Optimal Predictive Inferences for Future Order Statistics via a Specific Loss Function

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Abstract— **The purpose of this paper is to develop technique appropriate for construction of frequentist prediction intervals or prediction bounds for order statistics in future samples when only the functional form of the underlying distribution is specified, but some or all of its parameters are unspecified. Prediction intervals for order statistics are widely used for reliability problems and other related problems. The determination of these intervals has been extensively investigated. But the optimal properties of these intervals in the sense of minimal area for a given probability content or maximal probability content for a given area have not been fully explored. In this paper, in order to discuss this problem, a specific loss function is introduced to compare prediction intervals. In particular, within-sample prediction based on the early observed data from a current experiment (i.e., when for predicting the future observation in a sample there are available the early observed data only from that sample), new-sample prediction based on a previous sample (i.e., when for predicting the future observation in a new sample there are available the data only from a previous sample), and new-within-sample prediction based on both the early-failure data from that sample and the data from a previous sample (i.e., when for predicting the future failure time of an unit in a new sample there are available both the early-failure data from that sample and the data from a previous sample). We restrict attention to families of distributions invariant under location and/or scale changes. The technique used here for optimization of prediction intervals based on censored data emphasizes pivotal quantities relevant for obtaining ancillary statistics. It allows one to solve the optimization problems in a simple way. Illustrative examples are given.**

Index Terms—**Order statistic, prediction interval, loss function, optimization**

I. Introduction

PREDICTION is perhaps one of the most commonly undertaken activities in the physical, the engineering, and the biological sciences. In the econometric and the social sciences, prediction generally goes under the name of *forecasting*, and in the actuarial and the assurance sciences under the label *life-length assessment*. Automatic process control, filtering, and quality control, are some of the engineering techniques that use prediction as a basis of their modus operandus. Statistical techniques play a key role in prediction, with regression, time series analysis, and dynamic linear models (also known as state space models) being the predominant tools for producing forecasts. The importance of statistical methods in forecasting was underscored by Pearson [1] who claimed that prediction is the "fundamental problem of practical statistics." Patel [2] provides an extensive survey of literature on this topic. In the areas of reliability and life-testing, lifetime data are often modeled via the Exponential and the Weibull in order to make predictions about future observations. Prediction intervals are constructed to have a reasonably high probability of containing a specified number of such future observations. Interval prediction is an important part of the forecasting process aimed at enhancing the limited accuracy of point estimation. An interval forecast usually consists of an upper and a lower limit between which the future value is expected to lie with a prescribed probability. The limits are sometimes called prediction limits or prediction bounds. These limits may be helpful in establishing warranty policy, determining maintenance schedules, etc. For a very readable discussion of prediction limits and related intervals, see Hahn and Meeker [3].

Many authors have reported their efforts for constructing prediction limits for the Weibull and for the related extreme value distributions (see Patel [2]). Mann and Saunders [4] proposed prediction limits for the Weibull which make use of only two or three order statistics (see also Mann [5]). Antle and Rademaker [6] used simulation to produce a table of factors to use with ML estimates to obtain prediction limits. Lawless [7] proposed prediction limits based on a conditional confidence approach; his limits require both determination of the ML estimates and numerical integration. Engelhardt and Bain [8-9], and Fertig, Meyer and Mann [10] have proposed various approximate prediction limits for the Weibull. Mee and Kushary [11] provided a simulation based procedure for constructing prediction intervals for Weibull populations for Type II censored case. This procedure is based on maximum likelihood estimation and requires an iterative process to determine the percentile points. Bhaumik and Gibbons [12], and Krishnamoorthy et al. [13] proposed approximate methods for constructing upper prediction limits for a gamma distribution.

Consider the following examples of practical problems

Manuscript received October 7, 2011. This work was supported in part by Grant No. 06.1936, Grant No. 07.2036, Grant No. 09.1014, and Grant No. 09.1544 from the Latvian Council of Science and the National Institute of Mathematics and Informatics of Latvia.

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(Advance online publication: 27 February 2012)

which often require the computation of prediction bounds and prediction intervals for future values of random quantities: (i) a consumer purchasing a refrigerator would like to have a lower bound for the failure time of the unit to be purchased (with less interest in distribution of the population of units purchased by other consumers); (ii) financial managers in manufacturing companies need upper prediction bounds on future warranty costs; (iii) when planning life tests, engineers may need to predict the number of failures that will occur by the end of the test to predict the amount of time that it will be take for a specified number of units to fail.

Some applications require a two-sided prediction interval that will, with a specified high degree of confidence, contain the future random variable of interest. It is important to note that in the context of this paper, a prediction interval is not to be viewed as a *confidence interval*. The former is an estimate of a future observable value; the latter an estimate of some fixed but unknown (and often unobservable) parameter. Prediction intervals are produced via frequentist or Bayesian methods, whereas confidence intervals can only be constructed via a frequentist argument. The discussion of this paper revolves around prediction intervals produced by a frequentist approach; thus we are concerned here with *frequentist prediction intervals*. In many applications, however, interest is focused on either an upper prediction bound or a lower prediction bound (e.g., the maximum warranty cost is more important than the minimum, and the time of the early failures in a product population is more important than the last ones).

Conceptually, it is useful to distinguish between ‘within-sample’ prediction, ‘new-sample’ prediction, and ‘new-within-sample’ prediction.

For within-sample prediction, the problem is to predict future events in a sample or process based on early data from that sample or process. If, for example, m units are followed until t_r and there are r observed failures, $Y_1 < Y_2 < \cdots < Y_r$, one could be interested in predicting the time of the next failure, Y_{r+1}; time until l additional failures, Y_{r+l}; number of additional failures in a future interval (t_1,t_2).

For new-sample prediction, data from a previous sample are used to make predictions on a future unit or sample of units from the same process or population. For example, based on previous (possibly censored) life test data, one could be interested in predicting the time to failure of a new unit, time until r failures in a future sample of m units, or number of failures by time t in a future sample of m units.

For new-within-sample prediction, the problem is to predict future events in a sample or process based on the early data from that sample or process as well as on a previous data sample from the same process or population. For example, if m units are followed until t_r and there are the r observed failures, $Y_1, \ldots, Y_r$, from a current experiment as well as the k observed failures, $X_1, \ldots, X_k$, from a previous data sample (possibly censored), one could be interested in predicting the time of the next failure Y_{r+1}; time until l additional failures, Y_{r+l}; number of additional failures in a future interval (t_1,t_2).

In general, to predict a future realization of a random quantity one needs the following:

1) *A statistical model to describe the population or process of interest*. This model usually consists of a distribution depending on a vector of parameters $\boldsymbol{\theta}$. In this paper, attention is restricted to families of distributions which are invariant under location and/or scale changes. In particular, the case may be considered where a previously available complete or type II censored sample is from a continuous distribution with cdf $F((y-\mu)/\sigma)$, where $F(\cdot)$ is known but both the location (μ) and scale (σ) parameters are unknown. For such family of distributions the decision problem remains invariant under a group of transformations (a subgroup of the full affine group) which takes μ (the location parameter) and σ (the scale) into $c\mu + b$ and $c\sigma$, respectively, where b lies in the range of μ, $c > 0$. This group acts transitively on the parameter space.

2) *Information on the values of components of the parametric vector* $\boldsymbol{\theta}$. It is assumed that only the functional form of the distribution is specified, but some or all of its parameters are unspecified. In such cases ancillary statistics and pivotal quantities, whose distribution does not depend on the unknown parameters, are used.

The technique used here for constructing prediction intervals (or bounds) emphasizes pivotal quantities relevant for obtaining ancillary statistics. It represents a simple procedure that can be utilized by non-statisticians, and which provides easily computable explicit expressions for both prediction bounds and prediction intervals. The technique is a special case of the method of invariant embedding of sample statistics into a performance index (see, e.g., Nechval et al. [14-23]) applicable whenever the statistical problem is invariant under a group of transformations, which acts transitively on the parameter space.

II. Within-Sample Prediction Problem

For within-sample prediction, the problem is to predict future events in a sample or process based on early data from that sample or process. For example, if m units are followed until Y_r and there are r observed failures, $Y_1, \ldots, Y_r$, one could be interested in predicting the time of the next failure Y_{r+1}; time until l additional failures, Y_{r+l}; number of additional failures in a future interval.

A. Location-Scale Family of Density Functions

Consider a situation described by a location-scale family of density functions, indexed by the vector parameter $\boldsymbol{\theta}=(\mu,\sigma)$, where μ and σ (>0) are respectively parameters of location and scale. For this family, invariant under the group G of positive linear transformations: $y \rightarrow ay+b$ with a>0, we shall assume that there is obtainable (from some informative experiment) the first r order statistics $Y_1<Y_2< \cdots <Y_r$ from a random sample of size m with cumulative distribution function

$$F(y \mid \mu,\sigma) \equiv F\left(\frac{y-\mu}{\sigma}\right),$$

$$(-\infty\,)\mu < y < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0. \qquad (1)$$

If Y_s is a future observation (sth order statistic) from the same sample of size m, then $W=(Y-Y_r)/S_r$ (or $W=(Y_s-Y_r)/Y_r$) is an ancillary statistic, the distribution of which does not depend on (μ,σ); S_r is a sufficient statistic (or a maximum likelihood estimator $\hat{\sigma}_k$) for σ based on $\mathbf{Y}=(Y_1, \ldots, Y_r)$.

B. Piecewise-Linear Loss Function

We shall consider the interval prediction problem for the sth order statistic Y_s, $r<s\le m$, in the same sample of size m for the situation where the first r observations $Y_1 < Y_2 < \cdots < Y_r$, $1\le r<m$, have been observed. Suppose that we assert that an interval $\mathbf{d}=(d_1,d_2)$ contains Y_s. If, as is usually the case, the purpose of this interval statement is to convey useful information we incur penalties if d_1 lies above Y_s or if d_2 falls below Y_s. Suppose that these penalties are $c_1(d_1- Y_s)$ and $c_2(Y_s-d_2)$, losses proportional to the amounts by which Y_s escapes the interval. Since c_1 and c_2 may be different the possibility of differential losses associated with the interval overshooting and undershooting the true μ is allowed. In addition to these losses there will be a cost attaching to the length of interval used. For example, it will be more difficult and more expensive to design or plan when the interval $\mathbf{d}=(d_1,d_2)$ is wide. Suppose that the cost associated with the interval is proportional to its length, say $c(d_2-d_1)$. In the specification of the loss function, σ is clearly a 'nuisance parameter' and no alteration to the basic decision problem is caused by multiplying all loss factors by $1/\sigma$. Thus we are led to investigate the piecewise-linear loss function

$$r(\boldsymbol{\theta},\mathbf{d})=\begin{cases}\dfrac{c_1(d_1-Y_s)}{\sigma}+\dfrac{c(d_2-d_1)}{\sigma} & (Y_s<d_1),\\ \dfrac{c(d_2-d_1)}{\sigma} & (d_1\le Y_s\le d_2),\\ \dfrac{c(d_2-d_1)}{\sigma}+\dfrac{c_2(Y_s-d_2)}{\sigma} & (Y_s>d_2).\end{cases} \quad (2)$$

The decision problem specified by the informative experiment distribution function (1) and the loss function (2) is invariant under the group G of transformations. Thus, the problem is to find the best invariant interval predictor of Y_s,

$$\mathbf{d}^*=\arg\min_{\mathbf{d}\in\mathcal{D}} R(\boldsymbol{\theta},\mathbf{d}), \quad (3)$$

where $\mathcal{D}$ is a set of invariant interval predictors of Y_s, $R(\boldsymbol{\theta},\mathbf{d})=E_{\boldsymbol{\theta}}\{r(\boldsymbol{\theta},\mathbf{d})\}$ is a risk function.

C. Transformation of the Loss Function

It follows from (2) that the invariant loss function, $r(\boldsymbol{\theta},\mathbf{d})$, can be transformed as follows:

$$r(\boldsymbol{\theta},\mathbf{d})=\ddot{r}(\mathbf{V},\boldsymbol{\eta}), \quad (4)$$

where

$$\ddot{r}(\mathbf{V},\boldsymbol{\eta})=\begin{cases}c_1(-V_1+\eta_1V_2)+c(\eta_2-\eta_1)V_2 & (V_1<\eta_1V_2),\\ c(\eta_2-\eta_1)V_2 & (\eta_1V_2\le V_1\le\eta_2V_2),\\ c_2(V_1-\eta_2V_2)+c(\eta_2-\eta_1)V_2 & (V_1>\eta_2V_2),\end{cases} \quad (5)$$

$$\mathbf{V}=(V_1,V_2),\quad V_1=(Y_s-Y_r)/\sigma,\quad V_2=S_r/\sigma;$$

$$\boldsymbol{\eta}=(\eta_1,\eta_2),\quad \eta_1=(d_1-Y_r)/S_r,\quad \eta_2=(d_2-Y_r)/S_r. \quad (6)$$

D. Risk Function

It follows from (5) that the risk associated with $\mathbf{d}$ and $\boldsymbol{\theta}$ can be expressed as

$$R(\boldsymbol{\theta},\mathbf{d})=E_{\boldsymbol{\theta}}\{r(\boldsymbol{\theta},\mathbf{d})\}=E\{\ddot{r}(\mathbf{V},\boldsymbol{\eta})\}$$

$$=c_1\int_0^{\infty}\int_0^{\eta_1v_2}(-v_1+\eta_1v_2)f(v_1,v_2)dv_1dv_2$$

$$+c_2\int_0^{\infty}\int_{\eta_2v_2}^{\infty}(v_1-\eta_2v_2)f(v_1,v_2)dv_1dv_2$$

$$+c(\eta_2-\eta_1)\int_0^{\infty}\int_0^{\infty}v_2f(v_1,v_2)dv_1dv_2, \quad (7)$$

which is constant on orbits when an invariant predictor (decision rule) $\mathbf{d}$ is used, where $f(v_1,v_2)$ is defined by the joint probability density of the first r observations $Y_1 < Y_2 < \cdots < Y_r$ and Y_s,

$$f(y_1,\ldots,y_r,y_s\mid\mu,\sigma)=\frac{m!}{(s-r-1)!(m-s)!}$$

$$\times[F(y_s\mid\mu,\sigma)-F(y_r\mid\mu,\sigma)]^{s-r-1}[1-F(y_r\mid\mu,\sigma)]^{m-s}$$

$$\times\prod_{i=1}^{r}f(y_i\mid\mu,\sigma)f(y_s\mid\mu,\sigma). \quad (8)$$

E. Risk Minimization and Invariant Prediction Rules

The following theorem gives the central result in this section.

Theorem 1 (*Optimal predictor of* Y_s *based on* $\mathbf{Y}$). Suppose that (u_1, u_2) is a random vector having density function

$$u_2f(u_1,u_2)\left[\int_0^{\infty}\int_0^{\infty}u_2f(u_1,u_2)du_1du_2\right]^{-1}\quad (u_1,u_2>0), \quad (9)$$

where f is defined by $f(v_1,v_2)$, and let Q be the probability distribution function of u_1/u_2.

(i) If $c/c_1+c/c_2<1$ then the optimal invariant linear-loss interval predictor of Y_s based on $\mathbf{Y}$ is $\mathbf{d}^*=(Y_r+\eta_1S_r,\ Y_r+\eta_2S_r)$, where

$$Q(\eta_1)=c/c_1,\quad Q(\eta_2)=1-c/c_2. \quad (10)$$

(ii) If $c/c_1+c/c_2\ge1$ then the optimal invariant linear-loss interval predictor of Y_s based on $\mathbf{Y}$ degenerates into a point predictor $Y_r+\eta\, S_r$, where

$$Q(\eta) = c_2/(c_1 + c_2). \tag{11}$$

Proof. From (7)

$$\frac{\partial E\{\ddot{r}(\mathbf{V},\boldsymbol{\eta})\}}{\partial \eta_1}$$

$$= c_1 \int_0^\infty \int_0^{\eta_1 v_2} v_2 f(v_1, v_2) dv_1 dv_2 - c \int_0^\infty \int_0^\infty v_2 f(v_1, v_2) dv_1 dv_2$$

$$= \int_0^\infty \int_0^\infty v_2 f(v_1, v_2) dv_1 dv_2 [c_1 Q(\eta_1) - c], \tag{12}$$

and

$$\frac{\partial E\{\ddot{r}(\mathbf{V},\boldsymbol{\eta})\}}{\partial \eta_2}$$

$$= \int_0^\infty \int_0^\infty v_2 f(v_1, v_2) dv_1 dv_2 [-c_2(1 - Q(\eta_2)) + c], \tag{13}$$

where

$$Q(\eta) = \int_0^\eta q(w) dw, \tag{14}$$

$$q(w) = \frac{\int_0^\infty v_2^2 f(wv_2, v_2) dv_2}{\int_0^\infty \int_0^\infty v_2 f(v_1, v_2) dv_1 dv_2}, \tag{15}$$

$$W = V_1/V_2. \tag{16}$$

Now $\partial E\{\ddot{r}(\mathbf{V},\boldsymbol{\eta})\}/\partial \eta_1 = \partial E\{\ddot{r}(\mathbf{V},\boldsymbol{\eta})\}/\partial \eta_2 = 0$ if and only if (10) hold. Thus, $E\{\ddot{r}(\mathbf{V},\boldsymbol{\eta})\}$ provided (10) has a solution with $\eta_1 < \eta_2$ and this is so if $1 - c/c_2 > c/c_1$. It is easily confirmed that this $\boldsymbol{\eta} = (\eta_1, \eta_2)$ gives the minimum value of $E\{\ddot{r}(\mathbf{V},\boldsymbol{\eta})\}$. Thus (i) is established.

If $c/c_1 + c/c_2 \geq 1$ then the minimum of $E\{\ddot{r}(\mathrm{v},\eta)\}$ in the region $\eta_2 \geq \eta_1$ occurs where $\eta_1 = \eta_2 = \eta$, η being determined by setting

$$\partial E\{\ddot{r}(V,(\eta,\eta))\}/\partial \eta = 0 \tag{17}$$

and this reduces to

$$c_1 Q(\eta) - c_2[1 - Q(\eta)] = 0, \tag{18}$$

which establishes (ii). □

Corollary 1.1 (*Minimum risk of the optimal invariant predictor of Y_s based on* $\mathbf{Y}$). The minimum risk is given by

$$R(\boldsymbol{\theta}, \mathbf{d}^*) = E_{\boldsymbol{\theta}}\{r(\boldsymbol{\theta}, \mathbf{d}^*)\} = E\{\ddot{r}(\mathbf{V},\boldsymbol{\eta})\}$$

$$= -c_1 \int_0^\infty \int_0^{\eta_1 v_2} v_1 f(v_1, v_2) dv_1 dv_2 + c_2 \int_0^\infty \int_{\eta_2 v_2}^\infty v_1 f(v_1, v_2) dv_1 dv_2 \tag{19}$$

for case (i) with $\boldsymbol{\eta} = (\eta_1, \eta_2)$ as given by (10) and for case (ii) with $\eta_1 = \eta_2 = \eta$ as given by (11).

Proof. These results are immediate from (7) when use is made of $\partial E\{\ddot{r}(\mathbf{V},\boldsymbol{\eta})\}/\partial \eta_1 = \partial E\{\ddot{r}(\mathbf{V},\boldsymbol{\eta})\}/\partial \eta_2 = 0$ in case (i) and $\partial E\{\ddot{r}(\mathbf{V},(\eta,\eta))\}/\partial \eta = 0$ in case (ii). □

The underlying reason why $c/c_1 + c/c_2$ acts as a separator of interval and point prediction is that for $c/c_1 + c/c_2 \geq 1$ every interval predictor is inadmissible, there existing some point predictor with uniformly smaller risk.

Theorem 2 (*Optimal invariant predictor of Y_s based on Y_r*). Suppose that $\mu = 0$ and

$$\mathbf{V} = (V_1, V_2), \quad V_1 = (Y_s - Y_r)/\sigma, \quad V_2 = Y_r/\sigma;$$

$$\boldsymbol{\eta} = (\eta_1, \eta_2), \quad \eta_1 = (d_1 - Y_r)/Y_r, \quad \eta_2 = (d_2 - Y_r)/Y_r. \tag{20}$$

Let us assume that (u_1, u_2) is a random vector having density function

$$u_2 f_0(u_1, u_2) \left[\int_0^\infty \int_0^\infty u_2 f_0(u_1, u_2) du_1 du_2 \right]^{-1} \quad (u_1, u_2 > 0), \tag{21}$$

where f_0 is defined by $f_0(v_1, v_2)$, and let Q_0 be the probability distribution function of u_1/u_2.

(i) If $c/c_1 + c/c_2 < 1$ then the optimal invariant linear-loss interval predictor of Y_s based on Y_r is $\mathbf{d}^* = ((1+\eta_1)Y_r, (1+\eta_2)Y_r)$, where

$$Q_0(\eta_1) = c/c_1, \quad Q_0(\eta_2) = 1 - c/c_2. \tag{22}$$

(ii) If $c/c_1 + c/c_2 \geq 1$ then the optimal invariant linear-loss interval predictor of Y_s based on Y_r degenerates into a point predictor $(1+\eta)\,Y_r$, where

$$Q_0(\eta) = c_2/(c_1 + c_2). \tag{23}$$

Proof. For the proof we refer to Theorem 1. □

Corollary 2.1 (*Minimum risk of the optimal invariant predictor of Y_s based on Y_r*). The minimum risk is given by

$$R(\boldsymbol{\theta}, \mathbf{d}^*) = E_{\boldsymbol{\theta}}\{r(\boldsymbol{\theta}, \mathbf{d}^*)\} = E\{\ddot{r}(\mathbf{V},\boldsymbol{\eta})\}$$

$$= -c_1 \int_0^\infty \int_0^{\eta_1 v_2} v_1 f_0(v_1, v_2) dv_1 dv_2 + c_2 \int_0^\infty \int_{\eta_2 v_2}^\infty v_1 f_0(v_1, v_2) dv_1 dv_2 \tag{24}$$

for case (i) with $\boldsymbol{\eta} = (\eta_1, \eta_2)$ as given by (22) and for case (ii) with $\eta_1 = \eta_2 = \eta$ as given by (23).

Proof. For the proof we refer to Corollary 1.1. □

III. Equivalent Confidence Coefficient

For case (i) when we obtain an interval predictor for Y_s we may regard the interval as a confidence interval in the conventional sense and evaluate its confidence coefficient. The general result is contained in the following theorem.

Theorem 3 (*Equivalent confidence coefficient for* $\mathbf{d}^*$ *based on* $\mathbf{Y}$). Suppose that $\mathbf{V}=(V_1,V_2)$ is a random vector having density function $f(v_1,v_2)$ $(v_1,v_2>0)$ where f is defined by (8) and let H be the distribution function of $W=V_1/V_2$, i.e., the probability density function of W is given by

$$h(w)=\int_0^\infty v_2 f(wv_2,v_2)dv_2. \quad (25)$$

Then the confidence coefficient associated with the optimum prediction interval $\mathbf{d}^*=(d_1,d_2)$, where $d_1=Y_r+\eta_1S_r$, $d_2=Y_r+\eta_2S_r$, is

$$\Pr\{\mathbf{d}^*: d_1<Y_s<d_2 \mid \mu,\sigma\}$$

$$=H[Q^{-1}(1-c/c_2)]-H[Q^{-1}(c/c_1)]. \quad (26)$$

Proof. The confidence coefficient for $\mathbf{d}^*$ corresponding to (μ,σ) is given by

$$\Pr\{(Y_r,S_r): Y_r+\eta_1S_r<Y_s<Y_r+\eta_2S_r \mid \mu,\sigma\}$$

$$=\Pr\{(v_1,v_2):\eta_1<v_1/v_2<\eta_2\}$$

$$=H(\eta_2)-H(\eta_1)=H[Q^{-1}(1-c/c_2)]-H[Q^{-1}(c/c_1)]. \quad (27)$$

This is independent of (μ,σ). □

Theorem 4 (*Equivalent confidence coefficient for* $\mathbf{d}^*$ *based on* Y_r). Suppose that $\mathbf{V}=(V_1,V_2)$ is a random vector having density function $f_0(v_1,v_2)$ (v_1 real, $v_2>0$), where f_0 is defined by

$$f(y_r,y_s \mid \mu,\sigma)=\frac{1}{\mathrm{B}(r,s-r)\mathrm{B}(s,m-s+1)}$$

$$\times[F(y_r\mid\mu,\sigma)]^{s-1}[F(y_s\mid\mu,\sigma)-F(y_r\mid\mu,\sigma)]^{s-r-1}$$

$$\times[1-F(y_s\mid\mu,\sigma)]^{m-s} f(y_r\mid\mu,\sigma)f(y_s\mid\mu,\sigma), \quad (28)$$

and let H_0 be the distribution function of $W=V_1/V_2$, i.e., the probability density function of W is given by

$$h_0(w)=\int_0^\infty v_2 f_0(wv_2,v_2)dv_2. \quad (29)$$

Then the confidence coefficient associated with the optimum prediction interval $\mathbf{d}^*=(d_1,d_2)$, where $d_1=(1+\eta_1)Y_r$, $d_2=(1+\eta_2)Y_r$, is

$$\Pr\{\mathbf{d}^*: d_1<X_r<d_2 \mid \mu,\sigma\}$$

$$=H_0[Q_0^{-1}(1-c/c_2)]-H_0[Q_0^{-1}(c/c_1)]. \quad (30)$$

Proof. For the proof we refer to Theorem 3. □

The way in which (26) (or (30)) varies with c, c_1 and c_2, and the fact that c_1 and c_2 are the factors of proportionality associated with losses from overshooting and undershooting relative to loss involved in increasing the length of interval, provides an interesting interpretation of confidence interval prediction.

IV. New-Sample Prediction Problem

For new-sample prediction, data from a past sample are used to make predictions on a future unit or sample of units from the same process or population. For example, based on previous (possibly censored) life test data, one could be interested in predicting the time to failure of a new item, time until l failures in a future sample of m units, or number of failures by time $t_\bullet$ in a future sample of m units.

A. Location-Scale Family of Density Functions

Consider a situation described by a location-scale family of density functions, indexed by the vector parameter $\boldsymbol{\theta}=(\mu,\sigma)$, where μ and σ (>0) are respectively parameters of location and scale. For this family, invariant under the group of positive linear transformations: $x\to ax+b$ with $a>0$, we shall assume that there is obtainable from some informative experiment (the first k order statistics $X_1<X_2<\cdots<X_k$ from a previous random sample of size n) a sufficient statistic (M_k,S_k) (or a maximum likelihood estimator $(\widehat{\mu}_k,\widehat{\sigma}_k)$) for (μ,σ) based on $\mathbf{X}=(X_1,\ldots,X_k)$ with density function

$$p(m_k,s_k\mid\mu,\sigma)=\sigma^{-2}p_0[(m_k-\mu)/\sigma, s_k/\sigma]$$

$$-\infty<m_k<\infty,\ 0<s_k<\infty,\ -\infty<\mu>\infty,\ \sigma>0. \quad (31)$$

We are thus assuming that for the family of density functions an induced invariance holds under the group G of transformations: $m_k\to am_k+b$, $s_k\to as_k$ or $\widehat{\mu}_k\to a\widehat{\mu}_k+b$, $\widehat{\sigma}_k\to a\widehat{\sigma}_k$ $(a>0)$. The family of density functions satisfying the above conditions is, of course, the limited one of normal, negative exponential, Weibull and gamma (with known index) density functions. The structure of the problem is, however, more clearly seen within the general framework.

Let Y_s be an independent future observation (sth order statistic) from a new sample. If Y_s is invariantly predictable then $W=(Y_s-M_k)/S_k$ (or $W=(Y_s-\widehat{\mu}_k)/\widehat{\sigma}_k$) is a maximal invariant pivotal, conditional on $\mathbf{X}$.

B. Piecewise-Linear Loss Function

We shall consider the interval prediction problem for the sth order statistic Y_s, $1\le s\le m$, in a future sample of size m for the situation where the first k observations $X_1<X_2<\cdots<X_k$, $1\le k<n$, from a past sample of size n have been observed.

Suppose that we deal with the piecewise-linear loss function (2).

The decision problem specified by the informative experiment density function (31) and the loss function (2) is invariant under the group G of transformations. Thus, the problem is to find the optimal interval predictor of Y_s,

$$\mathbf{d}^* = \arg\min_{\mathbf{d}\in\mathscr{D}} R^\circ(\boldsymbol{\theta},\mathbf{d}), \tag{32}$$

where $\mathscr{D}$ is a set of invariant interval predictors of Y_s, $R^\circ(\boldsymbol{\theta},\mathbf{d})=E_{\boldsymbol{\theta}}\{r(\boldsymbol{\theta},\mathbf{d})\}$ is a risk function.

C. Transformation of the Loss Function

It follows from (2) that the invariant loss function, $r(\boldsymbol{\theta},\mathbf{d})$, can be transformed as follows:

$$r(\boldsymbol{\theta},\mathbf{d}) = \ddot{r}(\mathbf{V}^\circ,\boldsymbol{\eta}^\circ), \tag{33}$$

where

$$\ddot{r}(\mathbf{V}^\circ,\boldsymbol{\eta}^\circ)=\begin{cases} c_1(-V_1^\circ+\eta_1^\circ V_2^\circ)+c(\eta_2^\circ-\eta_1^\circ)V_2^\circ & (V_1^\circ<\eta_1^\circ V_2^\circ),\\ c(\eta_2^\circ-\eta_1^\circ)V_2^\circ & (\eta_1^\circ V_2^\circ\le V_1^\circ\le\eta_2^\circ V_2^\circ),\\ c_2(V_1^\circ-\eta_2^\circ V_2^\circ)+c(\eta_2^\circ-\eta_1^\circ)V_2^\circ & (V_1^\circ>\eta_2^\circ V_2^\circ),\end{cases} \tag{34}$$

$$\mathbf{V}^\circ=(V_1^\circ,V_2^\circ),\quad V_1^\circ=(Y_s-M_k)/\sigma,\quad V_2^\circ=S_k/\sigma;$$

$$\boldsymbol{\eta}^\circ=(\eta_1^\circ,\eta_2^\circ),\quad \eta_1^\circ=(d_1-M_k)/S_k,\quad \eta_2^\circ=(d_2-M_k)/S_k. \tag{35}$$

D. Risk Function

It follows from (34) that the risk associated with $\mathbf{d}$ and $\boldsymbol{\theta}$ can be expressed as

$$R^\circ(\boldsymbol{\theta},\mathbf{d})=E_{\boldsymbol{\theta}}\{r(\boldsymbol{\theta},\mathbf{d})\}=E\{\ddot{r}(\mathbf{V}^\circ,\boldsymbol{\eta}^\circ)\}$$

$$=c_1\int_0^\infty\int_{-\infty}^{\eta_1^\circ v_2^\circ}(-v_1^\circ+\eta_1^\circ v_2^\circ)f^\circ(v_1^\circ,v_2^\circ)dv_1^\circ dv_2^\circ$$

$$+c_2\int_0^\infty\int_{\eta_2^\circ v_2^\circ}^\infty(v_1^\circ-\eta_2^\circ v_2^\circ)f^\circ(v_1^\circ,v_2^\circ)dv_1^\circ dv_2^\circ$$

$$+c(\eta_2^\circ-\eta_1^\circ)\int_0^\infty\int_{-\infty}^\infty v_2^\circ f^\circ(v_1^\circ,v_2^\circ)dv_1^\circ dv_2^\circ, \tag{36}$$

which is constant on orbits when an invariant predictor (decision rule) $\mathbf{d}$ is used, where $f^\circ(v_1^\circ,v_2^\circ)$ is defined by the joint probability density of the first k observations $X_1 < X_2 < \cdots < X_k$ from the past random sample of size n and the sth order statistic Y_s in the future sample of size m,

$$f(x_1,x_2,\ldots,x_k,y_s\mid\mu,\sigma)=\frac{n!}{(n-k)!}\frac{m!}{(s-1)!(m-s)!}$$

$$\times\prod_{i=1}^k f(x_i\mid\mu,\sigma)[1-F(x_k\mid\mu,\sigma)]^{n-k}$$

$$\times[F(y_s\mid\mu,\sigma)]^{s-1}[1-F(y_s\mid\mu,\sigma)]^{m-s}f(y_s\mid\mu,\sigma). \tag{37}$$

E. Risk Minimization and Invariant Prediction Rules

The following theorem gives the central result in this section.

Theorem 5 (*Optimal invariant predictor of Y_s based on* $\mathbf{X}$). Suppose that (u_1, u_2) is a random vector having density function

$$u_2 f^\circ(u_1,u_2)\left[\int_0^\infty\int_{-\infty}^\infty u_2 f^\circ(u_1,u_2)du_1du_2\right]^{-1}\ (u_1 \text{ real},\ u_2>0), \tag{38}$$

where f° is defined by $f^\circ(v_1^\circ,v_2^\circ)$, and let Q° be the probability distribution function of u_1/u_2.

(i) If $c/c_1+c/c_2<1$ then the optimal invariant linear-loss interval predictor of Y_s based on $\mathbf{X}$ is $\mathbf{d}^*=(M_k+\eta_1^\circ S_k, M_k+\eta_2^\circ S_k)$, where

$$Q^\circ(\eta_1^\circ)=c/c_1,\quad Q^\circ(\eta_2^\circ)=1-c/c_2. \tag{39}$$

(ii) If $c/c_1+c/c_2\ge1$ then the optimal invariant linear-loss interval predictor of Y_s based on $\mathbf{X}$ degenerates into a point predictor $M_k+\eta^\circ S_k$, where

$$Q^\circ(\eta^\circ)=c_2/(c_1+c_2). \tag{40}$$

Proof. For the proof we refer to Theorem 1. □

Corollary 5.1 (*Minimum risk of the optimal invariant predictor of Y_s based on* $\mathbf{X}$). The minimum risk is given by

$$R^\circ(\boldsymbol{\theta},\mathbf{d}^*)=E_{\boldsymbol{\theta}}\{r(\boldsymbol{\theta},\mathbf{d}^*)\}=E\{\ddot{r}(\mathbf{V}^\circ,\boldsymbol{\eta}^\circ)\}$$

$$=-c_1\int_0^\infty\int_{-\infty}^{\eta_1^\circ v_2^\circ}v_1^\circ f^\circ(v_1^\circ,v_2^\circ)dv_1^\circ dv_2^\circ+c_2\int_0^\infty\int_{\eta_2^\circ v_2^\circ}^\infty v_1^\circ f^\circ(v_1^\circ,v_2^\circ)dv_1^\circ dv_2^\circ \tag{41}$$

for case (i) with $\boldsymbol{\eta}^\circ=(\eta_1^\circ,\eta_2^\circ)$ as given by (39) and for case (ii) with $\eta_1^\circ=\eta_2^\circ=\eta$ as given by (40).

Proof. For the proof we refer to Corollary 1.1. □

Theorem 6 (*Equivalent confidence coefficient for* $\mathbf{d}^*$ *based on* $\mathbf{X}$). Suppose that $\mathbf{V}^\circ=(V_1^\circ,V_2^\circ)$, is a random vector having density function $f^\circ(v_1^\circ,v_2^\circ)$ (v_1° real, $v_2^\circ>0$) where f° is defined by (38) and let H° be the distribution function of $W^\circ=V_1^\circ/V_2^\circ$, i.e., the probability density function of W° is given by

$$h^{\circ}(w^{\circ}) = \int_0^{\infty} v_2^{\circ} f^{\circ}(w^{\circ}v_2^{\circ}, v_2^{\circ}) dv_2^{\circ}. \tag{42}$$

Then the confidence coefficient associated with the optimum prediction interval $\mathbf{d}^*=(d_1,d_2)$, where $d_1=M_k+\eta_1^{\circ} S_k$, $d_2=M_k+\eta_2^{\circ} S_k$, is

$$\Pr\{\mathbf{d}^* : d_1 < Y_s < d_2 \mid \mu,\sigma\}$$

$$= H^{\circ}[Q^{\circ(-1)}(1-c/c_2)] - H^{\circ}[Q^{\circ(-1)}(c/c_1)]. \tag{43}$$

Proof. For the proof we refer to Theorem 3. □

V. New-Within-Sample Prediction Problem

For new-within-sample prediction, the problem is to predict future events in a sample or process based on early data from that sample or process as well as on a previous data sample from the same process or population. For example, if m units are followed until t_r and there are the r observed failures, $Y_1, \ldots, Y_r$, from a current experiment as well as the k observed failures, $X_1, \ldots, X_k$, from a previous data sample (possibly censored), one could be interested in predicting the time of the next failure Y_{r+1}; time until l additional failures, Y_{r+l}; number of additional failures in a future interval (t_1,t_2).

A. Location-Scale Family of Density Functions

Consider a situation described by a location-scale family of density functions, indexed by the vector parameter $\boldsymbol{\theta}=(\mu,\sigma)$, where μ and σ (>0) are respectively parameters of location and scale. For this family, invariant under the group of positive linear transformations: $x \to ax+b$ with $a>0$, we shall assume that there is obtainable from both some current informative experiment (the first r order statistics $Y_1<Y_2< \cdots <Y_r$ from a random sample of size m) and the k observed failures, $X_1, \ldots, X_k$, from a previous censored data sample of size n a sufficient statistic (M^+,S^+) (or a maximum likelihood estimator $(\widehat{\mu}^+, \widehat{\sigma}^+)$) for (μ,σ) based on $\mathbf{Y}=(Y_1,\ldots, Y_r)$ and $\mathbf{X}=(X_1,\ldots, X_k)$ with density function

$$p(m^+, s^+ \mid \mu,\sigma) = \sigma^{-2} p_0[(m^+ - \mu)/\sigma, s^+/\sigma]$$

$$-\infty < m^+ < \infty,\ \ 0 < s^+ < \infty,\ \ -\infty < \mu > \infty,\ \ \sigma > 0. \tag{44}$$

We are thus assuming that for the family of density functions an induced invariance holds under the group G of transformations: $m^+ \to am^+ + b$, $s^+ \to as^+$ or $\widehat{\mu}^+ \to a\widehat{\mu}^+ + b$, $\widehat{\sigma}^+ \to a\widehat{\sigma}^+$ ($a>0$). The family of density functions satisfying the above conditions is, of course, the limited one of normal, negative exponential, Weibull and gamma (with known index) density functions. The structure of the problem is, however, more clearly seen within the general framework.

Let Y_s be a future observation (sth order statistic) from the same sample of size m. If Y_s is invariantly predictable then $W^+ = (Y_s - M^+)/S^+$ (or $W^+ = (Y_s - \widehat{\mu}^+)/\widehat{\sigma}^+$) is a maximal invariant pivotal, conditional on $(\mathbf{X},\mathbf{Y})$.

B. Piecewise-Linear Loss Function

We shall consider the interval prediction problem for the sth order statistic Y_s, $1 \le s \le m$, in the same sample of size m for the situation where both the first k observations $X_1 < X_2 < \cdots < X_k$, $1 \le k < n$, from the previous data sample of size n and the early observed r order statistics $Y_1<Y_2< \cdots <Y_r$ from a random sample of size m are available. Suppose that we deal with the piecewise-linear loss function (2).

The decision problem specified by the informative experiment density function (31) and the loss function (2) is invariant under the group G of transformations. Thus, the problem is to find the optimal interval predictor of Y_s,

$$\mathbf{d}^* = \arg\min_{\mathbf{d} \in \mathscr{D}} R^+(\boldsymbol{\theta},\mathbf{d}), \tag{45}$$

where $\mathscr{D}$ is a set of invariant interval predictors of Y_s, $R^+(\boldsymbol{\theta},\mathbf{d} = E_{\boldsymbol{\theta}}\{r(\boldsymbol{\theta},\mathbf{d})\}$ is a risk function.

C. Transformation of the Loss Function

It follows from (2) that the invariant loss function, $r(\boldsymbol{\theta},\mathbf{d})$, can be transformed as follows:

$$r(\boldsymbol{\theta},\mathbf{d}) = \ddot{r}(\mathbf{V}^+, \boldsymbol{\eta}^+), \tag{46}$$

where

$$\ddot{r}(\mathbf{V}^+, \boldsymbol{\eta}^+)$$

$$= \begin{cases} c_1(-V_1^+ + \eta_1^+ V_2^+) + c(\eta_2^+ - \eta_1^+)V_2^+ & (V_1^+ < \eta_1^+ V_2^+), \\ c(\eta_2^+ - \eta_1^+)V_2^+ & (\eta_1^+ V_2^+ \le V_1^+ \le \eta_2^+ V_2^+), \\ c_2(V_1^+ - \eta_2^+ V_2^+) + c(\eta_2^+ - \eta_1^+)V_2^+ & (V_1^+ > \eta_2^+ V_2^+), \end{cases} \tag{47}$$

$$\mathbf{V}^+ = (V_1^+, V_2^+), \quad V_1^+ = (Y_s - M^+)/\sigma, \quad V_2^+ = S^+/\sigma;$$

$$\boldsymbol{\eta}^+ = (\eta_1^+, \eta_2^+), \quad \eta_1^+ = (d_1 - M^+)/S^+, \quad \eta_2^+ = (d_2 - M^+)/S^+. \tag{48}$$

D. Risk Function

It follows from (47) that the risk associated with $\mathbf{d}$ and $\boldsymbol{\theta}$ can be expressed as

$$R^+(\boldsymbol{\theta},\mathbf{d}) = E_{\boldsymbol{\theta}}\{r(\boldsymbol{\theta},\mathbf{d})\} = E\{\ddot{r}(\mathbf{V}^+, \boldsymbol{\eta}^+)\}$$

$$= c_1 \int_0^{\infty} \int_{-\infty}^{\eta_1^+ v_2^+} (-v_1^+ + \eta_1^+ v_2^+) f^+(v_1^+, v_2^+) dv_1^+ dv_2^+$$

$$+ c_2 \int_0^{\infty} \int_{\eta_2^+ v_2^+}^{\infty} (v_1^+ - \eta_2^+ v_2^+) f^+(v_1^+, v_2^+) dv_1^+ dv_2^+$$

$$+c(\eta_2^+-\eta_1^+)\int_0^\infty\int_{-\infty}^\infty v_2^+ f^+(v_1^+,v_2^+)dv_1^+dv_2^+, \quad (49)$$

which is constant on orbits when an invariant predictor (decision rule) **d** is used, where $f^+(v_1^+,v_2^+)$ is defined by the joint probability density of the first k observations $X_1 < X_2 < \cdots < X_k$ from the previous random sample of size n, the early observed r order statistics $Y_1<Y_2<\cdots<Y_r$ from a random sample of size m, and the future sth order statistic Y_s ($s>r$) in the same sample of size m,

$$f(x_1,x_2,...,x_k,y_1,...,y_r,y_s\mid\mu,\sigma)$$

$$=\frac{n!}{(n-k)!}\frac{m!}{(s-r-1)!(m-s)!}$$

$$\times\prod_{i=1}^{k} f(x_i\mid\mu,\sigma)[1-F(x_k\mid\mu,\sigma)]^{n-k}$$

$$\times[F(y_s\mid\mu,\sigma)-F(y_r\mid\mu,\sigma)]^{s-r-1}[1-F(y_r\mid\mu,\sigma)]^{m-s}$$

$$\times\prod_{i=1}^{r} f(y_i\mid\mu,\sigma)f(y_s\mid\mu,\sigma). \quad (50)$$

E. Risk Minimization and Invariant Prediction Rules

The following theorem gives the central result in this section.

Theorem 7 (*Optimal invariant predictor of Y_s based on* (**X**,**Y**)). Suppose that (u_1, u_2) is a random vector having density function

$$u_2 f^+(u_1,u_2)\left[\int_0^\infty\int_{-\infty}^\infty u_2 f^+(u_1,u_2)du_1du_2\right]^{-1} \ (u_1 \text{ real},\ u_2>0), \quad (51)$$

where f^+ is defined by $f^+(v_1^+,v_2^+)$, and let Q^+ be the probability distribution function of u_1/u_2.

(i) If $c/c_1+c/c_2<1$ then the optimal invariant linear-loss interval predictor of Y_s based on (**X**,**Y**) is $\mathbf{d}^*=(M^+ +\eta_1^+S^+,\ M^+ +\eta_2^+S^+)$, where

$$Q^+(\eta_1^+)=c/c_1,\quad Q^+(\eta_2^+)=1-c/c_2. \quad (52)$$

(ii) If $c/c_1+c/c_2\geq 1$ then the optimal invariant linear-loss interval predictor of Y_s based on (**X**,**Y**) degenerates into a point predictor $M^+ +\eta^+S^+$, where

$$Q^+(\eta^+)=c_2/(c_1+c_2). \quad (53)$$

Proof. For the proof we refer to Theorem 1. □

Corollary 5.1 (*Minimum risk of the optimal invariant predictor of Y_s based on* (**X**,**Y**)). The minimum risk is given by

$$R^+(\boldsymbol{\theta},\mathbf{d}^*)=E_{\boldsymbol{\theta}}\{r(\boldsymbol{\theta},\mathbf{d}^*)\}=E\{\ddot{r}(\mathbf{V}^+,\boldsymbol{\eta}^+)\}$$

$$=-c_1\int_0^\infty\int_{-\infty}^{\eta_1^+v_2^+} v_1^+ f^+(v_1^+,v_2^+)dv_1^+dv_2^+ + c_2\int_0^\infty\int_{\eta_2^+v_2^+}^\infty v_1^+ f^+(v_1^+,v_2^+)dv_1^+dv_2^+$$

(54)

for case (i) with $\boldsymbol{\eta}^+=(\eta_1^+,\eta_2^+)$ as given by (52) and for case (ii) with $\eta_1^+=\eta_2^+=\eta$ as given by (53).

Proof. For the proof we refer to Corollary 1.1. □

Theorem 8 (*Equivalent confidence coefficient for* $\mathbf{d}^*$ *based on* (**X**,**Y**)). Suppose that $\mathbf{V}^+=(V_1^+,V_2^+)$, is a random vector having the probability density function $f^+(v_1^+,v_2^+)$ (v_1^+ real, $v_2^+>0$), where f^+ is defined by (50), and let H^+ be the probability distribution function of $W^+=V_1^+/V_2^+$, where the probability density function of W^+ is given by

$$h^+(w^+)=\int_0^\infty v_2^+ f^+(w^+v_2^+,v_2^+)dv_2^+. \quad (55)$$

Then the confidence coefficient associated with the optimum prediction interval $\mathbf{d}^*=(d_1,d_2)$, where $d_1=M^+ +\eta_1^+S^+$, $d_2=M^+ +\eta_2^+S^+$, is

$$\Pr\{\mathbf{d}^*: d_1<Y_s<d_2\mid\mu,\sigma\}$$

$$=H^+[Q^{+(-1)}(1-c/c_2)]-H^+[Q^{+(-1)}(c/c_1)]. \quad (56)$$

Proof. For the proof we refer to Theorem 3. □

VI. EXAMPLES

A. Within-Sample Prediction of Order Statistic

Exponential distribution. Let $Y_1< Y_2< \cdots <Y_m$ be order statistics of size m from the exponential distribution with the density

$$f(y\mid\sigma)=\frac{1}{\sigma}\exp\left(-\frac{y}{\sigma}\right),\quad y>0,\quad \sigma>0. \quad (57)$$

Consider the prediction problem of Y_s for the situation where the first r observations $Y_1< Y_2< \cdots <Y_r$, $1\leq r < s \leq m$, have been observed. We are now concerned with optimization of the prediction interval for Y_s under the loss function (2).

Let $\mathbf{Y}=(Y_1, \ldots, Y_r)$ and $Y_s > Y_r$ for $s \leq m$. Then the joint probability density function of **Y** and Y_s is given by

$$f(y_1,...,y_r,y_s\mid\sigma)=\frac{m!}{(s-r-1)!(m-s)!}$$

$$\times[F(y_s|\sigma)-F(y_r|\sigma)]^{s-r-1}[1-F(x_r|\sigma)]^{m-s}$$

$$\times\prod_{i=1}^{r} f(y_i|\sigma)f(y_s|\sigma)$$

$$=\frac{m!}{(s-r-1)!(m-s)!}\frac{1}{\sigma^{r+1}}\exp\left(-\frac{\sum_{i=1}^{r}y_i+(m-r)y_r}{\sigma}\right)$$

$$\times\left[1-\exp\left(-\frac{y_s-y_r}{\sigma}\right)\right]^{s-r-1}\left[\exp\left(-\frac{y_s-y_r}{\sigma}\right)\right]^{m-s+1}. \quad (58)$$

Let

$$V_1=\frac{Y_s-Y_r}{\sigma} \quad (59)$$

and

$$V_2=\frac{S_r}{\sigma}=\frac{\sum_{i=1}^{r}Y_i+(m-r)Y_r}{\sigma}. \quad (60)$$

Using the invariant embedding technique [14-23], we then find in a straightforward manner that the joint density of V_1, V_2 is

$$f(v_1,v_2)=f_1(v_1)f_2(v_2), \quad (61)$$

where

$$f_1(v_1)=\frac{[1-e^{-v_1}]^{s-r-1}[e^{-v_1}]^{m-s+1}}{\mathrm{B}(s-r,m-s+1)}$$

$$=\frac{1}{\mathrm{B}(s-r,m-s+1)}\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}(-1)^j e^{-v_1(m-s+1+j)},$$

$$v_1>0, \quad (62)$$

and

$$f_2(v_2)=\frac{1}{\Gamma(r)}v_2^{r-1}e^{-v_2}, \quad v_2>0. \quad (63)$$

It follows from (15) and (61) that

$$q(w)=\frac{\int_0^\infty v_2^2 f(wv_2,v_2)dv_2}{\int_0^\infty\int_0^\infty v_2 f(v_1,v_2)dv_1dv_2}=\frac{1}{r}\int_0^\infty v_2^2 f_1(wv_2)f_2(v_2)dv_2$$

$$=\frac{r+1}{\mathrm{B}(s-r,m-s+1)}$$

$$\times\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}(-1)^j\frac{1}{[1+w(m-s+1+j)]^{r+2}}. \quad (64)$$

It follows from (25) and (61) that

$$h(w)=\int_0^\infty v_2 f(wv_2,v_2)dv_2=\int_0^\infty v_2 f_1(wv_2)f_2(v_2)dv_2$$

$$=\frac{r}{\mathrm{B}(s-r,m-s+1)}$$

$$\times\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}(-1)^j\frac{1}{[1+w(m-s+1+j)]^{r+1}}. \quad (65)$$

If $c/c_1+c/c_2<1$ then the optimal invariant linear-loss interval predictor of Y_s based on **Y** is given by

$$\mathbf{d}^*=(Y_r+\eta_1 S_r,\ Y_r+\eta_2 S_r), \quad (66)$$

where

$$\eta_1=\arg\left(\int_0^{\eta_1}q(w)dw=\frac{c}{c_1}\right) \quad (67)$$

and

$$\eta_2=\arg\left(\int_0^{\eta_2}q(w)dw=1-\frac{c}{c_2}\right). \quad (68)$$

The confidence coefficient associated with the optimum prediction interval $\mathbf{d}^*=(d_1,d_2)$, where $d_1=Y_r+\eta_1 S_r$, $d_2=Y_r+\eta_2 S_r$, is given by

$$\Pr\{\mathbf{d}^*: d_1<Y_s<d_2|\mu,\sigma\}$$

$$=H[\eta_2]-H[\eta_1]=\int_{\eta_1}^{\eta_2}h(w)dw. \quad (69)$$

B. New-Sample Prediction of Order Statistic

Exponential distribution. Consider the problem of prediction of the *s*th order statistic Y_s, $1 \le s \le m$, in a new (future) sample of size m from the exponential distribution with the probability density function (57) for the situation where the first k observations $X_1 < X_2 < \cdots < X_k$, $1\le k\le n$, from the previous data sample of size n have been observed. We are now concerned with optimization of the prediction interval for Y_s under the loss function (2).

Let $\mathbf{X}=(X_1, X_2, \ldots, X_k)$ for $k \le n$. Then the joint probability density function of $X_1, X_2, \ldots, X_k$ is given by

$$f(x_1,x_2,\ldots,x_k|\sigma)=\frac{n!}{(n-k)!}\prod_{i=1}^{k}f(x_i|\sigma)[1-F(x_k|\sigma)]^{n-k}$$

$$=\frac{n!}{(n-k)!}\frac{1}{\sigma^k}\exp\left(-\frac{\sum_{i=1}^{k}x_i+(n-k)x_k}{\sigma}\right). \quad (70)$$

The probability density function of the sth order statistic Y_s in the future sample of size m is given by

$$f(y_s|\sigma)=\frac{m!}{(s-1)!(m-s)!}$$

$$\times[F(y_s|\sigma)]^{s-1}[1-F(y_s|\sigma)]^{m-s}f(y_s|\sigma)$$

$$=\frac{1}{\mathrm{B}(s,m-s+1)}\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j\exp\left(-\frac{(j+s)y_s}{\sigma}\right)\frac{1}{\sigma}. \quad (71)$$

Let

$$V_1^\circ=\frac{Y_s}{\sigma} \quad (72)$$

and

$$V_2^\circ=\frac{S_k}{\sigma}=\frac{\sum_{i=1}^{k}X_i+(n-k)X_k}{\sigma}. \quad (73)$$

Using the invariant embedding technique [14-23], we then find in a straightforward manner that the joint density of V_1, V_2 is

$$f^\circ(v_1^\circ,v_2^\circ)=f_1^\circ(v_1^\circ)f_2^\circ(v_2^\circ), \quad (74)$$

where

$$f_1^\circ(v_1^\circ)=\frac{1}{\mathrm{B}(s,m-s+1)}\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j\exp\left(-(j+s)v_1^\circ\right),$$

$$v_1^\circ>0, \quad (75)$$

and

$$f_2^\circ(v_2^\circ)=\frac{1}{\Gamma(k)}(v_2^\circ)^{k-1}e^{-v_2^\circ},\quad v_2^\circ>0. \quad (76)$$

It follows from (15) and (74) that

$$q^\circ(w^\circ)=\frac{\int_0^\infty(v_2^\circ)^2 f^\circ(w^\circ v_2^\circ,v_2^\circ)dv_2^\circ}{\int_0^\infty\int_0^\infty v_2^\circ f^\circ(v_1^\circ,v_2^\circ)dv_1^\circ dv_2^\circ}$$

$$=\frac{1}{k}\int_0^\infty(v_2^\circ)^2 f_1^\circ(w^\circ v_2^\circ)f_2^\circ(v_2^\circ)dv_2^\circ$$

$$=\frac{k+1}{\mathrm{B}(s,m-s+1)}\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j\frac{1}{[1+w^\circ(s+j)]^{k+2}}. \quad (77)$$

It follows from (25) and (74) that

$$h^\circ(w^\circ)=\int_0^\infty v_2^\circ f^\circ(w^\circ v_2^\circ,v_2^\circ)dv_2^\circ=\int_0^\infty v_2^\circ f_1(w^\circ v_2^\circ)f_2(v_2^\circ)dv_2^\circ$$

$$=\frac{k}{\mathrm{B}(s,m-s+1)}\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j\frac{1}{[1+w^\circ(s+j)]^{k+1}}. \quad (78)$$

If $c/c_1+c/c_2<1$ then the optimal invariant linear-loss interval predictor of Y_s based on $\mathbf{X}$ is given by

$$\mathbf{d}^*=(X_k+\eta_1^\circ S_k,\ X_k+\eta_2^\circ S_k), \quad (79)$$

where

$$\eta_1^\circ=\arg\left(\int_0^{\eta_1^\circ}q^\circ(w^\circ)dw^\circ=\frac{c}{c_1}\right) \quad (80)$$

and

$$\eta_2^\circ=\arg\left(\int_0^{\eta_2^\circ}q^\circ(w^\circ)dw^\circ=1-\frac{c}{c_2}\right). \quad (81)$$

The confidence coefficient associated with the optimum prediction interval $\mathbf{d}^*=(d_1,d_2)$, where $d_1=X_k+\eta_1^\circ S_k$, $d_2=X_k+\eta_2^\circ S_k$, is given by

$$\Pr\{\mathbf{d}^*: d_1<Y_s<d_2|\sigma\}$$

$$=H^\circ[\eta_2^\circ]-H^\circ[\eta_1^\circ]=\int_{\eta_1^\circ}^{\eta_2^\circ}h^\circ(w^\circ)dw^\circ. \quad (82)$$

C. New-Within-Sample Prediction of Order Statistic

Exponential distribution. Consider the prediction problem of the sth order statistic Y_s, $1 \le s \le m$, in a new sample of size m from the exponential distribution with the probability density function (57) for the situation where both the first r observations $Y_1< Y_2< \cdots <Y_r$, $1\le r < s \le m$, from the new data sample and the first k observations $X_1 < X_2 < \cdots< X_k$, $1\le k\le n$, from the previous data sample of size n have been observed. We are now in need to optimize the prediction interval for Y_s under the loss function (2).

Let $\mathbf{X}=(X_1, X_2, \ldots, X_k)$ for $k \le n$. Then the joint probability density function of $X_1, X_2, \ldots, X_k$ is given by

$$f(x_1,x_2,\ldots,x_k|\sigma)=\frac{n!}{(n-k)!}\prod_{i=1}^{k}f(x_i|\sigma)[1-F(x_k|\sigma)]^{n-k}$$

$$=\frac{n!}{(n-k)!}\frac{1}{\sigma^k}\exp\left(-\frac{\sum_{i=1}^{k}x_i+(n-k)x_k}{\sigma}\right). \quad (83)$$

Let $\mathbf{Y}=(Y_1, \ldots, Y_r)$ and $Y_s > Y_r$ for $s \le m$. Then the joint probability density function of $\mathbf{Y}$ and Y_s is given by

$$f(y_1,\ldots,y_r,y_s|\sigma)=\frac{m!}{(s-r-1)!(m-s)!}$$

$$\times[F(y_s|\sigma)-F(y_r|\sigma)]^{s-r-1}[1-F(x_r|\sigma)]^{m-s}$$

$$\times\prod_{i=1}^{r} f(y_i|\sigma)f(y_s|\sigma)$$

$$=\frac{m!}{(s-r-1)!(m-s)!}\frac{1}{\sigma^{r+1}}\exp\left(-\frac{\sum_{i=1}^{r}y_i+(m-r)y_r}{\sigma}\right)$$

$$\times\left[1-\exp\left(-\frac{y_s-y_r}{\sigma}\right)\right]^{s-r-1}\left[\exp\left(-\frac{y_s-y_r}{\sigma}\right)\right]^{m-s+1}. \quad (84)$$

Let

$$V_1^+=\frac{Y_s-Y_r}{\sigma} \quad (85)$$

and

$$V_2^+=\frac{S^+}{\sigma}=\frac{\sum_{i=1}^{k}X_i+(n-k)X_k+\sum_{i=1}^{r}Y_i+(m-r)Y_r}{\sigma}. \quad (86)$$

Using the invariant embedding technique [14-23], we then find in a straightforward manner that the joint density of V_1^+, V_2^+ is

$$f^+(v_1^+,v_2^+)=f_1^+(v_1^+)f_2^+(v_2^+), \quad (87)$$

where

$$f_1^+(v_1^+)=\frac{[1-e^{-v_1^+}]^{s-r-1}[e^{-v_1^+}]^{m-s+1}}{\mathrm{B}(s-r,m-s+1)}$$

$$=\frac{1}{\mathrm{B}(s-r,m-s+1)}\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}(-1)^j e^{-v_1^+(m-s+1+j)},$$

$$v_1^+>0, \quad (88)$$

and

$$f_2^+(v_2^+)=\frac{1}{\Gamma(k+r)}v_2^{k+r-1}e^{-v_2}, \quad v_2>0. \quad (89)$$

It follows from (15) and (87) that

$$q^+(w^+)=\frac{\int_0^\infty (v_2^+)^2 f(w^+v_2^+,v_2^+)dv_2^+}{\int_0^\infty\int_0^\infty v_2^+ f(v_1^+,v_2^+)dv_1^+dv_2^+}$$

$$=\frac{1}{k+r}\int_0^\infty v_2^2 f_1(w^+v_2^+)f_2(v_2^+)dv_2^+$$

$$=\frac{k+r+1}{\mathrm{B}(s-r,m-s+1)}$$

$$\times\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}(-1)^j\frac{1}{[1+w^+(m-s+1+j)]^{k+r+2}}. \quad (90)$$

It follows from (25) and (87) that

$$h^+(w^+)=\int_0^\infty v_2^+ f^+(w^+v_2^+,v_2^+)dv_2^+=\int_0^\infty v_2^+ f_1^+(w^+v_2^+)f_2^+(v_2^+)dv_2^+$$

$$=\frac{r}{\mathrm{B}(s-r,m-s+1)}$$

$$\times\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}(-1)^j\frac{1}{[1+w^+(m-s+1+j)]^{r+1}}. \quad (91)$$

If $c/c_1+c/c_2<1$ then the optimal invariant linear-loss interval predictor of Y_s based on $(\mathbf{X},\mathbf{Y})$ is given by

$$\mathbf{d}^*=(Y_r+\eta_1^+S^+,\ Y_r+\eta_2^+S^+), \quad (92)$$

where

$$\eta_1^+=\arg\left(\int_0^{\eta_1^+}q^+(w^+)dw^+=\frac{c}{c_1}\right) \quad (93)$$

and

$$\eta_2^+=\arg\left(\int_0^{\eta_2^+}q^+(w^+)dw^+=1-\frac{c}{c_2}\right). \quad (94)$$

The confidence coefficient associated with the optimum prediction interval $\mathbf{d}^*=(d_1,d_2)$, where $d_1=Y_r+\eta_1^+S^+$, $d_2=Y_r+\eta_2^+S^+$, is given by

$$\Pr\{\mathbf{d}^*: d_1<Y_s<d_2|\sigma\}$$

$$=H^+[\eta_2^+]-H^+[\eta_1^+]=\int_{\eta_1^\circ}^{\eta_2^+}h^+(w^+)dw^+. \quad (95)$$

VII. Conclusion and Directions for Future Research

Traditionally, methods that are developed and implemented are point forecasting methods, i.e. they provide a single estimated value for a given horizon. Interval prediction is an important part of the forecasting process aimed at enhancing the limited accuracy of point estimation. Prediction intervals are needed to quantify prediction uncertainty in, for example, warranty prediction. In this paper, we have developed prediction techniques appropriate for constructing frequentist prediction intervals or prediction bounds for order statistics in future samples via a specific loss function when only the functional form of the underlying distribution is specified, but some or all of its parameters are unspecified. The within-sample prediction technique takes into account only current early-failure data, the new-sample prediction technique takes into account only previous independent observations, and the new-within-sample prediction technique takes into account both previous and current early-failure data. The techniques can be used to construct prediction intervals for arbitrary sth failure time in a sample of future observations. The results can be used to predict the total duration time in a Type II censoring life testing experiment, and to predict the lifetime of an k-out-of-n:F system. The computation procedure can be easily programmed and implemented for practical use. It will be noted that some inferences considered in this paper can be obtained through simulation, but simulation results are unstable; they vary from one to another. From theoretical as well as practical points of view, analytical solutions should be used if they are available. The results of this paper provide such analytical solutions. In many statistical decision problems it is reasonable co confine attention to rules that are invariant with respect to a certain group of transformations. If a given decision problem admits a sufficient statistic, it is well known that the class of invariant rules based on the sufficient statistic is essentially complete in the class of all invariant rules under some assumptions. This result may be used to show that if there exists a minimax invariant rule among invariant rules based on sufficient statistic, it is minimax among all invariant rules. In this paper, we consider statistical prediction problems which are invariant with respect to a certain group of transformations and construct the optimal invariant interval predictors. The method used is that of the invariant embedding of sample statistics in a loss function in order to form pivotal quantities which allow one to eliminate unknown parameters from the problem. This method is a special case of more general considerations applicable whenever the statistical problem is invariant under a group of transformations, which acts transitively on the parameter space. The techniques proposed in this paper can be applied to constructing prediction intervals for any other location-scale distributions.

ACKNOWLEDGMENT

This research was supported in part by Grant No. 06.1936, Grant No. 07.2036, Grant No. 09.1014, and Grant No. 09.1544 from the Latvian Council of Science and the National Institute of Mathematics and Informatics of Latvia.

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