Existence and Uniqueness of a Periodic Solution for Third-order Delay Differential Equation with Two Deviating Arguments

A. M. A. Abou-El-Ela, A. I. Sadek and A. M. Mahmoud

***Abstract*—In this paper we use the continuation theorem of coincidence degree theory and analysis techniques. We establish the existence and uniqueness of a T-periodic solution for the third-order delay differential equation with two deviating arguments.**

A new result on the existence and uniqueness of a periodic solution of this equation is obtained. In addition an example is given to illustrate the main result.

***Index Terms*—Existence and uniqueness, Coincidence degree, Third-order delay differential equations, Deviating arguments.**

I. INTRODUCTION

TO our knowledge, most exciting results on the existence of periodic solutions of delay differential equations are usually obtained by the technique of bifurcation, by fixed-point theorems or by coincidence degree theory. In general it is more difficult to study the uniqueness of the periodic solutions.

Continuation theorem of coincidence degree theory plays a significant role in the investigation of the existence and uniqueness of periodic solutions of differential equations with delay.

In recent years, by using continuation theorem of coincidence degree theory, the existence and uniqueness of periodic solutions for some types of first and second-order delay differential equations with a deviating argument or two deviating arguments were studied, for example, $[7-10, 12-16]$, etc. But the corresponding problem of third-order delay differential equations with a deviating argument or two deviating arguments was discussed far less often, for example, $[3, 11, 16]$. In 2006, Gui [3] established criteria for existence of positive periodic solutions to the following third-order neutral delay differential equation with a deviating argument

$$x^{(3)}(t) + a\ddot{x}(t) + g(\dot{x}(t-\tau(t))) + f(x(t-\tau(t))) = p(t),$$

where a is a positive constant; g, f and p are real continuous functions and are defined on R; $\tau(t), p(t)$ are periodic with period ω.

The main purpose of this paper is to provide new sufficient conditions for guaranteeing the existence and uniqueness of a T-periodic solution to third-order delay differential equation with two deviating arguments

$$\begin{aligned} &x^{(3)}(t) + \psi(\dot{x}(t))\ddot{x}(t) + f(x(t))\dot{x}(t) \\ &\quad + g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) = p(t), \end{aligned} \tag{1}$$

where $\psi, f, \tau_1, \tau_2, p : R \to R$ and $g_1, g_2 : R \times R \to R$ are continuous functions; τ_1, τ_2 and p are T-periodic, g_1 and g_2 are T-periodic in their first argument and $T > 0$.

II. PRELIMINARIES

For convenience we define

$|x|_k = (\int_0^T |x(t)|^k dt)^{\frac{1}{k}}, k \geq 1,\ |x|_\infty = \max_{t\in[0,T]} |x(t)|,$
$|p|_\infty = \max_{t\in[0,T]} |p(t)| \quad and \quad \bar{p} = \frac{1}{T}\int_0^T p(t)dt.$

Let

$$X = \{x | x \in C^2(R,R), x(t+T) = x(t),\ forall\ t \in R\}$$

and

$$Y = \{y | y \in C(R,R), y(t+T) = y(t),\ forall\ t \in R\}$$

be two Banach spaces with the norms

$$\|x\|_X = \max\{|x|_\infty, |\dot{x}|_\infty, |\ddot{x}|_\infty\}\ and\ \|y\|_Y = |y|_\infty.$$

Define a linear operator $L : D(L) \subset X \to Y$ by setting

$$D(L) = \{x | x \in X, x^{(3)}(t) \in C(R,R)\},$$

and for $x \in D(L)$,

$$Lx = x^{(3)}(t). \tag{2}$$

We also define a nonlinear operator $N : X \to Y$by setting

$$\begin{aligned} Nx = &-\psi(\dot{x}(t))\ddot{x}(t) - f(x(t))\dot{x}(t) \\ &-g_1(t, x(t-\tau_1(t))) - g_2(t, x(t-\tau_2(t))) + p(t). \end{aligned} \tag{3}$$

Then we notice that $KerL = R$, $dim(KerL) = 1$; $ImL = \{y | y \in Y,\ \int_0^T y(s)ds = 0\}$ is a subset of Y and $dim(Y/ImL) = 1$, which imply $codim(ImL) = dim(KerL)$.
Thus the operator L is a Fredholm operator with index zero. Define the continuous projectors $P : X \to KerL$ and $Q : Y \to Y/ImL$ by

$$Px(t) = x(0) = x(T) \quad and \quad Qy(t) = \frac{1}{T}\int_0^T y(s)ds,$$

hence $ImP = ImQ = KerL = R$ and $KerQ = ImL$. Let $L_P := L_{D(L)\cap KerP} : D(L) \cap KerP \to ImL$, then L_P has a continuous inverse L_P^{-1} on ImL defined by

$$(L_P^{-1}y)(t) = \int_0^T G(s,t)y(s)ds,$$
$$G(s,t) = \begin{cases} -\frac{s}{T}(T-t), & 0 \leq s \leq t; \\ -\frac{t}{T}(T-s), & t \leq s \leq T. \end{cases}$$

A. M. A. Abou-El-Ela is with the Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt. (A_El−Ela@aun.edu.eg).

A. I. Sadek is with the Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt. (Sadeka1961@hotmail.com).

A. M. Mahmoud is with the Department of Science and Mathematics, Faculty of Education, Assiut University, New Valley, El-khargah 72111, Egypt. (math_ayman27@yahoo.com).

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By using Ascoli-Arzela theorem we have from the above equation, that $L_P^{-1}(I-Q)N(\bar{\Omega})$ is compact. On the other hand, $QN(\bar{\Omega})$ is bounded by continuity of function QN. Thus N is L-compact on $\bar{\Omega}$, where Ω is an open bounded subset in X.
In view of (2) and (3) the operator equation

$$Lx = \lambda Nx,\ \lambda \in (0,1);$$

is equivalent to the following equation

$$\begin{aligned} &x^{(3)}(t) + \lambda\{\psi(\dot{x}(t))\ddot{x}(t) + f(x(t))\dot{x}(t) \\ &+g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t)))\} = \lambda p(t). \end{aligned} \tag{4}$$

The following Mawhin's continuation theorem is useful in obtaining the existence of T-periodic solution of (1) [2].

Theorem 2.1: ***(Mawhin's continuation theorem)*** Let X and Y be two Banach spaces. Suppose that $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero and $N : X \to Y$ is L-compact on $\bar{\Omega}$, where Ω is an open bounded subset in X. Moreover assume that all the following conditions are satisfied:

(a) $Lx \neq \lambda Nx,\ forall\ x \in \partial\Omega \cap D(L), \lambda \in (0,1);$
(b) $Nx \notin ImL,\ forall\ x \in \partial\Omega \cap KerL;$
(c) The Brower degree

$$deg\{QN, \Omega \cap KerL, 0\} \neq 0.$$

Then $Lx = Nx$ has at least one solution on $\bar{\Omega} \cap D(L)$.

Lemma 2.1: It is convenient to introduce the following assumptions:

(i) Assume that there exist non-negative constants a_1, a_2, b_1, b_2, c_1 and c_2 such that

$$|\psi(y)| \leq a_1,\ \ |\psi(y_1) - \psi(y_2)| \leq a_2|y_1 - y_2|$$

for all $y, y_1, y_2 \in R$,

$$|f(x)| \leq c_1,\ \ |f(x_1) - f(x_2)| \leq c_2|x_1 - x_2|$$

for all $x, x_1, x_2 \in R$ and

$$|g_i(t,u) - g_i(t,v)| \leq b_i|u - v|$$

for all $t, u, v \in R,\ i = 1, 2$.

(ii) Suppose that the following conditions are satisfied:
(H_1) One of the following conditions holds
(1) $(g_i(t,u) - g_i(t,v))(u-v) > 0$ for all $t, u, v \in R, u \neq v, i = 1, 2,$
(2) $(g_i(t,u) - g_i(t,v))(u-v) < 0$ for all $t, u, v \in R, u \neq v, i = 1, 2;$
(H_2) There exists $d > 0$ such that one of the following conditions holds
(1) $x\{g_1(t,x) + g_2(t,x) - \bar{p}\} > 0$ for all $t \in R, |x| > d,$
(2) $x\{g_1(t,x) + g_2(t,x) - \bar{p}\} < 0$ for all $t \in R, |x| > d;$
If $x(t)$ is a T-periodic solution of (4), then

$$|x|_\infty \leq d + \tfrac{1}{2}\sqrt{T}|\dot{x}|_2. \tag{5}$$

(iii) Assume that (i) and (ii) hold such that

$$a_1\tfrac{T}{2} + c_1\tfrac{T^2}{4} + (b_1 + b_2)\tfrac{T^3}{8} < 1. \tag{6}$$

If $x(t)$ is a T-periodic solution of (1), then

$$\begin{aligned} &|\ddot{x}|_\infty \\ &\leq \frac{[(b_1+b_2)d + \max\{|g_1(t,0)| + |g_2(t,0)| : 0 \leq t \leq T\} + |p|_\infty]T}{2\{1 - a_1\frac{T}{2} - c_1\frac{T^2}{4} - (b_1+b_2)\frac{T^3}{8}\}} \\ &:= k. \end{aligned} \tag{7}$$

(iv) Suppose that $(i) - (iii)$ hold. Also let the following condition holds

$$\begin{aligned} &a_1\tfrac{T}{2} + \left(a_2 k + c_1\right)\tfrac{T^2}{4} \\ &\quad + \left(b_1 + b_2 + c_2 k\tfrac{T}{2}\right)\tfrac{T^3}{8} < 1. \end{aligned} \tag{8}$$

Then (1) has at most one T-periodic solution.

Proof of $(\boldsymbol{ii})$. Let $x(t)$ be an arbitrary T-periodic solution of (4). Then by integrating (4) from 0 to T we have

$$\begin{aligned} &\textstyle\int_0^T \{g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) \\ &\quad -p(t)\}dt = 0, \end{aligned} \tag{9}$$

which implies that there exists $t_1 \in R$ such that

$$\begin{aligned} &g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) \\ &\quad -\bar{p} = 0. \end{aligned} \tag{10}$$

Next we show that the following claim is true.

Claim. If $x(t)$ is a T-periodic solution of (4), then there exists $t_2 \in R$ such that

$$|x(t_2)| \leq d. \tag{11}$$

Assume by the way of contradiction that (11) does not hold. Then

$$|x(t)| > d\ \ forall\ t \in R, \tag{12}$$

which together with $(H_1), (H_2)$ and (10) imply that one of the following four relations holds:

$$x(t_1-\tau_1(t_1)) > x(t_1-\tau_2(t_1)) > d, \tag{13}$$

$$x(t_1-\tau_2(t_1)) > x(t_1-\tau_1(t_1)) > d, \tag{14}$$

$$x(t_1-\tau_1(t_1)) < x(t_1-\tau_2(t_1)) < -d, \tag{15}$$

$$x(t_1-\tau_2(t_1)) < x(t_1-\tau_1(t_1)) < -d. \tag{16}$$

Suppose that (13) holds, in view of $(H_1)(1), (H_1)(2), (H_2)(1)$ and $(H_2)(2)$ we consider four cases as follows:

Case I: If $(H_1)(1)$ and $(H_2)(1)$ hold, according to (13) we have

$$\begin{aligned} 0 &< g_1(t_1, x(t_1-\tau_2(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - \bar{p} \\ &< g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - \bar{p}, \end{aligned}$$

which contradicts (10). Thus (11) is true.

Case II: If $(H_1)(2)$ and $(H_2)(1)$ hold, according to (13) we have

$$\begin{aligned} 0 &< g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_1(t_1))) - \bar{p} \\ &< g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - \bar{p}, \end{aligned}$$

which contradicts (10). Thus (11) is true.

Case III: If $(H_1)(1)$ and $(H_2)(2)$ hold, according to (13) we have

$$\begin{aligned} 0 &> g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_1(t_1))) - \bar{p} \\ &> g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - \bar{p}, \end{aligned}$$

which contradicts (10). Thus (11) is true.

Case IV: If $(H_1)(2)$ and $(H_2)(2)$ hold, according to (13) we have

$$\begin{aligned} 0 &> g_1(t_1, x(t_1-\tau_2(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - \bar{p} \\ &> g_1(t_1, x(t_1-\tau_1(t_1))) + g_2(t_1, x(t_1-\tau_2(t_1))) - \bar{p}, \end{aligned}$$

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which contradicts (10). Thus (11) is true.
Suppose (14) $(or(15), or(16))$ holds; using the methods similarly to those used in Case I-IV we can show that (11) is true.
This completes the proof of the above claim.
Let $t_2 = mT + t_0$, where $t_0 \in [0, T]$ and m is an integer, then from (11) and for any $t \in [t_0, t_0 + T]$ we obtain

$$\begin{aligned}|x(t)| &= |x(t_0) + \int_{t_0}^{t} \dot{x}(s)ds| \\ &< d + \int_{t_0}^{t} |\dot{x}(s)|ds.\end{aligned}$$

Also

$$\begin{aligned}|x(t)| &= |x(t_0 + T) + \int_{t_0+T}^{t} \dot{x}(s)ds| \\ &\le d + \int_{t}^{t_0+T} |\dot{x}(s)|ds.\end{aligned}$$

By combining the above two inequalities we find

$$|x(t)| \le d + \frac{1}{2}\int_0^T |\dot{x}(s)|ds.$$

Using the Cauchy-Schwarz inequality yields

$$\begin{aligned}|x(t)| &\le d + \frac{1}{2}\sqrt{T}(\int_0^T |\dot{x}(s)|^2 ds)^{\frac{1}{2}} \\ &= d + \frac{1}{2}\sqrt{T}|\dot{x}|_2.\end{aligned}$$

Therefore we have

$$\begin{aligned}|x|_\infty &= \max_{t\in[0,T]} |x(t)| \\ &\le d + \frac{1}{2}\sqrt{T}|\dot{x}|_2.\end{aligned}$$

This completes the proof of condition (ii) in Lemma 2.1.
Proof of $(\boldsymbol{iii})$. Let $x(t)$ be a T-periodic solution of (1) for a certain $\lambda \in (0, 1)$. Multiplying (1) by $x^{(3)}(t)$ and then integrating it over $[0, T]$ and by using condition (i), we find

$$\begin{aligned}&|x^{(3)}(t)|_2^2 \le a_1 \int_0^T |\ddot{x}(t)||x^{(3)}(t)|dt \\ &+c_1 \int_0^T |\dot{x}(t)||x^{(3)}(t)|dt + \int_0^T |p(t)||x^{(3)}(t)|dt \\ &+ \int_0^T \{|g_1(t, x(t-\tau_1)) - g_1(t, 0)| + |g_1(t, 0)|\}|x^{(3)}(t)|dt \\ &+ \int_0^T \{|g_2(t, x(t-\tau_2)) - g_2(t, 0)| + |g_2(t, 0)|\}|x^{(3)}(t)|dt.\end{aligned}$$

Then we get

$$\begin{aligned}&|x^{(3)}(t)|_2^2 \le a_1 \int_0^T |\ddot{x}(t)||x^{(3)}(t)|dt \\ &+c_1 \int_0^T |\dot{x}(t)||x^{(3)}(t)|dt \\ &+b_1 \int_0^T |x(t-\tau_1(t))||x^{(3)}(t)|dt \\ &+ \int_0^T |g_1(t, 0)||x^{(3)}(t)|dt \\ &+b_2 \int_0^T |x(t-\tau_2(t))||x^{(3)}(t)|dt \\ &+ \int_0^T |g_2(t, 0)||x^{(3)}(t)|dt + \int_0^T |p(t)||x^{(3)}(t)|dt \\ &\quad \le a_1 \int_0^T |\ddot{x}(t)||x^{(3)}(t)|dt \\ &+c_1 \int_0^T |\dot{x}(t)||x^{(3)}(t)|dt \\ &+b_1|x|_\infty \int_0^T |x^{(3)}(t)|dt + |p|_\infty \int_0^T |x^{(3)}(t)|dt \\ &+b_2|x|_\infty \int_0^T |x^{(3)}(t)|dt \\ &+ \max\{|g_1(t, 0)| + |g_2(t, 0)| : 0 \le t \le T\} \int_0^T |x^{(3)}(t)|dt.\end{aligned}$$

Thus from (5) we obtain

$$\begin{aligned}&|x^{(3)}(t)|_2^2 \le a_1 \int_0^T |\ddot{x}(t)||x^{(3)}(t)|dt \\ &+c_1 \int_0^T |\dot{x}(t)||x^{(3)}(t)|dt \\ &+\frac{1}{2}(b_1 + b_2)\sqrt{T}|\dot{x}|_2 \int_0^T |x^{(3)}(t)|dt \\ &+[(b_1 + b_2)d + \max\{|g_1(t, 0)| + |g_2(t, 0)| : 0 \le t \le T\} \\ &\quad +|p|_\infty] \int_0^T |x^{(3)}(t)|dt.\end{aligned}$$

By using Cauchy-Schwarz inequality we find

$$\begin{aligned}|x^{(3)}(t)|_2^2 &\le a_1|\ddot{x}|_2|x^{(3)}|_2 + c_1|\dot{x}|_2|x^{(3)}|_2 \\ &+\frac{1}{2}(b_1 + b_2)T|\dot{x}|_2|x^{(3)}|_2 + [(b_1 + b_2)d \\ &+ \max\{|g_1(t, 0)| + |g_2(t, 0)| : 0 \le t \le T\} \\ &+|p|_\infty]\sqrt{T}|x^{(3)}|_2.\end{aligned} \tag{17}$$

Since $x(0) = x(T)$, there exists a constant $\xi \in [0, T]$ such that $\dot{x}(\xi) = 0$ and

$$\begin{aligned}|\dot{x}(t)| &= |\dot{x}(\xi) + \int_\xi^t \ddot{x}(s)ds| \\ &\le \int_\xi^t |\ddot{x}(s)|ds,\ t \in [\xi, T + \xi].\end{aligned} \tag{18}$$

Again

$$\begin{aligned}|\dot{x}(t)| &= |\dot{x}(\xi + T) + \int_{\xi+T}^t \ddot{x}(s)ds| \\ &\le |\dot{x}(\xi + T)| + \int_t^{\xi+T} |\ddot{x}(s)|ds \\ &= \int_t^{\xi+T} |\ddot{x}(s)|ds, \quad t \in [0, T].\end{aligned} \tag{19}$$

The inequalities (18) and (19) imply

$$\begin{aligned}2|\dot{x}(t)| &\le \int_\xi^t |\ddot{x}(s)|ds + \int_t^{\xi+T} |\ddot{x}(s)|ds \\ &= \int_0^T |\ddot{x}(s)|ds.\end{aligned}$$

Therefore by using Cauchy-Schwarz inequality we have

$$|\dot{x}(t)| \le \frac{1}{2}\sqrt{T}(\int_0^T |\ddot{x}(s)|^2 ds)^{\frac{1}{2}},\ forall\ t \in [0, T], \tag{20}$$

so

$$|\dot{x}|_\infty \le \frac{1}{2}\sqrt{T}|\ddot{x}|_2, \tag{21}$$

$$\begin{aligned}|\dot{x}|_2 &\le \sqrt{T}\max_{t\in[0,T]} |\dot{x}(s)| \\ &\le \frac{1}{2}T(\int_0^T |\ddot{x}(s)|^2 ds)^{\frac{1}{2}} \\ &= \frac{1}{2}T|\ddot{x}|_2.\end{aligned} \tag{22}$$

Since $x(t)$ is periodic function for $t \in [0, T]$ and by using the above similar technique, we find

$$|\ddot{x}(t)| \le \frac{1}{2}\int_0^T |x^{(3)}(t)|dt.$$

Which together with Cauchy-Schwarz inequality imply

$$\begin{aligned}|\ddot{x}|_\infty &\le \frac{1}{2}\sqrt{T}(\int_0^T |x^{(3)}(s)|^2 ds)^{\frac{1}{2}} \\ &= \frac{1}{2}\sqrt{T}|x^{(3)}|_2,\end{aligned} \tag{23}$$

$$\begin{aligned}|\ddot{x}|_2 &\le \sqrt{T}\max_{t\in[0,T]} |\ddot{x}(s)| \\ &\le \frac{1}{2}\sqrt{T}\int_0^T |x^{(3)}(s)|ds \\ &\le \frac{1}{2}T|x^{(3)}|_2.\end{aligned} \tag{24}$$

By substituting from (24) in (22) we get

$$|\dot{x}|_2 \le \frac{1}{4}T^2|x^{(3)}|_2. \tag{25}$$

By substituting from (24) in (21) we have

$$|\dot{x}|_\infty \le \frac{1}{4}T^{\frac{3}{2}}|x^{(3)}|_2. \tag{26}$$

From (5) and (25) we obtain

$$|x|_\infty \le d + \frac{1}{8}T^{\frac{5}{2}}|x^{(3)}|_2. \tag{27}$$

Then by substituting from (24) and (25) in (17) we find

$$\begin{aligned}&|x^{(3)}|_2^2 \le a_1\frac{T}{2}|x^{(3)}|_2^2 + c_1\frac{T^2}{4}|x^{(3)}|_2^2 + (b_1 + b_2)\frac{T^3}{8}|x^{(3)}|_2^2 \\ &+[(b_1 + b_2)d + \max\{|g_1(t, 0)| + |g_2(t, 0)| : 0 \le t \le T\} \\ &+|p|_\infty]\sqrt{T}|x^{(3)}|_2.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}&|x^{(3)}|_2 \\ &\le \frac{[(b_1+b_2)d+\max\{|g_1(t,0)|+|g_2(t,0)|:0\le t\le T\}+|p|_\infty]\sqrt{T}}{1-a_1\frac{T}{2}-c_1\frac{T^2}{4}-(b_1+b_2)\frac{T^3}{8}}.\end{aligned} \tag{28}$$

By substituting from (28) in (23) and (26) we find

$$\begin{aligned}|\ddot{x}|_\infty &\le \frac{[(b_1+b_2)d+\max\{|g_1(t,0)|+|g_2(t,0)|:0\le t\le T\}+|p|_\infty]T}{2\{1-a_1\frac{T}{2}-c_1\frac{T^2}{4}-(b_1+b_2)\frac{T^3}{8}\}} \\ &:= k,\end{aligned} \tag{29}$$

$$\begin{array}{l} |\dot{x}|_{\infty} \\ \leq \frac{[(b_1+b_2)d+\max\{|g_1(t,0)|+|g_2(t,0)|:0\leq t\leq T\}+|p|_{\infty}]T^2}{4\{1-a_1\frac{T}{2}-c_1\frac{T^2}{4}-(b_1+b_2)\frac{T^3}{8}\}} \\ := \frac{T}{2}k. \end{array} \quad (30)$$

This completes the proof of condition (iii) in Lemma 2.1.

Proof of $(\boldsymbol{iv})$. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1), then we have

$$\begin{array}{l} x_1^{(3)}(t)-x_2^{(3)}(t)+\psi(\dot{x}_1(t))\ddot{x}_1(t)-\psi(\dot{x}_2(t))\ddot{x}_2(t) \\ \quad +f(x_1(t))\dot{x}_1(t)-f(x_2(t))\dot{x}_2(t) \\ \quad +g_1(t,x_1(t-\tau_1(t)))-g_1(t,x_2(t-\tau_1(t))) \\ \quad +g_2(t,x_1(t-\tau_2(t)))-g_2(t,x_2(t-\tau_2(t)))=0. \end{array}$$

Set

$$z(t)=x_1(t)-x_2(t).$$

Then we find

$$\begin{array}{l} z^{(3)}(t)+\{\psi(\dot{x}_1(t))\ddot{x}_1(t)-\psi(\dot{x}_2(t))\ddot{x}_2(t)\} \\ +\{f(x_1(t))\dot{x}_1(t)-f(x_2(t))\dot{x}_2(t)\} \\ +\{g_1(t,x_1(t-\tau_1(t)))-g_1(t,x_2(t-\tau_1(t)))\} \\ +\{g_2(t,x_1(t-\tau_2(t)))-g_2(t,x_2(t-\tau_2(t)))\}=0. \end{array} \quad (31)$$

Since $x_1(t)$ and $x_2(t)$ are two T-periodic solutions, by integrating (31) over $[0,T]$ we obtain

$$\begin{array}{l} \int_0^T\{g_1(t,x_1(t-\tau_1(t)))-g_1(t,x_2(t-\tau_1(t))) \\ \quad +g_2(t,x_1(t-\tau_2(t)))-g_2(t,x_2(t-\tau_2(t)))\}dt=0. \end{array}$$

By using the integral mean-value theorem, it follows that there exists a constant $\gamma\in[0,T]$ such that

$$\begin{array}{l} g_1(\gamma,x_1(\gamma-\tau_1(\gamma)))-g_1(\gamma,x_2(\gamma-\tau_1(\gamma))) \\ +g_2(\gamma,x_1(\gamma-\tau_2(\gamma)))-g_2(\gamma,x_2(\gamma-\tau_2(\gamma)))=0. \end{array} \quad (32)$$

But (H_1) and (32) imply

$$\begin{array}{l} z(\gamma-\tau_1(\gamma))z(\gamma-\tau_2(\gamma)) \\ =(x_1(\gamma-\tau_1(\gamma))-x_2(\gamma-\tau_1(\gamma)))(x_1(\gamma-\tau_2(\gamma)) \\ \quad -x_2(\gamma-\tau_2(\gamma)))\leq 0. \end{array}$$

Since $z(t)=x_1(t)-x_2(t)$ is a continuous function in R, it follows that there exists $\xi\in R$ such that

$$z(\xi)=0.$$

Set $\xi=nT+\bar{\gamma}$, where $\bar{\gamma}\in[0,T]$ and n is an integer then we get

$$z(\bar{\gamma})=z(nT+\bar{\gamma})=z(\xi)=0. \quad (33)$$

Thus for any $t\in[\bar{\gamma},\bar{\gamma}+T]$ we obtain

$$\begin{array}{rl} |z(t)| & =|z(\bar{\gamma})+\int_{\bar{\gamma}}^t\dot{z}(s)ds| \\ & \leq\int_{\bar{\gamma}}^t|\dot{z}(s)|ds, \end{array}$$

and

$$\begin{array}{rl} |z(t)| & =|z(\bar{\gamma}+T)+\int_{\bar{\gamma}+T}^t\dot{z}(s)ds| \\ & \leq\int_t^{\bar{\gamma}+T}|\dot{z}(s)|ds. \end{array}$$

Combining these two inequalities and using Cauchy-Schwarz inequality yield

$$\begin{array}{rl} 2|z(t)| & \leq\int_{\bar{\gamma}}^{\bar{\gamma}+T}|\dot{z}(s)|ds \\ & =\int_0^T|\dot{z}(s)|ds \\ & \leq\sqrt{T}(\int_0^T|\dot{z}(s)|^2ds)^{\frac{1}{2}} \\ & =\sqrt{T}|\dot{z}|_2. \end{array}$$

Therefore

$$|z|_{\infty}\leq\frac{1}{2}\sqrt{T}|\dot{z}|_2. \quad (34)$$

Multiplying (31) by $z^{(3)}(t)$ and then by integrating it over $[0,T]$ it follows

$$\begin{array}{l} |z^{(3)}(t)|_2^2=-\int_0^T\{\psi(\dot{x}_1(t))\ddot{x}_1(t)-\psi(\dot{x}_2(t))\ddot{x}_2(t)\}z^{(3)}(t)dt \\ -\int_0^T\{f(x_1(t))\dot{x}_1(t)-f(x_2(t))\dot{x}_2(t)\}z^{(3)}(t)dt \\ -\int_0^T\{g_1(t,x_1(t-\tau_1(t)))-g_1(t,x_2(t-\tau_1(t)))\}z^{(3)}(t)dt \\ -\int_0^T\{g_2(t,x_1(t-\tau_1(t)))-g_2(t,x_2(t-\tau_1(t)))\}z^{(3)}(t)dt. \end{array}$$

From (i) we get

$$\begin{array}{l} |z^{(3)}(t)|_2^2\leq\int_0^T|\psi(\dot{x}_1(t))||\ddot{x}_1(t)-\ddot{x}_2(t))||z^{(3)}(t)|dt \\ +\int_0^T|\psi(\dot{x}_1(t))-\psi(\dot{x}_2(t))||\ddot{x}_2(t))||z^{(3)}(t)|dt \\ +\int_0^T|f(x_1(t))||\dot{x}_1(t)-\dot{x}_2(t))||z^{(3)}(t)|dt \\ +\int_0^T|f(x_1(t))-f(x_2(t))||\dot{x}_2(t))||z^{(3)}(t)|dt \\ +b_1\int_0^T|x_1(t-\tau_1(t))-x_2(t-\tau_1(t))||z^{(3)}(t)|dt \\ +b_2\int_0^T|x_1(t-\tau_2(t))-x_2(t-\tau_2(t))||z^{(3)}(t)|dt \\ \leq a_1\int_0^T|\ddot{z}(t)||z^{(3)}(t)|dt+a_2\int_0^T|\dot{z}(t)||\ddot{x}_2(t)||z^{(3)}(t)|dt \\ +c_1\int_0^T|\dot{z}(t)||z^{(3)}(t)|dt+c_2\int_0^T|z(t)||\dot{x}_2(t)||z^{(3)}(t)|dt \\ +b_1\int_0^T|z(t-\tau_1(t))||z^{(3)}(t)|dt \\ +b_2\int_0^T|z(t-\tau_2(t))||z^{(3)}(t)|dt. \end{array}$$

By using Cauchy-Schwarz inequality we have

$$\begin{array}{l} |z^{(3)}|_2^2\leq a_1|\ddot{z}|_2|z^{(3)}|_2+a_2|\dot{z}|_2|\ddot{x}_2|_{\infty}|z^{(3)}|_2 \\ \quad +c_1|\dot{z}|_2|z^{(3)}|_2+c_2|z|_{\infty}|\dot{x}_2|_{\infty}\sqrt{T}|z^{(3)}|_2 \\ \quad +b_1|z|_{\infty}\sqrt{T}|z^{(3)}|_2+b_2|z|_{\infty}\sqrt{T}|z^{(3)}|_2. \end{array}$$

From (25), (26), (27), (29) and (30) we obtain

$$\begin{array}{l} |z^{(3)}|_2^2\leq\frac{1}{2}a_1T|z^{(3)}|_2^2+\frac{1}{4}a_2kT^2|z^{(3)}|_2^2+\frac{1}{4}c_1T^2|z^{(3)}|_2^2 \\ \quad +\frac{1}{16}c_2kT^4|z^{(3)}|_2^2+\frac{1}{8}(b_1+b_2)T^3|z^{(3)}|_2^2 \\ \leq\{a_1\frac{T}{2}+(a_2k+c_1)\frac{T^2}{4}+(b_1+b_2+c_2k\frac{T}{2})\frac{T^3}{8}\}|z^{(3)}|_2^2. \end{array}$$

Since $z(t),\dot{z}(t),\ddot{z}(t)$ and $z^{(3)}(t)$ are continuous T-periodic functions, by (iv), (22), (33), (34) and the above inequality we have

$$z(t)\equiv\dot{z}(t)\equiv\ddot{z}(t)\equiv z^{(3)}(t)=0,\ forall\ t\in R.$$

Thus $x_1(t)\equiv x_2(t)$, for all $t\in R$. Hence (1) has at most one T-periodic solution.

This completes the proof of condition (iv) in Lemma 2.1.

So the proof of Lemma 2.2 is now complete.

III. Main Result

Theorem 3.1: Suppose that (i) and (iv) hold, then (1) has a unique T-periodic solution.

Proof. Condition (iv) of Lemma 2.1 states that (1) has at most one T-periodic solution. Thus to prove Theorem 3.1 it suffices to show that (1) has at least one T-periodic solution. To do this, we shall apply Theorem 2.1.

First we will claim that the set of all possible T-periodic solutions of (4) is bounded.

Let $x(t)$ be a T-periodic solution of (4). Multiplying (4) by $x^{(3)}(t)$ and then by integrating it over $[0,T]$ we obtain

$$\begin{array}{l} \int_0^T|x^{(3)}(t)|^2dt=-\lambda\int_0^T\psi(\dot{x}(t))\ddot{x}(t)x^{(3)}(t)dt \\ \quad -\lambda\int_0^T f(x(t))\dot{x}(t)x^{(3)}(t)dt \\ \quad -\lambda\int_0^T g_1(t,x(t-\tau_1(t)))x^{(3)}(t)dt \\ \quad -\lambda\int_0^T g_2(t,x(t-\tau_2(t)))x^{(3)}(t)dt+\lambda\int_0^T p(t)x^{(3)}(t)dt. \end{array}$$

In view of $(i), (5)$ and the inequality of Cauchy-Schwarz we have

$$\begin{aligned}|x^{(3)}|_2^2 &\le \{a_1\tfrac{T}{2}+c_1\tfrac{T^2}{4}+(b_1+b_1)\tfrac{T^3}{8}\}|x^{(3)}|_2^2\\ &+[(b_1+b_1)d+\max\{|g_1(t,0)|+|g_2(t,0)|:0\le t\le T\}\\ &+|p|_\infty]\sqrt{T}|x^{(3)}|_2,\end{aligned}$$

which together with condition (iii) imply that there exists $M_0>0$ such that

$$|x^{(3)}|_2 < M_0.$$

This together with $(23), (26)$ and (27) leads to

$$\begin{aligned}|\ddot{x}|_\infty &\le \tfrac{1}{2}\sqrt{T}M_0,\\ |\dot{x}|_\infty &\le \tfrac{1}{4}T^{\frac{3}{2}}M_0,\\ |x|_\infty &\le d+\tfrac{1}{8}T^{\frac{5}{2}}M_0.\end{aligned}$$

Let $M=\max\{d+\frac{1}{8}T^{\frac{5}{2}}M_0,\ \frac{1}{4}T^{\frac{3}{2}}M_0,\ \frac{1}{2}\sqrt{T}M_0\}$, then we have $\Omega=\{x|x\in X, \|x\|<M\}$ as a non-empty open bounded subset of X.
So condition (a) in Theorem 2.1 holds.
In view of $(H_2)(1)$ and $(H_2)(2)$ we will consider two cases:
Case (i): If $(H_2)(1)$ holds. Since

$$\begin{aligned}QNx = -\tfrac{1}{T}\int_0^T\{&\psi(\dot{x}(t))\ddot{x}(t)+f(x(t))\dot{x}(t)\\ &+g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-\bar{p}\}dt,\end{aligned}$$

for any $x\in\partial\Omega\cap KerL=\partial\Omega\cap R$, then x is a constant with $x(t)=M$ or $x(t)=-M$. Then

$$\begin{aligned}QN(M) &= -\tfrac{1}{T}\int_0^T\{g_1(t,M)+g_2(t,M)-\bar{p}\}dt\\ &< 0,\\ QN(-M) &= -\tfrac{1}{T}\int_0^T\{g_1(t,-M)+g_2(t,-M)\\ &-\bar{p}\}dt>0,\end{aligned}\tag{35}$$

which implies that condition (b) of Theorem 2.1 is satisfied.
Furthermore define a continuous function $H(x,\mu)$ by setting

$$\begin{aligned}H(x,\mu) &= -\mu x+(1-\mu)QNx\\ &= -\mu x-(1-\mu)\tfrac{1}{T}\int_0^T\{\psi(\dot{x}(t))\ddot{x}(t)+f(x(t))\dot{x}(t)\\ &+g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-\bar{p}\}dt,\end{aligned}$$

in view of (35) we get $xH(x,\mu)<0$ for all $x\in\partial\Omega\cap KerL$ and $\mu\in[0,1]$.
Thus $H(x,\mu)$ is a homotopic transformation.
By using the homotopy invariance theorem we have

$$deg\{QN,\Omega\cap KerL,0\}=deg\{-x,\Omega\cap KerL,0\}\ne 0,$$

so condition (c) of Theorem 2.1 is satisfied.
Case (ii): If $(H_2)(2)$ holds. Since

$$\begin{aligned}QNx = -\tfrac{1}{T}\int_0^T\{&\psi(\dot{x}(t))\ddot{x}(t)+f(x(t))\dot{x}(t)\\ &+g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-\bar{p}\}dt,\end{aligned}$$

for any $x\in\partial\Omega\cap KerL=\partial\Omega\cap R$, $x(t)=M$ or $x(t)=-M$, we obtain

$$\begin{aligned}QN(M) &= -\tfrac{1}{T}\int_0^T\{g_1(t,M)+g_2(t,M)-\bar{p}\}dt\\ &> 0,\\ QN(-M) &= -\tfrac{1}{T}\int_0^T\{g_1(t,-M)+g_2(t,-M)\\ &-\bar{p}\}dt<0,\end{aligned}\tag{36}$$

which implies that condition (b) of Theorem 2.1 is satisfied.
Define

$$\begin{aligned}H(x,\mu) &= \mu x+(1-\mu)QNx\\ &= \mu x-(1-\mu)\tfrac{1}{T}\int_0^T\{\psi(\dot{x}(t))\ddot{x}(t)+f(x(t))\dot{x}(t)\\ &+g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-\bar{p}\}dt,\end{aligned}$$

in view of (36) we get $xH(x,\mu)>0$ for all $x\in\partial\Omega\cap KerL$ and $\mu\in[0,1]$.
Thus $H(x,\mu)$ is a homotopic transformation.
By using the homotopy invariance theorem we have

$$deg\{QN,\Omega\cap KerL,0\}=deg\{x,\Omega\cap KerL,0\}\ne 0,$$

so condition (c) of Theorem 2.1 is satisfied. Therefore it follows from Theorem 2.1 that (1) has at least one T-periodic solution. This completes the proof.

IV. Example

Example 4.1. In this section we apply the main result obtained in the previous section to an example.

Consider the existence and uniqueness of a π-periodic solution of the third-order delay differential equation with two deviating arguments

$$\begin{aligned}x^{(3)}(t) &+ \tfrac{1}{20}(\sin\dot{x})\ddot{x}(t)+\tfrac{1}{16}(\cos x)\dot{x}(t)\\ &+g_1(t,x(t-\cos 2t))+g_2(t,x(t-\sin 2t))\\ &= \tfrac{1}{\pi}\cos 2t,\end{aligned}\tag{37}$$

where

$$\begin{aligned}&T=\pi,\ \tau_1(t)=\cos 2t,\ \tau_2(t)=\sin 2t,\\ &g_1(t,x)=\tfrac{1}{80\pi(1+\cos^2 t)}\tan^{-1}x,\\ &g_2(t,x)=\tfrac{1}{120\pi}(1+\sin^2 t)\tan^{-1}x\ \ and\\ &p(t)=\tfrac{1}{\pi}\cos 2t.\end{aligned}$$

By (37) we find

$$\begin{aligned}&a_1=a_2=\tfrac{1}{20},\ b_1=\tfrac{1}{80\pi},\ b_2=\tfrac{1}{60\pi},\\ &c_1=c_2=\tfrac{1}{16},\end{aligned}$$

noticing that

$$\begin{aligned}&\bar{p}=\tfrac{1}{T}\int_0^T p(t)dt=\tfrac{1}{\pi}\int_0^\pi\tfrac{1}{\pi}\cos 2tdt=0,\\ &|p|_\infty=\tfrac{1}{\pi},\end{aligned}$$

we can get $d=\frac{1}{10}$, (d is an arbitrary small positive constant).
Then we can obtain

$$\begin{aligned}&\tfrac{[(b_1+b_2)d+\max\{|g_1(t,0)|+|g_2(t,0)|:0\le t\le T\}+|p|_\infty]T}{2\{1-a_1\frac{T}{2}-c_1\frac{T^2}{4}-(b_1+b_2)\frac{T^3}{8}\}}\\ &=\tfrac{\{(\frac{1}{80\pi}+\frac{1}{60\pi})\frac{1}{10}+\frac{1}{\pi}\}\pi}{2\{1-\frac{1}{20}\frac{\pi}{2}-\frac{1}{16}\frac{\pi^2}{4}-(\frac{1}{80\pi}+\frac{1}{60\pi})\frac{\pi^3}{8}\}}\\ &:=k=0.68,\end{aligned}$$

$$\begin{aligned}&a_1\frac{T}{2}+\left(a_2k+c_1\right)\frac{T^2}{4}+\left(b_1+b_2+c_2k\frac{T}{2}\right)\frac{T^3}{8}\\ &=\frac{1}{20}\frac{\pi}{2}+\left(\frac{0.68}{20}+\frac{1}{16}\right)\frac{\pi^2}{4}+\left(\frac{1}{80\pi}+\frac{1}{60\pi}+\frac{0.68}{16}\frac{\pi}{2}\right)\frac{\pi^3}{8}\\ &=0.62<1.\end{aligned}$$

It is obvious that all the assumptions (ii)-(iv) hold.
Hence by Theorem 3.1, equation (37) has a unique π-periodic solution.

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