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New Solutions for Synchronized Quadromineering

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***Abstract*—In synchronized games players make their moves simultaneously rather than alternately. Synchronized Quadromineering is the synchronized version of Quadromineering, a variants of a classical two-player combinatorial game called Domineering. New theoretical results for the $n \times 7$ and $n \times 11$ boards are presented.**

***Index Terms*—combinatorial game, synchronized game, Synchronized Quadromineering.**

I. Introduction

THE game of Domineering is a typical two-player game with perfect information, proposed around 1973 by Göran Andersson [3], [13], [14]. The two players, usually denoted by Vertical and Horizontal, take turns in placing dominoes (2×1 tile) on a checkerboard. Vertical is only allowed to place its dominoes vertically and Horizontal is only allowed to place its dominoes horizontally on the board. Dominoes are not allowed to overlap and the first player that cannot find a place for one of its dominoes loses. After a time the remaining space may separate into several disconnected regions, and each player must choose into which region to place a domino.

Berlekamp [2] solved the general problem for $2 \times n$ board for odd n. The 8×8 board and many other small boards were recently solved by Breuker, Uiterwijk and van den Herik [5] using a computer search with a good system of transposition tables. Subsequently, Lachmann, Moore, and Rapaport solved the problem for boards of width 2, 3, 5, and 7 and other specific cases [15]. Finally, Bullock solved the 10×10 board [6].

The game of Triomineering was proposed in 2004 by Blanco and Fraenkel [4]. In Triomineering Vertical and Horizontal alternate in tiling with a straight triomino (3×1 tile) on a checkerboard. Blanco and Fraenkel calculated Triomineering and values for boards up to 6 squares and small rectangular boards.

The game of Quadromineering is a further extension of Domineering where Vertical and Horizontal alternate in tiling with a straight quadromino (4×1 tile) on a checkerboard.

II. Synchronized games

For the sake of self containment, we recall the previous results concerning synchronized games. Initially, the concept of synchronism was introduced in the games of Cutcake [8], Maundy Cake [9], Domineering [10], [1], and Triomineering [11], [7] in order to study combinatorial games where players make their moves simultaneously.

As a result, in the synchronized versions of these games there exist no zero-games (fuzzy-games), i.e., games where the winner depends exclusively on the player that makes the second (first) move. Moreover, there exists the possibility of a draw, which is impossible in a typical combinatorial game. In this work, we continue to investigate synchronized combinatorial games by focusing our attention on Quadromineering.

In the game of Synchronized Quadromineering [12], a general instance and the legal moves for Vertical and Horizontal are defined exactly in the same way as defined for the game of Quadromineering.

There is only one difference: Vertical and Horizontal make their legal moves simultaneously, therefore, quadrominoes are allowed to overlap if they have a 1×1 tile in common. We note that 1×1 overlap is only possible within a simultaneous move.

At the end, if both players cannot make a move, then the game ends in a draw, else if only one player can still make a move, then he/she is the winner.

For each player there exist 3 possible outcomes:

1) The player has a *winning strategy* (ws) independently of the opponent's strategy, or
2) The player has a *drawing strategy* (ds), i.e., he/she can always get a draw in the worst case, or
3) The player has a *losing strategy* (ls), i.e., he/she does not have a strategy either for winning or for drawing.

Table I shows all the possible cases. It is clear that if one player has a winning strategy, then the other player has neither a winning strategy nor a drawing strategy. Therefore, the cases $ws - ws$, $ws - ds$, and $ds - ws$ never happen. As a consequence, if G is an instance of Synchronized Quadromineering, then we have 6 possible legal cases:

1) $G = D$ if both players have a drawing strategy, and the game will always end in a draw under perfect play, or
2) $G = V$ if Vertical has a winning strategy, or
3) $G = H$ if Horizontal has a winning strategy, or
4) $G = VD$ if Vertical can always get a draw in the worst case, but he/she could be able to win if Horizontal makes an unlucky move, or
5) $G = HD$ if Horizontal can always get a draw in the worst case, but he/she could be able to win if Vertical makes an unlucky move, or
6) $G = VHD$ if both players have a losing strategy and the outcome is totally unpredictable.

TABLE I
The possible outcomes in Synchronized Quadromineering

	Horizontal ls	Horizontal ds	Horizontal ws
Vertical ls	$G = VHD$	$G = HD$	$G = H$
Vertical ds	$G = VD$	$G = D$	-
Vertical ws	$G = V$	-	-

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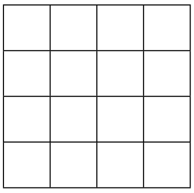
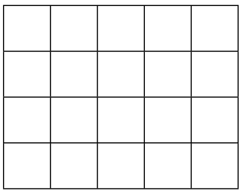
III. Examples of Synchronized Quadromineering

The game



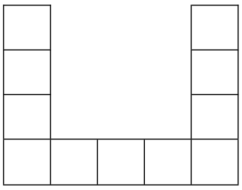
always ends in a draw, therefore $G = D$.

In the game

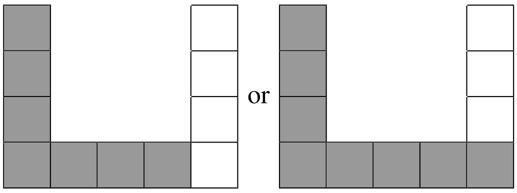


Vertical has a winning strategy moving in the central column, therefore $G = V$.

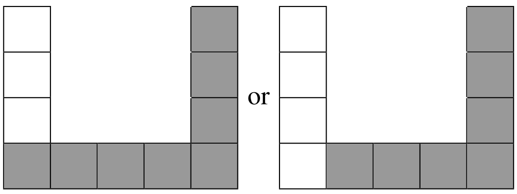
In the game



if Vertical moves in the first column we have two possibilities

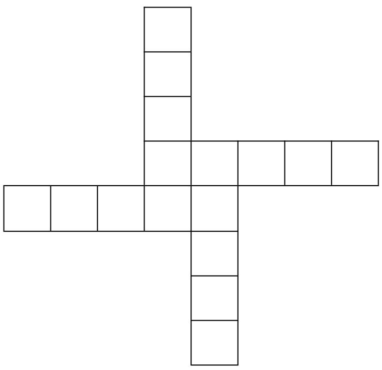


therefore, either Vertical wins or the game ends in a draw. Symmetrically, if Vertical moves in the third column we have two possibilities



therefore, either Vertical wins or the game ends in a draw. It follows $G = VD$.

In the game



each player has 4 possible moves. For every move of Vertical, Horizontal can win or draw (and sometimes lose); likewise,

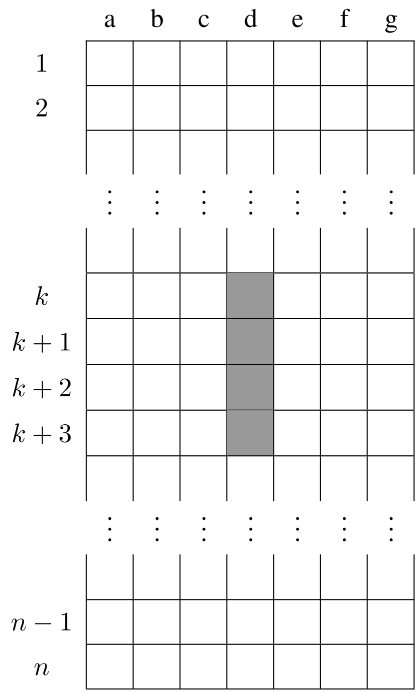


Fig. 1. Vertical strategy on the $n \times 7$ board of Synchronized Quadromineering

for every move by Horizontal, Vertical can win or draw (and sometimes lose). As a result, it follows that $G = VHD$.

IV. New Solutions

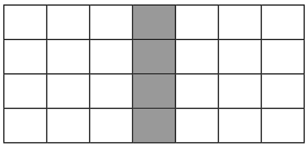
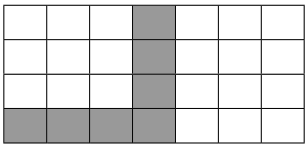
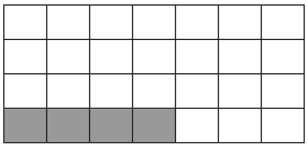
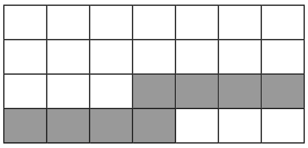
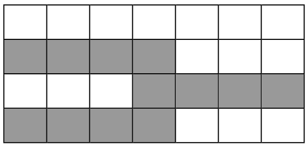
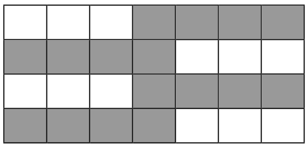
Theorem 1: Let $G = [n, 7]$ be a rectangle of Synchronized Quadromineering with $n \geq 8$. Then, Vertical has a winning strategy.

Proof: In the beginning, Vertical will always move into the central column of the board, i.e., $(k, d), (k+1, d), (k+2, d), (k+3, d)$ where $k \equiv 1 \pmod 4$, as shown in Fig. 1. When Vertical cannot move anymore into the central column, let us imagine that we divide the main rectangle into 4×7 sub-rectangles starting from the top of the board (by using horizontal cuts). Of course, if $n \not\equiv 0 \pmod 4$, then the last sub-rectangle will be of size 1×7, 2×7 or 3×7, and Horizontal will be able to make respectively one more move, two more moves or three more moves. We can classify all these sub-rectangles into 6 different classes according to:

- The number of vertical quadrominoes already placed in the sub-rectangle (vq),
- The number of horizontal quadrominoes already placed in the sub-rectangle (hq),
- The number of moves that Vertical is able to make in the worst case, in all the sub-rectangles of that class (vm),
- The number of moves that Horizontal is able to make in the best case, in all the sub-rectangles of that class (hm),

as shown in Table II. We denote with $|A|$ the number of sub-rectangles in the A class, with $|B|$ the number of sub-rectangles in the B class, and so on. The value of vm in all the sub-rectangles belonging to the class C is $|C|$. The last statement is true under the assumption that Vertical moves first into the sub-rectangles of class C as long as they exist, and finally into the sub-rectangles of the other classes.

TABLE II
THE 6 CLASSES FOR 4×7 SUB-RECTANGLES

Class	vq	hq	vm	hm	Example
A	1	0	$6\|A\|$	0	
B	1	1	$3\|B\|$	0	
C	0	1	$\|C\|$	$3\|C\|$	
D	0	2	0	$2\|D\|$	
E	0	3	0	$\|E\|$	
F	0	4	0	0	

When Vertical cannot move anymore into the central column, both Vertical and Horizontal have placed the same number of quadrominoes, therefore

$$|A| = |C| + 2|D| + 3|E| + 4|F| \quad (1)$$

Let us prove by contradiction that Vertical can make a larger number of moves than Horizontal. Assume therefore $moves(V) \leq moves(H)$ using the data in Table II

$$6|A| + 3|B| + |C| \leq 3|C| + 2|D| + |E| + 3$$

and applying Equation 1

$$3|A| + 3|B| + |C| + 4|D| + 8|E| + 12|F| \leq 3$$

which is false because

$$|A| + |B| + |C| + |D| + |E| + |F| = \lfloor n/4 \rfloor$$

and by hypothesis $n \geq 8$. Therefore, $moves(V) \leq moves(H)$ does not hold and consequently $moves(H) < moves(V)$. We observe that if $n \equiv 2 \pmod 4$, then the theorem holds for $n \geq 6$, if $n \equiv 1 \pmod 4$, then the theorem holds for $n \geq 5$, and if $n \equiv 0 \pmod 4$, then the theorem holds for $n \geq 4$. ∎

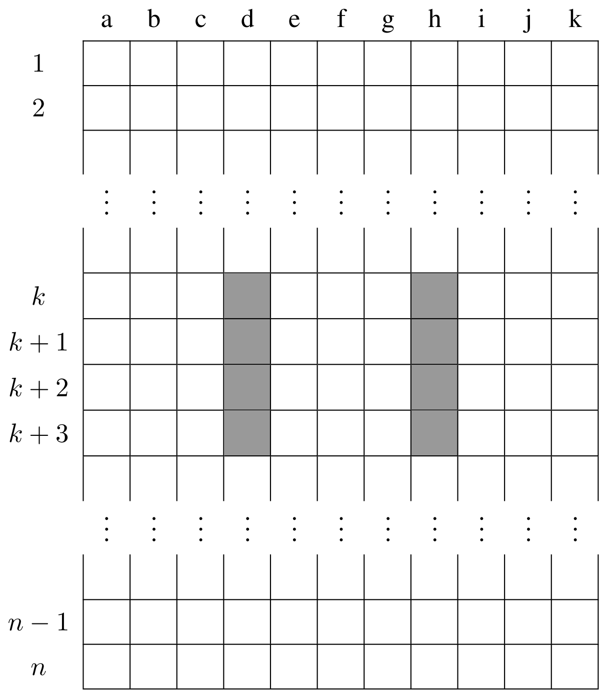


Fig. 2. Vertical strategy on the $n \times 11$ board of Synchronized Quadromineering

Theorem 2: Let $G = [n, 11]$ be a rectangle of Synchronized Quadromineering with $n \geq 31$. Then, Vertical has a winning strategy.

Proof: In the beginning, Vertical will always move into the column d and h of the board, i.e., $(k, d), (k+1, d), (k+2, d), (k+3, d), (k, h), (k+1, h), (k+2, h), (k+3, h)$ where $k \equiv 1 \pmod 4$, as shown in Fig. 2. When Vertical cannot move anymore into the column d and h, let us imagine that we divide the main rectangle into 4×11 sub-rectangles starting from the top of the board (by using horizontal cuts). Of course, if $n \not\equiv 0 \pmod 4$, then the last sub-rectangle will be of size 1×11, 2×11 or 3×11, and Horizontal will be able to make respectively two more move, four more moves or six more moves. We can classify all these sub-rectangles into 15 different classes according to:

- The number of vertical quadrominoes already placed in the sub-rectangle (vq),
- The number of horizontal quadrominoes already placed in the sub-rectangle (hq),
- The number of moves that Vertical is able to make in the worst case, in all the sub-rectangles of that class (vm),
- The number of moves that Horizontal is able to make in the best case, in all the sub-rectangles of that class (hm),

as shown in Table III. We denote with $|A|$ the number of sub-rectangles in the A class, with $|B|$ the number of sub-rectangles in the B class, and so on. The value of vm in all the sub-rectangles belonging to the class D, E, and I, considered as a group, is $4|D| + |E| + |I|$. The last statement is true under the assumption that Vertical moves first into the sub-rectangles of class D and E as long as they exist, second into the sub-rectangles of class I as long as they exist, and finally into the sub-rectangles of the other classes.

TABLE III
THE 15 CLASSES FOR 4×11 SUB-RECTANGLES

Class	vq	hq	vm	hm
A	2	0	$9\|A\|$	0
B	2	1	$6\|B\|$	0
C	2	2	$3\|C\|$	0
D	1	1	$4\|D\|$	$3\|D\|$
E	1	2	$\|E\|$	$3\|E\|$
F	1	3	0	$2\|F\|$
G	1	4	0	$\|G\|$
H	1	5	0	0
I	0	2	$\|I\|$	$6\|I\|$
J	0	3	0	$5\|J\|$
K	0	4	0	$4\|K\|$
L	0	5	0	$3\|L\|$
M	0	6	0	$2\|M\|$
N	0	7	0	$\|N\|$
O	0	8	0	0

When Vertical cannot move anymore into the column d and h, both Vertical and Horizontal have placed the same number of quadrominoes, therefore

$$\begin{aligned} 2|A| + |B| &= |E| + 2|F| + 3|G| + 4|H| + 2|I| + 3|J| + \\ &\quad 4|K| + 5|L| + 6|M| + 7|N| + 8|O| \end{aligned} \tag{2}$$

Let us prove by contradiction that Vertical can make a larger number of moves than Horizontal. Assume therefore $moves(V) \leq moves(H)$ using the data in Table III

$$\begin{aligned} &9|A| + 6|B| + 3|C| + \\ &\quad 4|D| + |E| + |I| \leq 3|D| + 3|E| + 2|F| + |G| + \\ &\quad 6|I| + 5|J| + 4|K| + 3|L| + \\ &\quad 2|M| + |N| + 6 \end{aligned}$$

and applying Equation 2

$$\begin{aligned} &3|A| + 3|B| + 3|C| + |D| + |E| + 4|F| + \\ &\quad 8|G| + 12|H| + |I| + 4|J| + 8|K| + \\ &\quad 12|L| + 16|M| + 20|N| + 24|O| \leq 6 \end{aligned}$$

which is false because

$$\begin{aligned} &|A| + |B| + |C| + |D| + |E| + |F| + |G| + \\ &|H| + |I| + |J| + |K| + |L| + |M| + |N| + |O| = \lfloor n/4 \rfloor \end{aligned}$$

and by hypothesis $n \geq 31$. Therefore, $moves(V) \leq moves(H)$ does not hold and consequently $moves(H) < moves(V)$. We observe that if $n \equiv 2 \pmod 4$, then the theorem holds for $n \geq 22$, if $n \equiv 1 \pmod 4$, then the theorem holds for $n \geq 13$, and if $n \equiv 0 \pmod 4$, then the theorem holds for $n \geq 4$. ■

By symmetry the following two theorems hold.

Theorem 3: Let $G = [7, n]$ be a rectangle of Synchronized Quadromineering with $n \geq 8$. Then, Horizontal has a winning strategy.

Theorem 4: Let $G = [11, n]$ be a rectangle of Synchronized Quadromineering with $n \geq 31$. Then, Horizontal has a winning strategy.

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