se^u,Y)F(du).
\end{aligned} \tag{15}$$

with G the distribution function of the r.v. V_1 under $\mathbb{P}$.

Proof: See Appendix (B). ■

Remark 2: In the above PIDE it is important to note the term $\left(\frac{S_t-Y_t}{t}\right)$. This plays an important role in the Upwinding technique used for numerical computation. The appearance of this term is precisely due to assumption $Y_t = \frac{1}{t}\int_0^t S_u du$. It is interesting to note that, for example, [32] defined $Y_t = \int_0^t S_u du$ in order to compute the pricing PDE of an Asian option when stock price follows geometric Brownian motion.

Remark 3: Note that the integro differential operator (15) is not well defined at $t=0$. It is easy to check that we can take the following transformation

$$A_t = tY_t = \int_0^t S_u du$$

to remove the singularity. The corresponding integro differential operator $\mathcal{L}_1$ is given by:

$$\begin{aligned}
\mathcal{L}_1\phi &= \frac{\partial\phi}{\partial t} + \left(r+\mu+\beta^*\sigma^2\right) S\frac{\partial\phi}{\partial S} + S\frac{\partial\phi}{\partial A} \\
&+ \frac{\sigma^2}{2}S^2\frac{\partial^2\phi}{\partial S^2} - (r+\lambda)\phi \\
&+ \lambda\int_{-1}^{+\infty} e^{\beta^* v}\phi(t,S(1+v),A)G(dv) \\
&= \frac{\partial\phi}{\partial t} + \left(r+\mu+\beta^*\sigma^2\right) S\frac{\partial\phi}{\partial S} + S\frac{\partial\phi}{\partial A} \\
&+ \frac{\sigma^2}{2}S^2\frac{\partial^2\phi}{\partial S^2} - (r+\lambda)\phi \\
&+ \lambda\int_{-\infty}^{+\infty} e^{\beta^*(e^u-1)}\phi(t,Se^u,A)F(du).
\end{aligned}$$

B. Numerical computation of the PIDE

Despite the original domain is $0 \le S < \infty$, and $0 \le Y < \infty$, for computational sake we need to truncate the domain with $[0, S_{\max}] \times [0, Y_{\max}]$, with $S_{\max} = Y_{\max}$. In order to make the integral defined, it is necessary to approximate the solution outside the computational domain $S > S_{\max}$. Abide by [11], no boundary condition is required at $S=0, Y=0, Y=Y_{\max}$, which is because, either PIDE becomes ODE or the resulting degenerate parabolic PIDE has a normal hyperbolic term with outgoing characteristics. [11] ensured with the development of consistent, stable and monotone discrete scheme that no data outside the computational domain at $S=0$ is required.
For the boundary condition at $S=\infty$ two measure issues are of concern.
(a) No obvious Dirichlet type condition can be imposed.
(b) On any finite domain $[0, S_{\max}]$, the integral term appears to gather information from the outside domain. According to [11], $S_{\max}$ will be taken large so that the solution can be well approximated by a linear function of S in the region $[S_{\max}-\delta, S_{\max}]$. δ should be chosen carefully, so that integral term in the PIDE in $[0, S_{\max}-\delta]$ has sufficient data, for accurate computation. In the region $[S_{\max}-\delta, S_{\max}] \times [0, Y_{\max}]$ $\mathcal{L}$ is linear in S so the PIDE is reduced to the PDE.
We refer interested readers to [11] for the computational detail in which Semi-Lagrangian method coupled with Implicit-discretization is proposed. For the Integral term evaluation FFT is used and correction term δ has alleviated the wrap around effect.

V. Minimal variance hedging

Consider an economic agent who has sold at $t=0$ the contingent claim with terminal payoff $H = \varphi(S_T, Y_T)$ for the price c and decides to hedge the associated risk by trading in the risky asset S. Note that under the model assumptions, the payoff H is $(\mathcal{F}_T, \mathbb{P}^*)$-square integrable.
The predictable σ-algebra is the σ-algebra generated on $[0,T] \times \Omega$ by all adapted and left-continuous processes. A stochastic process $X : [0,T] \times \Omega \longrightarrow \mathbb{R}$ which is measurable with respect to the predictable σ-algebra is called *predictable process*. Any left-continuous process is, therefore, predictable (by definition). All predictable processes are generated from left-continuous processes. However, there are predictable processes which are not left-continuous.

1) The admissible strategy: A hedging strategy is a $\mathbb{R}^2$-valued adapted process $\mathcal{K} = \{(\Delta_t^0, \Delta_t), 0 \le t \le T\}$ such that Δ_t (resp. Δ_t^0) represents the number of shares held by the investor in the risky asset (resp. the riskless asst). The value at a time t of any strategy $\mathcal{K}$ is given by $V_t = \Delta_t^0 S_t^0 + \Delta_t S_t$. A strategy $\mathcal{K}$ is said to be self-financing if

$$dV_t = \Delta_t^0 dS_t^0 + \Delta_t d\hat{S}_t.$$

Taking account of jumps, the components (Δ_t^0) and (Δ_t) will be taken to be left-continuous with limits from the right (caglad). For any self-financing strategy $\mathcal{K}$ with initial value V_0, we have:

$$\hat{V}_t(\mathcal{K}) = V_0 + \int_0^t \Delta_u d\hat{S}_u. \tag{16}$$

In order to make Equation (16) meaningful, we shall restrict ourselves to the following class Θ of *admissible hedging strategies*, composed by all caglad and predictable processes $\Delta : \Omega \times [0,T] \longrightarrow \mathbb{R}$ such that:

$$\mathbb{E}^*\left[\left(\int_0^T \Delta_u d\hat{S}_u\right)^2\right] < \infty. \tag{17}$$

Remark 4: : Thanks to the isometry formula given by Proposition 8.8 of Cont and Tankov [8], the condition (17) is equivalent to:

$$\mathbb{E}^*\left[\int_0^T \Delta_t^2 \hat{S}_t^2 dt\right] < \infty. \tag{18}$$

It ensures that the discounted value $\hat{V}_t$ of any admissible strategy $\mathcal{K}$ is a $\mathbb{P}^*$-square integrable martingale.

The value at a time t of any admissible hedging strategy is given by:

$$\begin{aligned}\hat{V}_t(\mathcal{K}) &= V_0 + \int_0^t \Delta_u \hat{S}_u \left[\sigma dB_s^* + (\mu + \beta^* \sigma^2) dt\right] \\ &+ \sum_{j=1}^{N_t} \Delta_{\tau_j} V_j \hat{S}_{\tau_j^-} \end{aligned} \quad (19)$$

2) The optimal hedging ratio: Under market incompleteness, the contingent claim H cannot be perfectly replicated by a self-financing strategy. Therefore, our objective is naturally to hedge the claim H by the selection of a self-financing strategy and an initial endowment that minimize the variance of the error of replication over the set of all admissible strategies and all initial endowments. The residual hedging error associated to any admissible strategy $\Delta \in \Theta$ is given by:

$$\epsilon(c, \Delta) = c - \hat{H} + \int_0^T \Delta_u d\hat{S}_u = e^{-rT}(V_T(\mathcal{K}) - H), \quad (20)$$

where $\hat{H} = e^{-rT}H$. Following [34], we consider the following optimization problem to price and hedge the claim H under market incompleteness:

$$(\mathcal{P}) \qquad \inf_{(c,\Delta)\in\mathbb{R}\times\Theta} \mathbb{E}^* \left[\left(c - \hat{H} + \int_0^T \Delta_u d\hat{S}_u\right)^2\right]$$

$= \inf_{(c,\Delta)\in\mathbb{R}\times\Theta} \mathbb{E}^*[\epsilon(c, \Delta)^2]$
where the expectation is taken to be under $\mathbb{P}^*$.

Proposition 2: Let $H = \varphi(S_T, Y_T)$ be the payoff of a generalized Asian option as in the previous proposition. Suppose that the function ϕ, given by Equation (14), is in $C^{1,2,1}([0, T[\times\mathbb{R}_+ \times \mathbb{R}_+)$ and that assumptions $(\mathbf{A_2})$ and $(\mathbf{A_3})$ are satisfied. The optimal hedging ratio of the risky asset and initial endowment that allow the minimization of the quadratic risk over the set Θ are given by:

$$\begin{aligned}\bar{c} &= e^{-rT}\mathbb{E}^*[H], \quad (21)\\ \bar{\Delta}_t &= \bar{\Delta}(t, S_{t-}, Y_t),\end{aligned}$$

with

$$\begin{aligned}\bar{\Delta}(t, S, Y) &= \frac{1}{\sigma^2 + \int_{\mathbb{R}} e^{\beta^*(e^u-1)}(e^u - 1)^2 \nu_L(du)} \quad (22)\\ &\times \left[\sigma^2 \frac{\partial \phi}{\partial S}(u, S, Y) + \frac{1}{S}\int_{\mathbb{R}} e^{\beta^*(e^u-1)}(e^u - 1)\right.\\ &\times [\phi(t, Se^u, Y) - \phi(t, S, Y)]\nu_L(du)]\end{aligned}$$

Proof: See Appendix (C). ∎

VI. The MEMM and utility indifference pricing

The indifference price concept was first introduced by [22] in one dimensional Black and Scholes price model, where trading in the underlying involves a transaction cost. This concept was developed in order to define a price that could not be based any more on exact replication, since transaction costs where admitted, but was founded on agent preferences.

[17] was the first to use an alternative approach to compute indifference price, applying the general theory of convex analysis and duality results proven by [4]. He studied an incomplete market model driven by a d-dimensional semimartingale, where the set of admissible strategies was taken to be the set of predictable processes such that the corresponding wealth processes are uniformly bounded from below.

In a recent paper, [1] studies the problem of utility indifference pricing in a constrained financial market, using a utility function defined over the positive real line. He provides a dynamic programming equation associated with the risk measure, and characterize the last as a viscosity solution of this equation.

Let $U_\alpha(x), \alpha > 0$ be a CARA exponential utility function defined by

$$U_\alpha(x) = 1 - e^{-\alpha x}$$

and c be a real constant, Γ be a suitable set of trading strategies. The gain process of every strategy $\theta \in \Gamma$ is given by

$$G(\theta)_t = \int_0^t \theta_u dS_u,$$

Note that exponential utility implies constant absolute risk-aversion, with coefficient of absolute risk-aversion:

$$\frac{-U''(x)}{U(x)} = \alpha.$$

Though isoelastic utility, exhibiting constant relative risk-aversion, is considered more plausible, exponential utility is particularly convenient for many calculations.

We denote by $\mathcal{M}$ a convex subset of probability measures on $(\Omega, \mathbb{F}_T)$ which are absolutely continuous with respect to $\mathbb{P}$.
Let us take a *bounded* generalized Asian payoff $H = \varphi(S_T, Y_T)$ and consider the following assumption $(\mathbf{A_4})$:

$(\mathbf{A_4^1})$ The MEMM $\mathbb{P}^*$ belongs to $\mathcal{M}$ i.e.

$$\inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{H}(\mathbb{Q}|\mathbb{P}) = \mathbb{H}(\mathbb{P}^*|\mathbb{P}) < \infty \quad (23)$$

$(\mathbf{A_4^2})$ (Duality relation) [9], [4] : Duality relation holds.

$$\begin{aligned}&J_\alpha(c, H) = \sup_{\theta\in\Gamma} \mathbb{E}^{\mathbb{P}}[U_\alpha(c + G(\theta)_T - H)]\\ &= 1 - \exp\left[-\inf_{\mathbb{Q}\in\mathcal{M}} \left(\mathbb{H}(\mathbb{Q}|\mathbb{P}) + \alpha c - \mathbb{E}^{\mathbb{Q}}[\alpha H]\right)\right]\\ &J_\alpha(c, H)\\ &= 1 - e^{-\alpha c} \exp\left[\alpha \sup_{\mathbb{Q}\in\mathcal{M}} \left(\mathbb{E}^{\mathbb{Q}}[H] - \frac{1}{\alpha}\mathbb{H}(\mathbb{Q}|\mathbb{P})\right)\right]\end{aligned}$$

For the duality relation to hold [9] and [26] have considered locally bounded semimartingales. But the geometric jump-diffusion Lévy process $\hat{S}_t$ considered in this article does not necessarily possess this property. [4] has shown that for unbounded price process infimum of equation (23) may not be attained by an equivalent martingale measure. But [19] have established existence of the MEMM for the geometric Lévy process without using the assumption. So it would be natural to expect that Duality theorem hold. For details see [18]. Now we introduce utility indifference price $p_\alpha(c, H)$ [9]. The value $p_\alpha(c, H)$ which satisfies the following equation :

$$J_\alpha(c + p_\alpha(c, H), H) = J_\alpha(c, 0)$$

is called utility indifference price. Solution of above equation is written as

$$\begin{aligned} p_\alpha(c,H) &= \sup_{\mathbb{Q}\in\mathcal{M}}\left(\mathbb{E}^{\mathbb{Q}}[H]-\frac{1}{\alpha}\mathbb{H}(\mathbb{Q}|\mathbb{P})\right) \\ &+ \frac{1}{\alpha}\inf_{\mathbb{Q}\in\mathcal{M}}\mathbb{H}(\mathbb{Q}|\mathbb{P}) \end{aligned}$$

Remark that $p_\alpha(c,H)$ does not depend on c and will be denoted by $p_\alpha(H)$; i.e.

$$p_\alpha(H)=p_\alpha(0,H)=p_\alpha(c,H).$$

Let $\mathbb{Q}_\alpha^H$ be a probability measure in $\mathcal{M}$ such that

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}_\alpha^H}[H]-\frac{1}{\alpha}\mathbb{H}(\mathbb{Q}_\alpha^H|\mathbb{P}) \\ &> \sup_{\mathbb{Q}\in\mathcal{M}}\left(\mathbb{E}^{\mathbb{Q}}[H]-\frac{1}{\alpha}\mathbb{H}(\mathbb{Q}|\mathbb{P})\right)-\alpha. \end{aligned}$$

Theorem 2: **MEMM for bounded generalized Asian options**
Consider a generalized Asian options with bounded payoff $H=e^{-rT}\varphi(S_T,Y_T)$, (for example a fixed strike Asian put option with payoff $H=e^{-rT}\max(K-Y_T,0)$). Then following is true.

(1) $\lim_{\alpha\downarrow 0}\mathbb{H}(\mathbb{Q}_\alpha^H|\mathbb{P})=\mathbb{H}(\mathbb{P}^*|\mathbb{P})$.
(2) $\lim_{\alpha\downarrow 0}\|\mathbb{Q}_\alpha^H-\mathbb{P}^*\|_{var}=0$ where $\|.\|_{var}$ denotes total variation norm.
(3) $p_\alpha(H)\geq\mathbb{E}^{\mathbb{P}^*}[H]$ for any $\alpha>0$.
(4) If $0<\alpha<\beta$ then $p_\alpha(H)\leq p_\beta(H)$.
(5) $\lim_{\alpha\downarrow 0}p_\alpha(H)=\mathbb{E}^{\mathbb{P}^*}(H)$.

The proof of the theorem will exactly follow with the same lines of [19] for bounded contingent claim H. [5] has shown that the utility indifference price of any bounded claim lies within the interval of possible arbitrage-free valuations, i.e. **No-arbitrage consistency:** For $H\in L^\infty(\mathbb{P})$:

$$\inf_{\mathbb{Q}\in\text{EMM}(\mathbb{P})}\mathbb{E}\mathbb{Q}[H]\leq p_\alpha(H)\leq\sup_{\mathbb{Q}\in\text{EMM}(\mathbb{P})}\mathbb{E}^{\mathbb{Q}}[H].$$

It is also clear that the utility-indifference price tends to the super-replication price, which is the supremum of the expectations of H over all equivalent martingale measures, as the risk-aversion tends to infinity, price converges to the super-hedging price, that is, worst price under the viability assumption. On the other hand, as the absolute risk-aversion decreases to zero, the utility indifference price tends to a risk-neutral valuation under the minimal entropy martingale measure $\mathbb{P}^*$ which is the lower bound of the interval of arbitrage-free price i.e. infimum of expectation of H over all equivalent martingale measures. Above theorem for Asian put option and arguments portrays that MEMM $\mathbb{P}^*$ is the appropriate choice for pricing generalized Asian options.

VII. Concluding Remarks

In this paper, we studied generalized Asian option pricing and hedging problems in a financial market with jumps in the underlying under the minimal entropy martingale measure. Such measure is better than the method suggested by [16] in the presence of jumps: Their choice for semimartingale price process in the market measure, would end up with signed measure. But, it is well-known that this kind of measures are not robust with stopping times. Thus, an obvious extension of this theory to the case of Asian-American style options is not feasible to us.

We showed that the Asian option's price is subject to a time-dependent partial-integro differential equation. To solve this PIDE we need to customize boundary condition viewing the option constraint. The obtained PIDE is not very obvious to solve numerically and simulation techniques are also not very apparent. In subsection (IV-B) we tried to make some vague study of the numerical computation. Precise study of the numerical computation of the PIDE regarding consistency, stability and comparison with empirical data, this is what we intend to do in future works.

We also showed that the utility indifference price for bounded Asian option payoffs, converges to the risk-neutral valuation under the MEMM, as the risk-aversion goes to zero. For the case of unbounded generalized Asian options, the question remains open.

Appendix A
Proof of Lemma 1

Proof of lemma 1: Let $S_0>0$, $Y_0>0$ and $K_0=[0,T]\times[0,S_0]\times[0,Y_0]$. For $(t,S,Y),(t',S',Y')\in K_0$, we have taking into account assumption $(\mathbf{A_1})$:

$$\begin{aligned} & |\phi(t,S,Y)-\phi(t',S',Y')|\leq e^{-r(T-t)}\times \\ & \mathbb{E}^*\left[\left|\varphi\left(\frac{S}{S_0}S_{T-t},\frac{t}{T}Y+\frac{S}{TS_0}\int_0^{T-t}S_u du\right)\right.\right. \\ & \left.\left.-\varphi\left(\frac{S'}{S_0}S_{T-t},\frac{t}{T}Y'+\frac{S'}{TS_0}\int_0^{T-t}S_u du\right)\right|\right] \\ + & \left|e^{-r(T-t)}-e^{-r(T-t')}\right| \\ & \times\mathbb{E}^*\left[\left|\varphi\left(\frac{S'}{S_0}S_{T-t},\frac{t}{T}Y'+\frac{S'}{TS_0}\int_0^{T-t}S_u du\right)\right|\right] \\ \leq & \ \xi(T)\left[|S'-S|+|Y'-Y|\right]+re^{rT}|t'-t|\left(|\varphi(0,0)|\right. \\ + & \left.\xi(T)(|S_0|+|Y_0|)\right) \\ \leq & \ a_0[|S'-S|+|Y'-Y|+|t'-t|], \end{aligned}$$

where $\xi(T):=a\left(e^{rT}+\frac{1}{r}(e^{rT}-1)+1\right)$ and $a_0=\max\left\{\xi(T),re^{rT}\left(|\varphi(0,0)|+\xi(T)(|S_0|+|Y_0|)\right)\right\}$.

Appendix B
Proof of Proposition 1

Let $\hat{V}_t=e^{-rt}V_t=e^{-rt}\phi(t,S_t,Y_t):=\hat{\phi}(t,S_t,Y_t)$ be the discounted value of the option at time t. The continuity of the function ϕ derives from Lemma (1). We have

$$\begin{aligned} dS_t &= e^{rt}\left(d\hat{S}_t+r\hat{S}_t dt\right)=S_{t^-}\left(\sigma dB_t^*+\varsigma dt\right), \\ dY_t &= \frac{S_t-Y_t}{t}dt, \end{aligned}$$

with $\varsigma=r-\lambda\mathbb{E}^*[V_1]=r+\mu+\beta^*\sigma^2$, taking account of $(\mathbf{A_3})$, and $Y_0=\lim_{t\to 0}\frac{1}{t}\int_0^t S_u du=S_0$, a.s., Then by application of Itô's lemma on each time interval $[\tau_{j-1},\tau_j]$ to

$\hat{\phi}(t, S_t, Y_t)$, $t < T$, we get after addition

$$\begin{aligned}\hat{V}_t &= \phi(0, S_0, Y_0) - r\int_0^t e^{-ru}\phi(u, S_u, Y_u)du\\ &+ \int_0^t e^{-ru}\frac{\partial\phi}{\partial t}(u, S_u, Y_u)du\\ &+ \int_0^t e^{-ru}\frac{\partial\phi}{\partial S}(u, S_u, Y_u)S_{u^-}(\sigma dB_u^* + \varsigma du)\\ &+ \int_0^t e^{-ru}\frac{\partial\phi}{\partial Y}(u, S_u, Y_u)\frac{S_u - Y_u}{u}du\\ &+ \frac{1}{2}\int_0^t e^{-ru}\frac{\partial^2\phi}{\partial S^2}(u, S_u, Y_u)\,\sigma^2 S_{u^-}^2 du + \Sigma_t,\end{aligned}$$

where

$$\Sigma_t = \sum_{j=1}^{N_t}\left[\hat{\phi}(\tau_j, S_{\tau_j}, Y_{\tau_j}) - \hat{\phi}(\tau_j, S_{\tau_j^-}, Y_{\tau_j})\right].$$

Since $S_{\tau_j} = (1 + V_j)S_{\tau_j^-} = S_0 e^{U_j + X_{\tau_j^-}} = S_0 e^{\Delta X_{\tau_j} + X_{\tau_j^-}}$ with $X_t = L_t + rt$, we obtain:

$$\begin{aligned}&\Sigma_t =\\ &\sum_{j=1}^{N_t}\left[\hat{\phi}(\tau_j, S_0 e^{\Delta X_{\tau_j} + X_{\tau_j^-}}, Y_{\tau_j}) - \hat{\phi}(\tau_j, S_0 e^{X_{\tau_j^-}}, Y_{\tau_j})\right]\\ &= \int_0^t\int_{\mathbb{R}}\left[\hat{\phi}(s, S_0 e^{u + X_{s^-}}, Y_s) - \hat{\phi}(s, S_0 e^{X_{s^-}}, Y_s)\right]\\ &\quad\times J_L(ds, du)\\ &= \int_0^t\int_{\mathbb{R}}\left[\hat{\phi}(s, S_{s^-}e^u, Y_s) - \hat{\phi}(s, S_{s-}, Y_s)\right]\\ &\quad\times J_L(ds, du) = C_t + D_t,\end{aligned}$$

where

$$\begin{aligned}C_t &= \int_0^t\int_{\mathbb{R}}\left[\hat{\phi}(s, S_{s^-}e^u, Y_s) - \hat{\phi}(s, S_{s-}, Y_s)\right]\\ &\times\ (J_L - \nu_L^*)(ds, du)\\ D_t &= \int_0^t\int_{\mathbb{R}}\left[\hat{\phi}(s, S_{s^-}e^u, Y_s) - \hat{\phi}(s, S_{s-}, Y_s)\right]\\ &\times\ \nu_L^*(du)ds\\ &= \lambda\int_0^t\int_{\mathbb{R}} e^{\beta^*(e^u - 1)}\left[\hat{\phi}(s, S_{s^-}e^u, Y_s)\right.\\ &- \left.\hat{\phi}(s, S_{s-}, Y_s)\right]F(du)ds\\ &= \lambda\int_0^t\int_{-1}^{+\infty} e^{\beta^* v}\left[\hat{\phi}(s, S_{s^-}(1 + v), Y_s)\right.\\ &- \left.\hat{\phi}(s, S_{s-}, Y_s)\right]G(dv)ds,\end{aligned}$$

Hence,

$$\begin{aligned}\hat{V}_t &= \phi(0, S_0, Y_0) + \sigma\int_0^t e^{-ru}\frac{\partial\phi}{\partial S}(u, S_u, Y_u)S_u dB_u^*\\ &+ \int_0^t e^{-ru}k_u du + C_t\end{aligned}$$

where $k_.$ is the process defined by:

$$k_u = \mathcal{L}\phi(u, S_u, Y_u),$$

$\mathcal{L}$ being the parabolic-hyperbolic integro differential operator defined by Equation (15).

Since the process $\hat{V}_t$ is a martingale, as well as the process C_t, the process k vanishes almost surely and we consequently obtain:

$$\begin{aligned}\hat{V}_t - C_t &= \phi(0, S_0, Y_0)\\ &+ \sigma\int_0^t e^{-rt}\frac{\partial\phi}{\partial S}(u, S_u, Y_u)S_u dB_u^*.\end{aligned}\tag{24}$$

Appendix C
Proof of Proposition 2

Since for all admissible strategy $\Delta \in \Theta$, $\mathbb{E}^*\left[\int_0^T \Delta_u d\hat{S}_u\right] = 0$, one has :

$$\begin{aligned}\mathbb{E}^*[\epsilon(c, \Delta)^2] &= (c - \mathbb{E}^*[\hat{H}])^2\\ &+ \mathbb{E}^*\left[\left(\mathbb{E}^*[\hat{H}] - \hat{H} + \int_0^T \Delta_u d\hat{S}_u\right)^2\right].\end{aligned}$$

The last equation shows that the optimal initial capital for the problem $(\mathcal{P})$ is given by $\bar{c} = \mathbb{E}^*[\hat{H}]$. After replacing c by $\bar{c}$ in the expression of the quadratic criterion to be minimized, we obtain:

$$\mathbb{E}^*[\epsilon(\bar{c}, \Delta)^2] = \mathbb{E}^*\left[\left(\mathbb{E}^*[\hat{H}] - \hat{H} + \int_0^T \Delta_u d\hat{S}_u\right)^2\right].$$

Let $\hat{V}_t = e^{-rt}V_t = e^{-rt}\phi(t, S_t, Y_t)$ be the discounted value of the option at time t. Equation (24) is written as:

$$\begin{aligned}\hat{\phi}(t, S_t, Y_t) &= \phi(0, S_0, Y_0) + \sigma\int_0^t\frac{\partial\phi}{\partial S}(u, S_u, Y_u)\hat{S}_u dB_u^*\\ &+ C_t,\end{aligned}$$

where

$$\begin{aligned}C_t &= \int_0^t\int_{\mathbb{R}}\left[\hat{\phi}(s, S_{s^-}e^u, Y_s) - \hat{\phi}(s, S_{s-}, Y_s)\right]\\ &\times\ (J_L - \nu_L^*)(ds, du).\end{aligned}$$

We have

Lemma 2: The process C_t is a $\mathbb{P}^*$-square integrable martingale.

Proof: See Appendix (D). ∎

Then it comes from Proposition 8.8 of [8], that the compensated Poisson integral C_t is a $\mathbb{P}^*$-square integrable martingale.

Now, from Equation (19) we obtain:

$$\hat{H} - \hat{V}_T(\mathcal{K}) = \hat{\phi}(T, S_T, A_T) - \hat{V}_T(\mathcal{K}) = \alpha_T + \gamma_T,$$

where

$$\begin{aligned}\alpha_t &= \int_0^t\left(\frac{\partial\phi}{\partial S}(u, S_u, Y_u) - \Delta_u\right)\sigma\hat{S}_u dB_u^*\\ \gamma_t &= C_t - \sum_{j=1}^{N_t}\Delta_{\tau_j}V_j\hat{S}_{\tau_j^-} + \lambda\mathbb{E}^*[V_1]\int_0^t du\Delta_u\hat{S}_u.\\ &= C_t - \sum_{j=1}^{N_t}\Delta_{\tau_j}V_j\hat{S}_{\tau_j^-} - (\mu + \beta^*\sigma^2)\int_0^t du\Delta_u\hat{S}_u.\end{aligned}$$

since $\lambda\mathbb{E}^*[V_1] = -(\mu + \beta^*\sigma^2)$ by assumption $(\mathbf{A_3})$. We can easily verify that:

$$\sum_{j=1}^{N_t}\Delta_{\tau_j}V_j\hat{S}_{\tau_j^-} = \int_0^t\int_{\mathbb{R}}\Delta_s\hat{S}_{s^-}(e^u - 1)J_L(ds, du).$$

It follows that:

$$\begin{aligned}\gamma_t &= \int_0^t \int_{\mathbb{R}} \Big[\hat{\phi}(s, S_{s^-}e^u, Y_s) - \hat{\phi}(s, S_{s-}, Y_s) \\ &- \Delta_s \hat{S}_{s^-}(e^u - 1)\Big] (J_L - \nu_L^*)(ds, du).\end{aligned}$$

Since the processes α_t and γ_t are two $\mathbb{P}^*$-zero-mean square integrable martingales, it comes that $\alpha_t\gamma_t$ is a martingale null at time $t = 0$. Hence, according to the isometry formula given by Proposition 8.8 of [8] we get:

$$\begin{aligned}&\mathbb{E}^*\left[\left(\hat{H} - \hat{V}_T(\mathcal{K})\right)^2\right] = \mathbb{E}^*[\alpha_T^2] + \mathbb{E}^*[\gamma_T^2] \\ &= \mathbb{E}^*\left[\int_0^T \left(\frac{\partial\phi}{\partial S}(u, S_u, Y_u) - \Delta_u\right)^2 \sigma^2 \hat{S}_u^2 du\right] + \mathbb{E}^*[\gamma_T^2].\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}^*[\gamma_T^2] &= \mathbb{E}^*\Bigg[\int_0^T \int_{\mathbb{R}} [\hat{\phi}(t, S_{t^-}e^u, Y_t) \\ &- \hat{\phi}(t, S_{t-}, Y_t) - \Delta_t \hat{S}_{t^-}(e^u - 1)]^2 \nu_L^*(du)dt\Bigg].\end{aligned}$$

The quadratic risk is finally written as:

$$\begin{aligned}&R_T(\mathcal{K}) = \mathbb{E}^*\Bigg[\int_0^T \left(\frac{\partial\phi}{\partial S}(t, S_t, Y_t) - \Delta_t\right)^2 \sigma^2 \hat{S}_t^2 dt \\ &+ \int_0^T \int_{\mathbb{R}} [\hat{\phi}(t, S_{t^-}e^u, Y_t) - \hat{\phi}(t, S_{t-}, Y_t) \\ &-\Delta_t \hat{S}_{t^-}(e^u - 1)]^2 \nu_L^*(du)dt\Bigg].\end{aligned}$$

The optimal strategy $\bar{\mathcal{K}} = (\bar{\Delta}^0, \bar{\Delta})$ is obtained by differentiating the last quadratic expression, and should satisfy $\mathbb{P}^*$-a.s.

$$\begin{aligned}&\left(\frac{\partial\phi}{\partial S}(t, S_t, Y_t) - \bar{\Delta}_t\right) \sigma^2 \hat{S}_t^2 \\ &+ \hat{S}_{t^-} \int_{\mathbb{R}} (e^u - 1)[\hat{\phi}(t, S_{t^-}e^u, Y_t) \\ &-\hat{\phi}(t, S_{t-}, Y_t) - \bar{\Delta}_t \hat{S}_{t^-}(e^u - 1)]\nu_L^*(du) = 0,\end{aligned}$$

which is rewritten again as:

$$\begin{aligned}&\left(\frac{\partial\phi}{\partial S}(t, S_t, Y_t) - \bar{\Delta}_t\right) \sigma^2 {S_t}^2 \\ &+ S_{t^-} \int_{\mathbb{R}} e^{\beta(e^u - 1)}(e^u - 1)[\phi(t, S_{t^-}e^u, Y_t) \\ &-\phi(t, S_{t-}, Y_t) - \bar{\Delta}_t S_{t^-}(e^u - 1)]\nu_L(du) = 0.\end{aligned}$$

Since the process $(\bar{\Delta}_t)_{t\geq 0}$ is left-continuous, we obtain:

$$\bar{\Delta}_t = \bar{\Delta}(t, S_{t^-}, Y_t), \tag{25}$$

where $\bar{\Delta}$ is given by Equation (22).

Appendix D
Proof of Lemma 2

The process $(S_t)_{0\leq t\leq T}$ is $\mathbb{P}^*$-square integrable. Indeed,

$$\begin{aligned}\mathbb{E}^*[S_t^2] &= S_0^2 e^{2\mu_0 t}\mathbb{E}^*[e^{2\sigma B_t^*}]\mathbb{E}^*\left[\prod_{j=1}^{N_t}(1 + V_j)^2\right] \\ &= S_0^2 e^{2\mu_0 t + 2\sigma^2 t} \exp\left(\lambda t \left[\mathbb{E}^*[(1 + V_1)^2] - 1\right]\right) \\ &= S_0^2 \exp\left(2\mu_0 t + 2\sigma^2 t + 2\lambda t\mathbb{E}^*[V_1] + \lambda t\mathbb{E}^*[V_1^2]\right)\end{aligned}$$

Since $\lambda\mathbb{E}^*[V_1] = -(\mu + \beta^*\sigma^2)$ by assumption $(\mathbf{A_3})$ and $\mu_0 = \mu + \beta^*\sigma^2 + r$ it comes that

$$\mathbb{E}^*[S_t^2] = S_0^2 \exp\left(2(\sigma^2 + r)t + \lambda t\mathbb{E}^*[V_1^2]\right). \tag{26}$$

Denote:

$$\rho(t, u) := \phi(t, S_{t^-}e^u, Y_t) - \phi(t, S_{t-}, Y_t).$$

Taking account of assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$, we have for all $(t, S, S', Y) \in [0, T] \times \mathbb{R}^3$:

$$\begin{aligned}&|\phi(t, S, Y) - \phi(t, S', Y)| \\ &\leq \frac{a}{S_0}|S - S'|\mathbb{E}^*\left[S_{T-t} + \frac{1}{T}\int_0^{T-t} S_u du\right] \\ &\leq \frac{a}{S_0}|S - S'|\left(\mathbb{E}^*[S_{T-t}] + \frac{1}{T}\int_0^{T-t} \mathbb{E}^*[S_u]du\right) \\ &\leq b|S' - S|,\end{aligned}$$

where $b = a\left[e^{rT} + \frac{1}{rT}(e^{rT} - 1\right] < \infty$. Hence

$$\begin{aligned}&\mathbb{E}^*\left[\int_0^T \int_{\mathbb{R}} \rho^2(t, u)\nu_L^*(du)dt\right] \\ &\leq b^2\mathbb{E}^*\left(\int_0^T S_t^2 dt\right) \left|\int_{\mathbb{R}} (e^u - 1)^2 e^{\beta^*(e^u - 1)}\nu(du)\right| \\ &= b^2\left(\int_0^T \mathbb{E}^*[S_t^2]dt\right)\mathbb{E}^*[V_1^2] \\ &\leq b^2 T S_0^2 \exp\left(2(\sigma^2 + r)T + \lambda T\mathbb{E}^*[V_1^2]\right)\mathbb{E}^*[V_1^2] < \infty,\end{aligned}$$

where the last inequality is obtained thanks to (26).

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