A Hybrid Method for Solving Optimal Control Problems

Mohammad Keyanpour, Ali Mahmoudi

Abstract—**This paper presents a numerical technique for solving optimal control problems. Control variables are series constructed by Legendre polynomials. In this technique state variables are computed in terms of control variables by implementing Decomposition Method. Performance index is transformed by replacing new control and state variables. The system dynamics and the performance index are converted into some algebraic equations. Then the optimal control problem is reduced to constrained classical optimization problem. To show the efficiency of the purposed technique results and comparisons are given at the end of this paper. Also optimal state trajectory and optimal control policy graphs are included.**

Index Terms—**Legendre polynomials, Decomposition method, Optimal control problems, Optimization.**

I. Introduction

At the large number of problems arising in analysis, mechanics, geometry, and so forth, it is necessary to determine the maximal and minimal of a certain functional. Because of the important role of this subject in science and engineering, considerable attentions has been received on this kind of problems. It is well known that generally optimal control problems are difficult to solve. Particularly, their analytical solutions are in many cases out of the question. To overcome this difficulty numerical methods are purposed for solving many of these real world problems. Numerical methods for solving optimal control problems dated back nearly six decades. From that time to the present, the complexity of methods and corresponding complexity and variety of applications has increased tremendously making optimal control a discipline that is relevant to many branches of engineering. There are various numerical approaches for solving optimal control problems (for instance see [1], [2], [3]) In recent years, considerable attentions have been given to the use of spectral methods for solving nonlinear problems. The approach, known as the spectral method [4] is based on converting the differential equations into an integral equation through integration. The state and/or control involved in the equation are approximated by finite terms of orthogonal series and using an operational matrix of integration to eliminate the integral operations. At the beginning in [5] A numerical technique for solving nonlinear optimal control problems is introduced. The state and control variables are expanded in the Chebyshev series, and an algorithm is provided for approximating the system dynamics, boundary conditions, and performance index. Application of this method results in the transformation of differential and integral expressions into systems of algebraic or transcendental expressions in the Chebyshev coefficients. In [6] for directly solving a generic optimal control problem with state and control constraints Chebyshev pseudospectral method are implemented, This method employs N'th degree Lagrange polynomial approximations for the state and control variables with the values of these variables at the Chebyshev-Gauss-Lobatto points as the expansion coefficients. This process yields a Nonlinear Programming Problem (NLP) with the state and control values at the CGL points as unknown NLP parameters.

In [7] an alternative computational method presented for solving the controlled Duffing oscillator. The approach is a spectral method in which we approximate the control and state variables by Chebyshev series. An explicit formula for the Chebyshev polynomials in terms of arbitrary order of their derivatives is used to convert the system dynamics into an algebraic equation.

In [8] the usage of orthogonal polynomials for obtaining an analytical approximate solution to optimal control problems with a weighed quadratic cost function is proposed. The method consists of using the orthogonal polynomials for the expansion of the state variables and the control signal. And in [9] wavelet functions are utilized instead of orthogonal polynomials.

In the present paper normalized Legendre polynomials are used for constructing control variables, but state variables obtained in different manner, here we compute state variables employing Decomposition Method, afterwards control and state variables are replaced in objective function, considering control and state constraints, former optimal control problem converted into classical optimization problem. Optimizing obtained performance index, parameters which give us state trajectory and control policy, could be calculated.

The paper is organized as follows: In Section 2, Decomposition Method and Legendre orthogonal polynomials are reviewed. In Section 3, the problem is formulated. Section 4, purposed method is presented and in Section 5, convergency of the method is discussed, finally in section 6 Numerical examples are expressed.

A. Preliminaries

B. Legendre polynomials

The definition of orthogonal polynomials $\{\varphi_j(t)\}$ and some of their features are presented below:

$$\int_a^b w(t)\varphi_i(t)\ \varphi_j(t)\ dt = \begin{cases} \delta_i & i=j \\ 0 & i\neq j \end{cases}$$

In which $W(t)$ is the weight function.

The expansion of an arbitrary function $f(t)$ on the closed

Manuscript received April, 2010; revised May 9, 2012.

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interval $[0, t_f]$ is as follows:

$$f(t) = \sum_{i=0}^{N} C_i \varphi_i(t)$$

In which:

$$C_i = \frac{1}{\delta_i} \int_0^{t_f} t^k f(t) \varphi_i(t)\ dt$$

An important property of orthogonal polynomials presented in [10] is

$$\int \varphi(t) = k\varphi(t)$$

And

$$\int_0^{t_f} PP^T\ dt = \gamma$$

where

$$P(t) = [P_0, P_1, \ldots, P_n]^T$$

$$\gamma = diag[\gamma_0, \gamma_1, \ldots, \gamma_n]$$

$$\gamma_i = \int_0^{t_f} {P_i}^2(\mathrm{t})\ dt$$

In the linear space of all polynomials, with the inner product

$$\langle x, y \rangle = \int_{-1}^{1} x(t) y(t) dt$$

consider the infinite sequence $x_0, x_1, x_2, \ldots$ where $x(t) = t^n$. When the orthogonalization theorem is applied to this sequence it yields another sequence of polynomials $y_0, y_1, y_2, \ldots$ first encountered by the French mathematician *A. M. Legendre* (1752-1833) in his work on potential theory. The first few polynomials are easily calculated by the Gram-Schmidt process.

$$y_0(t) = 1$$

$$y_1(t) = t$$

$$y_2(t) = \frac{1}{2}\left(3t^2 - 1\right)$$

$$\vdots$$

The polynomials in the corresponding orthonormal sequence are given by

$$P_0(t) = \sqrt{\frac{1}{2}}$$

$$P_1(t) = \sqrt{\frac{3}{2}} t$$

$$P_2(t) = \frac{1}{2}\sqrt{\frac{5}{2}}(3t^2 - 1)$$

$$P_3(t) = \frac{1}{2}\sqrt{\frac{7}{2}}(5t^3 - 3t)$$

$$\vdots$$

given by $P_n = y_n / \|y_n\|$ are called *normalized Legendre polynomials*. Legendre polynomials are categorized as the Jacobi orthogonal polynomials whose interval of orthogonality is $[-1, 1]$.

C. Decomposition Method

In what follows Adomian's Decomposition Method for solving Initial Value Problems (IVP-s) is briefly introduced. For more details see [11] and [12]. Consider the following ODE modelling a dynamical system via independent variable t, $F(x(t)) = g(t)$ with $x(0) = a$;
Here F represents a general nonlinear ordinary differential operator involves both linear and nonlinear parts, in which linear part is decomposed as $L + R$, so that L is easily invertible and R remainder of the linear operator. For convenience, L may be taken as the highest order derivative whose inverse does not involve difficult integrations that results when complicated Green's functions are involved. Thus the above equation may be written as

$$Lx + Rx + Nx = g(t)$$

where Nu representing the nonlinear terms. Solving for above equation for Lu, we have

$$Lx = g - Rx - Nx$$

Since L is invertible, an equivalent form of the above equation is

$$L^{-1}Lx = L^{-1}g - L^{-1}Rx - L^{-1}Nx$$

If this corresponds to an IVP, the integral operator L^{-1} may be regarded as definite integration from 0 to s.
Former equation yields

$$x = a + L^{-1}g - L^{-1}Rx - L^{-1}Nx \tag{1}$$

The nonlinear term Nu is now approximated by $\sum_{k=0}^{\infty} A_k$ where A_k are special polynomials to be further elaborated. Thus we have the identity

$$Nx(t) = \sum_{i=0}^{\infty} A_i(A_0, A_1, \ldots, A_i)$$

Moreover, x is decomposed as $\sum_{k=0}^{\infty} x_k$ with x_0 identified as $a + L^{-1}g$ (considering our purpose, we get L^{-1} as one fold integration) accordingly, the above equation can be written as

$$\sum_{k=0}^{\infty} x_k = x_0 - L^{-1}R\sum_{k=0}^{\infty} x_k - L^{-1}\sum_{k=0}^{\infty} A_k$$

Consequently, we may readily derive the following set of equations

$$\begin{gathered} x_1 = -L^{-1}Rx_0 - L^{-1}A_0 \\ x_2 = -L^{-1}Rx_1 - L^{-1}A_1 \\ \vdots \end{gathered} \tag{2}$$

.

$$x_{k+1} = -L^{-1}Rx_k - L^{-1}A_k$$

Note that the polynomials A_k are generated for each nonlinear term Nu such that A_0 depends only on x_0; A_1 depends only on x_0 and x_1 ; A_2 depends on x_0, x_1 and x_2, and so forth, i.e. Adomian's polynomials are $A_0(x_0)$, $A_1(x_0,x_1)$, $A_2(x_0,x_1,x_2)$, ... so the components x_k are computable for all $k \geq 0$, and thus one computes $x = \sum_{k=0}^{\infty} x_k$.
It is well known that the series $\sum_{k=0}^{\infty} A_k$ are generalized Taylor series for $f(x_0)$; i.e. $\sum_{k=0}^{\infty} x_k$ is a functional Taylor series about the function x_0, and that the terms in the series approach zero with $1/(mk)!$ (see [13]), where *m* is highest order of differentiation in the linear differential operator. Since the series converges and does so very rapidly.
k term partial sum $S_k = \sum_{i=0}^{k-1} x_k$ may serve as a practical solution for a rather small *k* and we naturally have $\lim_{k\to\infty} S_k = x$. It is important to emphasize that A_k are readily computable for complicated nonlinearities of the form $f(x,\dot{x},\ddot{x},\ldots)$ or $f(g(x))$. Since the solutions are analytic (and verifiable by substitution), we can have insights into how the solution evolves. Let $\hat{g} = L^{-1}g$. Taylor series expansion of $\hat{g}$ with respect to $\tau = 0$ is

$$\hat{g}(\tau) = \hat{g}(t_0) + \frac{d\hat{g}}{d\tau}\tau + \frac{d\hat{g}^2}{d\tau^2}\frac{\tau^2}{2!} + \ldots$$

Thus the first series term is modified as $x_0 = a + \hat{g}(t_0)$ The other series terms (see (2)) are

$$x_{k+1} = \frac{d\hat{g}^k}{d\tau^k}\frac{\tau^k}{k!} - L^{-1}Rx_k - L^{-1}A_k \quad , k = 1, 2, \ldots$$

Then the solution (valid over the *i*th interval) is obtained as summation of x_k for $k = 0, 1, 2, \ldots$. Here $a = x(t_0)$.

II. Problem statement

We deal with the optimal control problem in which the optimal solution satisfies ordinary differential equation on the fixed time interval $I := [t_0, t_f]$ together with initial and final conditions while optimizes (i.e. maximize or minimize) performance index. in other word we want to find control function to optimize

$$\mathrm{J} = \int_{t_0}^{t_f} f_0(t, x(t), u(t))dt \tag{3}$$

subject to:

$$\dot{x} = f(t, x(t), u(t)) \tag{4}$$

and

$$x(0) = x^0,\ Gx(t_f) = x^1 \tag{5}$$

Where $x(t) = [x_1, x_2, x_3, \ldots, x_n]^T \in H^n$ and $u(t) = [u_1, u_2, u_3, \ldots, u_k]^T \in H^k$ are respectively, state and control vectors . $x^0 \in H^n$, $x^1 \in H^l$ are given vectors, and G is $l \times n$ matrix. $f_0 : H^{m+n+1} \to H$ is supposed to be a continuous function.
The aim of control theory is to find a control u such that performance index (3) gets it's optimum value.

III. Method of the solution

Since Legendre polynomials are defined on the time interval $[-1, 1]$, it is necessary to transform the time variable t in the optimal control problem from the time interval $[0, t_f]$ -for simplicity and without losing generality we get $t_0 = 0$- into the the time interval $[-1, 1]$. This transformation can be achieved by

$$\tau = \frac{2t}{t_f} - 1 \tag{6}$$

formula (6) transforms the optimal control problem (3)-(5) into the problem of minimizing

$$J = \frac{t_f}{2}\int_{-1}^{1} f_0(t, x(t), u(t))\ d\tau \tag{7}$$

Subject to:

$$\frac{dx}{d\tau} = \frac{t_f}{2} f(\tau, x(\tau), u(\tau)) \tag{8}$$

$$x(-1) = x^0,\ Gx(1) = x^1 \tag{9}$$

Where G is a non zero $l \times n$ matrix.
Let us approximate control function by normalized Legendre polynomials:

$$u(\tau) = \sum_{i=0}^{k} c_i P_i(\tau) \tag{10}$$

Where $P_i(\tau)$ $i = 1, 2, 3\ldots$ is normalized Legendre polynomials and $c_i \in \Re$, $i = 1, 2, 3\ldots$ are coefficients must be obtained as solutions of transformed optimization problem. Replacing (10) into (8), and supposing that coefficients $c_0,\ c_1, \ldots,\ c_k$ are constant, the following system of Ordinary Differential Equation (ODE) is acquired;

$$\frac{dx}{d\tau} = \frac{t_f}{2} f(\tau, x(\tau), \sum_{i=0}^{k} c_i P_i(\tau))$$

subject to

$$x(-1) = x^0 \tag{11}$$

In the followind IVP (11), is solved using Decomposition method introduced in previous section, as a result state variable x is obtained as a function in terms of $c_0,\ c_1,\ c_2, \ldots,\ c_k,\ \tau$ indeed one can write;

$$x = \psi(c_0,\ c_1,\ c_2, \ldots,\ c_k,\ \tau)$$

What remains is replacing x, and u by $\psi(c_0,\ c_1,\ c_2, \ldots,\ c_k,\ \tau)$ and $\sum_{i=0}^{k} c_i P_i(\tau)$ respectively, through (7);

$$J = \frac{t_f}{2}\int_{-1}^{1} F(c_0,\ c_1,\ c_2, \ldots,\ c_k,\ \tau))d\tau \tag{12}$$

After integration, (12) can be rewritten in the following format:

$$J = \varphi(c_0,\ c_1,\ c_2, \ldots,\ c_k,\ \tau)$$

Considering boundary condition (9), one can see that optimal control problem (3),(4),(5) is converted into classical optimization problem;

$$\min_{c_0,\ c_1,\ c_2, \ldots,\ c_k} \varphi(c_0,\ c_1,\ c_2, \ldots,\ c_k\ )$$

Subject to:

$$G\psi(c_0,\ c_1,\ c_2, \ldots,\ c_k,\ 1) = x^1$$

This optimization problem can be solved using conventional optimization Toolboxes, here in this study it is solved by MAPLE optimization Toolbox. Algorithm 1. illustrates present method strategy for solving optimal control problems;

Algorithm 1 .

Step 1 : Transform optimal control problem from the time interval $[0, t_f]$ *into the time interval* $[-1, 1]$.

Step 2 : Approximate control function using Legendre polynomials.

Step 3 : Use Decomposition method in order to obtain state in terms of control parameters.

Step 4 : Replace approximated control and obtained state into objective function and construct optimization problem.

Step 5 : Solve optimization problem using optimization Toolbox.

IV. Convergency of the method

Consider the following functional equation:

$$x - Nx = g \tag{13}$$

where N is a nonlinear operator from Hilbert space H into H, g is a given function in H. ADM gives the solution of the problem as follows:

$$x = \sum_{i=0}^{\infty} x_i$$

and Nx is replaced by series;

$$Nx = \sum_{i=0}^{\infty} A_i$$

where A_n for $n = 1, 2, \ldots$ are polynomials in terms of $x_0, x_1, ..., x_n$ is called Adomian polynomials. The method consist of the following scheme:

$$\begin{cases} x_0 = g \\ x_{n+1} = A_n(x_0, x_1, \ldots, x_n) \end{cases} \tag{14}$$

The Adomian technique is equivalent to determining the sequence

$$S_n = x_0 + x_1 + \ldots + x_n$$

by using the iterative scheme

$$S_0 = 0$$

$$S_n = N(x_0 + S_n)$$

Associated with the functional equation

$$S = N(x_0 + S)$$

Theorem 1: Let N be an operator from a Hilbert space H into H and x be the exact solution of (13), $\sum_{i=0}^{\infty} x_i$ which is obtained by (14), converges to x if there exists $0 \le \gamma < 1$, such that $\|x_{k+1}\| \le \gamma \, \|x_k\| \quad k \in N \cup \{0\}$.
proof: see [14] .

Definition 1:

$$\gamma_i = \begin{cases} \frac{\|y_{i+1}\|}{\|y_i\|}, & \|y_i\| \neq 0 \\ 0, & \|y_i\| = 0 \end{cases}$$

Corollary 1: In Theorem 1., $\sum_{i=0}^{\infty} x_i$ converges to exact solution x, when $0 \le \gamma_i < 1$, $i = 1, 2, 3, ...$ i.e. $\lim_{n\to\infty} S_n \ = x$.
When control u inserted through (13), we have $x - Nx = g(u)$ instead, reconstructing (14), one can rewrite corollary 1. as;

$$\lim_{n\to\infty} S_n(u) \ = x(u)$$

Definition 2: Pair (x, u) is called an admissible pair, if and only if (x, u) satisfies (4), (5).
In the present paper admissible pair that optimizes performance index (3) will be shown by (x^*, u^*) in which $x^* = x(u^*)$.
Note that for the special pair (x^*, u^*) corollary 1. implies $\lim_{n\to\infty} S_n(u^*) \ = x^*$.
Let us define the following sets:

Definition 3:
(i)

$$\xi_k := \{(x, u_k) | u_k = \sum_{i=0}^{k} c_i t^i\},$$

(ii)

$$\xi^n := \{(S_n(u), u) | S_n(u) = \sum_{i=0}^{n} x_i(u)\},$$

(iii)

$$\xi_k^n := \{(S_n(u_k), u_k) | u_k = \sum_{i=0}^{k} c_i t^i, S_n(u_k) = \sum_{i=0}^{n} x_i(u_k)\},$$

Where u belongs to an admissible pair, x is the solution of $x - Nx = g(u)$, $n, k \in \aleph$, $c_i \in \Re, i = 1, 2, \ldots$

Notation 1:
(i) $\inf_u J(x(u), u) \ := J(x(\bar{u}), \bar{u})$,
(ii) $J(x(\bar{u}_k), \bar{u}_k) \ := \alpha_k$,
(iii) $J(S_n(\bar{u}_k), \bar{u}_k) \ := \alpha_k^n$,

Lemma 1: For $n \in \aleph$ the following relationship exists:

$$\alpha_1^n \ge \alpha_2^n \ge \cdots \ge \alpha_k^n \ge \cdots \ge \alpha^n = J(S_n(u^*), u^*)$$

specially,

$$\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_k \ge \cdots \ge \ J(x(u^*), u^*)$$

holds.
Proof: Considering the definition of ξ_k^n one can easily verify that;

$$\xi_1^n \subseteq \xi_2^n \subseteq, \cdots, \subseteq \xi_k^n \cdots \subseteq \xi^n.$$

which proves the lemma.

Lemma 2: The following equation hold:

$$lim_{k\to\infty}|\bar{u}_k - u^*| = 0.$$

Proof: Since polynomials are dense in $C(I)$, in which $C(I)$ is set of continuous functions on I, and considering continuity of f and f_0 it is easy to verify the proof.

Lemma 3: Now we can show that:

$$\lim_{k\to\infty} J\left(x\left(\bar{u}_k\right), \bar{u}_k\right) = J(x(u^*), u^*)$$

Proof: From lemma 1. we know that the sequence $\{\alpha_n\}_n$ is non increasing and bounded from below, thus it converges, suppose that $\lim_{n\to\infty}\alpha_n = \widehat{\alpha}$ we have $\widehat{\alpha} \geq J(x(u^*), u^*)$. if $\widehat{\alpha} > J(x(u^*), u^*)$ then there exist α' and $\varepsilon' > 0$ such that:

$$\begin{aligned} &|\ \alpha' - J(x(u^*), u^*)\ | = \varepsilon', \\ &\widehat{\alpha} > \alpha' > J(x(u^*), u^*), \end{aligned} \tag{15}$$

Because of continuity of J and x for each $\varepsilon > 0$ there exist a $\delta > 0$ such that $|u - u^*| < \delta$ which implies;

$$|J(x(u), u) - J(x(u^*), u^*)| < \varepsilon$$

Getting $\varepsilon = \varepsilon'$ and using lemma 2. for each $\delta > 0$ there exists a $\widehat{u}$ such that:

$$|\widehat{u} - u^*| < \delta$$

And immediately

$$|J(x(\widehat{u}), \widehat{u}) - J(x(u^*), u^*)| < \varepsilon$$

Which implies

$$|J(x(\widehat{u}), \widehat{u}) - J(x(u^*), u^*)| < |\alpha' - J(x(u^*), u^*)|$$

This contradicts with (15), Thus lemma 3. is proved.

Theorem 4: Now, we can show

$$lim_{k\to\infty} lim_{n\to\infty} J(S_n(\bar{u}_k), \bar{u}_k) = J(x^*, u^*)$$

Proof: Continuity of J and corollary 1. give

$$lim_{k\to\infty} lim_{n\to\infty} J(S_n(\bar{u}_k), \bar{u}_k) = lim_{k\to\infty} J(x(\bar{u}_k), \bar{u}_k)$$

And lemma 3. proves the theorem.

V. Computational results

To show the efficiency and practical application of purposed method described in the previous section, we present two numerical examples. First example is a well known linear system with quadratic performance index and we just have an initial condition, The second example is nonlinear system with quadratic performance index, in which we have initial and final conditions. We compare numerical estimation which obtained by present method and answers obtained by other methods. All of calculations is checked by Maple, also obtained optimization problems are solved by Maple optimization Toolbox. Figures of optimal state and optimal control are presented.

Example 1 . Find the optimal control $u(t)$ that minimizes

$$J = \int_0^1 x^2 + u^2 dt$$

subject to:

$$\dot{x} = u$$

The initial condition is:

$$x\left(0\right) = 1$$

Where x is a state function and u is control function.
This problem is solved for the following control approximations:

$$u_0 = \sqrt{\frac{1}{2}} c_{_0}$$

$$u_1 = \sqrt{\frac{1}{2}} c_{_0} + \sqrt{\frac{3}{2}} c_1 t$$

$$u_2 = \sqrt{\frac{1}{2}} c_{_0} + \sqrt{\frac{3}{2}} c_1 t + \sqrt{\frac{5}{8}} c_2 (3t^2 - 1)$$

$$u_3 = \sqrt{\frac{1}{2}} c_{_0} + \sqrt{\frac{3}{2}} c_1 t + \sqrt{\frac{5}{8}} c_2 (3t^2 - 1) + \sqrt{\frac{7}{8}} c_3 (5t^3 - 3t)$$

And finally

$$u_4 = \sum_{k=0}^{4} c_k P_k(t)$$

Corresponding objective functions are shown in Fig 1.
Exact objective value is given in [15] with 7 decimal precision is: $J^* = 0.7615941$. Optimal state trajectory and optimal control history are illustrated in Fig 2 and Fig 3.

Example 2. Find the suitable control for the nonlinear optimal control problem.

$$MinJ(x, u) = \int_0^1 x_1\left(t\right)^2 + x_2\left(t\right)^2 dt$$

$$\dot{x}_1\left(t\right) = x_2(t)$$

$$\dot{x}_2\left(t\right) = 10 x_1(t)^3 + u$$

$$x_1\left(0\right) = 0, x_1\left(1\right) = 0.1$$

$$x_2\left(0\right) = 0, x_2\left(1\right) = 0.3$$

This problem solved for $u_0, \ldots, u_4$, as presented in Example 1. and objective values are shown in Fig 4.
This problem is solved in [16] with $J^* = 0.0135$, the result given in [17] for the problem is $J^* = 0.024$, in Table I these results are compared with present method.
The state trajectory and control history are shown in Fig 5 and Fig 6 respectively.
One can see from Fig 1 and Fig 4 that both for Example 1. and Example 2. objective value converges to optimal control so fast, results are accurate, and very little computations are needed to be done. Also relative error for Example 1. and Example 2. i.e. $|J^*_{k+1} - J^*_k|$, is presented in Table II, as one can see we obtain these accurate results with very little computation efforts.

TABLE I
COMPARING RESULTS FOR EXAMPLE 2.

implemented methods	J*
present method	0.0149
method of [16]	0.0135
Rubio's Method	0.024

TABLE II
COMPUTATION ERRORS FOR EXAMPLE 1. AND EXAMPLE 2.

k	Example 1	Example 2
0	5.07×10^{-2}	1.44×10^{-2}
1	1.68×10^{-4}	4.26×10^{-3}
2	6.72×10^{-8}	2.41×10^{-4}
3	1.00×10^{-9}	1.00×10^{-8}

VI. CONCLUSION

Legendre polynomials combined with ADM are used to solve optimal control problems. this polynomials are orthogonal in closed interval $[-1, 1]$, so optimal control problem transformed in order to take advantage of their orthogonality. Then ADM is implemented to solve obtained ODE. Many authors utilized ADM to deal with ODE and they got wonderful results, It is also important that the ADM does not require the discretization of the variables. It is not affected by computation round errors and one is not faced with necessity of large computer memory and time. In numerical examples solved in this paper, solutions are obtained just by calculating 3 or 4 terms with low memory consumption and low effort with rational precise. ADM is powerful method for solving nonlinear differential equations so it make present method a good tool dealing with nonlinear optimal control problems.

ACKNOWLEDGMENT

The authors would like to thank from anonymous reviewer of the journal of Numerical Algorithms for his (her) helpful comments.

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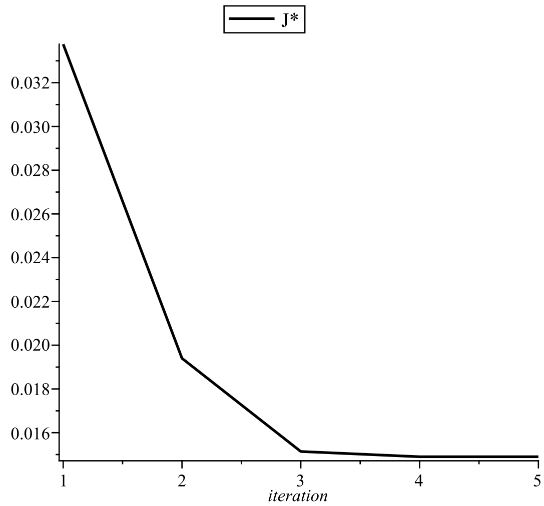

Fig. 1. Objective function for Example 1.

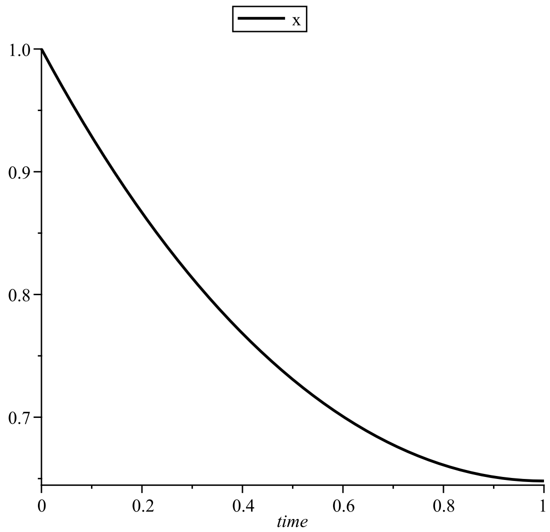

Fig. 2. Optimal state trajectory for Example 1.

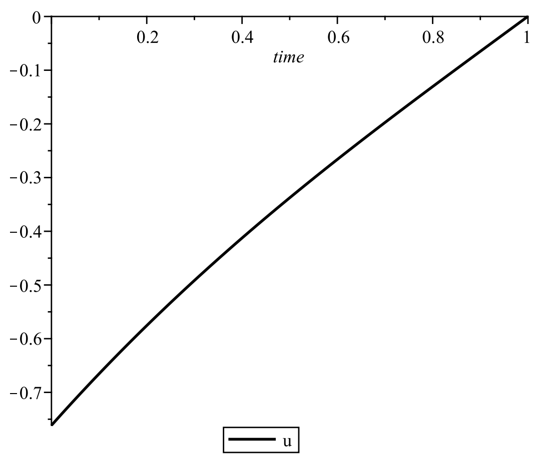

Fig. 3. Optimal control policy for Example 1.

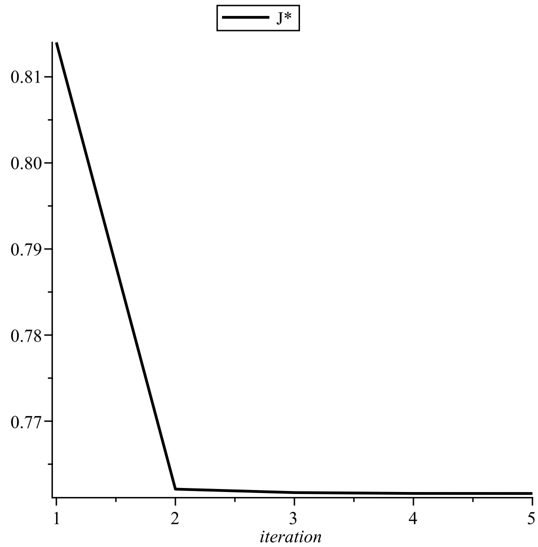

Fig. 4. Objective function for Example 2.

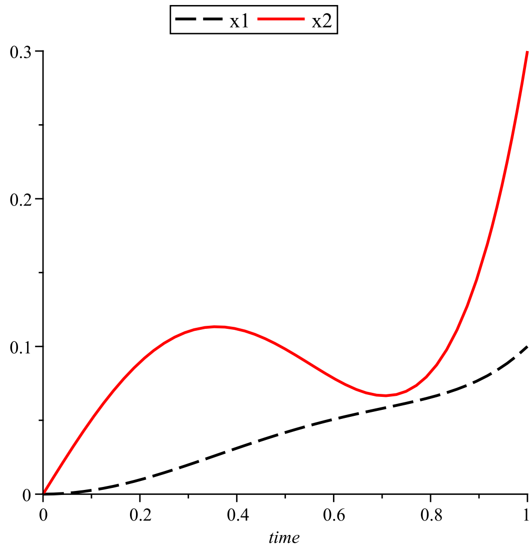

Fig. 5. Optimal state trajectory for Example 2.

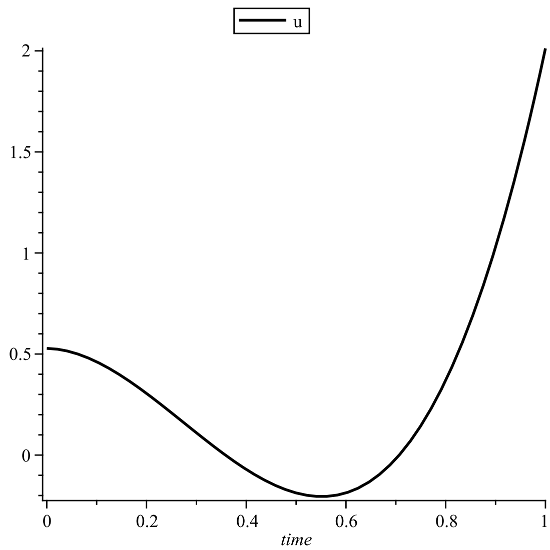

Fig. 6. Optimal control policy for Example 2.