Quadratic Hedging And Two-colours Rainbow American Options

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***Abstract*—The market-maker of a standard European option hedge by a single transaction on the financial market. We consider the (optimal control) problem (P) of selecting the best hedging time, initial investment and ratio for a quadratic criterion. We transforms (P) into an optimal stopping problem, which criterion depends only of the hedging (stopping) time and the corresponding stopped stock price. This last problem, is a standard optimal stopping one, with a time dependent payoff. In order to avoid this time dependence of the payoff, we approximate the risk value (up to an affine transformation), by the value of a fictitious two-colours Rainbow American option, unique solution of a two-dimensional stationary variational inequality. This allows us to use the spitting methods (analytical version of dynamic programming) followed by [19], for pricing two-colours Rainbow American options, in order to compute the risk value. Using the technical of localization and discretization with the finite difference method, we establish an iterative algorithm converging to the risk value.**

***Index Terms*—variance optimal hedging, optimal stopping, American options, variational inequalities, pricing two-colours Rainbow American options, dynamic programming.**

I. Introduction and summary

From the practical point of view, continuous trading is of course unrealistic. For example, in presence of transaction costs, the market-maker would like to hedge as little as possible. In practice he must follow a discrete trading strategy at stopping times. In [18], the problem of discrete hedging of European options in the otherwise continuous Black-Schloes model is studied. Given a fixed number N of rebalances, the problem of selecting the best hedging times and ratios for a quadratic criterion is studied. It is shown that in the case of a single hedge this problem is transformed into an optimal stopping one, where the criterion depends only of the rebalancing time and the stock price at this time.

The goal of this paper is to compute the value function of the last problem and, therefore, to compute the risk value, when only one hedge is permitted to the market maker.

We consider the financial market composed by two assets:

• a bond whose price is given by

$$S_0(t) = e^{rt}; \tag{1}$$

• a risky asset, whose price S_t^x is a continuous version of the flow of the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t; \tag{2}$$

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with the initial condition $S_0^x = x > 0$. Here r, μ and σ are some positive constants, with μ the instantaneous return rate, r the interest rate and σ the volatility of the stock price S_t^x. The process B is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we denote by $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ the $\mathbb{P}$-completion of the natural filtration of B, where $T > 0$, is the finite time-horizon.

In this paper, we shall study the case of a single hedge of the market-maker before the expiration date T. We suppose that his trading times run across the interval $[0, T_\nu]$, where $T_\nu := T - \nu$ with ν a positive constant small enough, such that $0 < \nu < T$. The justification of this restriction will be explained later.

Set

$$A_t^x := e^{-rt} S_t^x \tag{3}$$

$$L_t(r) := \exp\left[-\frac{\mu - r}{\sigma} B_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right] \tag{4}$$

$$\beta := r - \frac{\sigma^2}{2} \tag{5}$$

The process $W_t := B_t + \left(\frac{\mu - r}{\sigma}\right) t$ is a standard Brownian motion under the probability measure $\mathbb{Q}^r$ with density $L_T(r)$ with respect to P.

The probability $\mathbb{Q}^r$ is the unique equivalent martingale measure under which the process A_t^x is the martingale $A_t^x = x \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)$. Denote by $\mathbb{E}^r$ the expectation under the probability $\mathbb{Q}^r$.

In the following, a standard European option is considered, with payoff $H_T = f(S_T^x)$, expiration date $T > 0$, and underlying asset S_t^x, where f is a real-valued function satisfying the following hypothesis:

(H^0)

f is an absolutely continuous function, then almost everywhere differentiable, such that

$$|f(y)| + |f'(y)| \leq L(1 + |y|^\alpha), \text{ for almost all } y \in \mathbb{R}.$$

L and α are some given positive constants. The option value at time t is given by

$$\mathbb{E}^r\left[e^{-r(T-t)} f(S_T^x)/\mathcal{F}_t\right] = F(t, S_t^x, r) \tag{6}$$

where

$$F(t, x, r) = \mathbb{E}^r\left[e^{-r(T-t)} f\left(e^{\sigma(W_T - W_t) + \beta(T-t)}\right)\right] \tag{7}$$

Remark 1: Calls and puts satisfy this hypothesis [20]. Recall that since f has polynomial growth, then $F(.,.,r) \in C^{1,2}([0, T[\times\mathbb{R})$ with $F(T,.,r) = f(.)$ ([20], proposition 1.4.1).

We should deal in this paper with the problem of selecting the best hedging (stopping) time, initial investment, and ratio, for

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the following variance optimal hedging (quadratic criterion):

$$\text{(P)} \qquad V(x) = \inf_{(\tau,\xi,x_0)\in\hat{\Theta}_0\times\mathbb{R}_+} \mathbb{E}^r\left[\left(e^{-rT}f(S_T^x) - x_0 - \xi(A_T^x - A_\tau^x)\right)^2\right],$$

where

$$\begin{aligned}\hat{\Theta}_0 &:= \{(\tau,\xi); \tau\in\mathcal{T}_{0,T_\nu} \text{ and } \xi\in\mathcal{V}_\tau\}\\ \mathcal{V}_\tau &:= \{\text{ random variables } \xi<\infty\ P \text{ almost surely}\\ &\quad (a.s.) \text{ and } \mathcal{F}_\tau-\text{measurable}\}\end{aligned}$$

x_0 is the initial investment, ξ is the allocation at time τ in the risky asset and $\mathcal{T}_{0,T_\nu}$ is the set of all $\{\mathcal{F}_t\}$-stopping times τ such that $0\le\tau\le T_\nu$ $a.s.$ We have the following

Proposition 1: The optimal initial investment $x_0^*(x)$ (hedging price) is the Black-Scholes price under the risk-neutral probability, given by

$$x_0^* = \mathbb{E}^r\left[\frac{f(S_T^x)}{S_0(T)}\right] = e^{-rT}\mathbb{E}^r\left[f(S_T^x)\right].$$

Proof: Fix $(\tau,\xi)\in\hat{\Theta}_0$ and let

$$J(x_0,\tau,\xi) := \mathbb{E}^r\left[\left(e^{-rT}f(S_T^x) - x_0 - \xi(A_T^x - A_\tau^x)\right)^2\right].$$

Since A_t^x is a $\mathbb{Q}^r$-martingale, then $\mathbb{E}^r\left[\xi(A_T^x - A_\tau^x)\right] = \mathbb{E}^r\left[\xi\mathbb{E}^r\left((A_T^x - A_\tau^x)/\mathcal{F}_\tau\right)\right] = 0$. It follows that:

$$\begin{aligned}J(x_0,\tau,\xi) &= x_0^2 - 2e^{-rT}\mathbb{E}^r\left[f(S_T^x)\right]x_0\\ &+ \mathbb{E}^r\left[\left(e^{-rT}f(S_T^x) - \xi(A_T^x - A_\tau^x)\right)^2\right]\end{aligned}$$

It is then clear that $x_0^*(x) := e^{-rT}\mathbb{E}^r\left[f(S_T^x)\right]$ is a global minimum of $J(x_0,\tau,\xi)$. ■

The problem (P) is equivalently rewritten as:

$$\text{(P)} \qquad V(x) = \inf_{(\tau,\xi)\in\hat{\Theta}_0} \mathbb{E}^r\left[\left(m - \xi(A_T^x - A_\tau^x)\right)^2\right],$$

where

$$m = e^{-rT}[f(S_T^x) - \mathbb{E}^r[f(S_T^x)]].$$

We will show in the following section, that the problem (P) may be transformed into an optimal stopping one, with time dependent payoff ψ. It is then natural to try to apply the approach of [1] to optimal stopping, via a one-dimensional evolution variational inequality. Since the payoff ψ does not satisfy the necessary conditions required by this approach, this last can not followed.

In order to avoid this time dependence, we shall introduce the two dimensional diffusion $(X_s^1, X_s^2) := (s, \log(S_s))$, where S follows (2), which we will approximate by $X_s^\varepsilon = (X_s^{1,\varepsilon}, X_s^{2,\varepsilon}) = (\log(1-\varepsilon)+(1-\frac{\varepsilon^2}{2})s+\varepsilon B_s^0, \log(S_s))$, where B^0 is an independent Brownian motion, so that X^ε is an elliptic diffusion. Given $\varepsilon>0$ small enough, this allows us to use the theory of optimal stopping via two dimensional stationary variational inequalities, to approximate the risk value. On makes an approximation of the risk value $V(x)$ (up to an affine transformation) by the value of a fictitious two-colours Rainbow American option, which is the unique solution of a two-dimensional stationary variational inequality. This last is approximated using the splitting methods as in the approach of [19], for pricing two-colours Rainbow American options. Note that the splitting method can be viewed as an analytical version of dynamic programming.

On the topic of pricing multi-asset American-style options and the related numerical methods, the reader can consult among others [4], [5], [13], [15] and [16].

The remainder of the paper is arranged as follows: In Section 2, it is shown that the problem (P) is equivalently rewritten as

$$\text{(P)} \qquad V(x) = v(x) - \sup_{\tau\in\mathcal{T}_{0,T_\nu}} \mathbb{E}^r\left[\psi(\tau, S_\tau^x)\right],$$

where $\psi(.,.)$ is a continuous function on $\mathbb{R}^2$ and $v(x)$ is the variance of the random variable $e^{-rT}f(S_T^x)$ under $\mathbb{Q}^r$:

$$v(x) = e^{-2rT}\left\{\mathbb{E}^r\left[f^2(S_T^x)\right] - (\mathbb{E}^r\left[f(S_T^x)\right])^2\right\}. \qquad (8)$$

The continuity of the value function $V(x)$ is shown and the optimal stopping problem (P) is solved.

In Section 3, is introduced a fictitious market composed of the assets S^1 and S^2 modelled by $(S_s^1, S_s^2) := (e^s, S_s^x)$ $a.s.$, where $S_\cdot^x$ is given in (2). We have with the notations of [10], $(X_s^1, X_s^2) := (\log S_s^1, \log S_s^2) = (s, \log(S_s^x))$ $a.s.$ and the problem (P) is equivalently rewritten as:

$$\text{(P)} \qquad V(x) = v(x) - \sup_{\tau\in\mathcal{T}_{0,T_\nu}} \mathbb{E}^r\left[\psi\left(X_\tau^1, X_\tau^2\right)\right].$$

Since, the volatility of the couple $X = (X^1, X^2)$ is not elliptic, we can not apply standard results directly (e.g. [10] and [19]). To overcome this difficulty, we disrupt the couple $X = (X^1, X^2)$ by a parameter ε to obtain a couple $X^\varepsilon = (X^{1,\varepsilon}, X^{2,\varepsilon})$ converging $a.s.$ to $X = (X^1, X^2)$ as ε tends to zero and having an elliptic volatility. Then the results of [19], can be applied for ε small enough.

More precisely, we denote by $(X^{1,\varepsilon,t,x_1}, X^{2,\varepsilon,t,x_2})$ the couple X^ε starting at the time t, with the initial condition $X_t^\varepsilon = (x_1, x_2)$, and we consider a fictitious market in which there is no interest rate, with two risky assets, with prices $e^{X^{1,\varepsilon,t,x_1}}$ and $e^{X^{2,\varepsilon,t,x_2}}$. We show that the value of the American option with payoff ψ, maturity T and vector price $(e^{X^{1,\varepsilon,t,x_1}}, e^{X^{2,\varepsilon,t,x_2}})$ at time t is equal to $u_\varepsilon^*(t, X_t^{1,\varepsilon,t,x_1}, X_t^{2,\varepsilon,t,x_2})$, where $(t,x_1,x_2) \longrightarrow u_\varepsilon^*(t,x_1,x_2)$ is a continuous map. We solve the optimal stopping problem with boundary constraints related to the valuation of such option.

It is also shown that the function u_ε^* is continuous with respect to the volatility ε of X^{1,ε,t,x_1} on $\mathbb{R}$ and in particular that:

$$V(x) = v(x) - \lim_{\varepsilon\longrightarrow 0} u_\varepsilon^*(0,0,\log(x)).$$

In Section 4, $\varepsilon>0$ is taken such that u_ε^* and u_0^* are close for practical purposes. It is shown that the function u_ε^* corresponding to the case of a call or a put, is the unique solution of some variational inequality. We state this variational inequality, and we compute $u_\varepsilon^*(t,x_1,x_2)$ by the numerical method based on dynamic programming approach as in [19], which leads to the computation of the risk value $V(x)$, since

$$V(x) = v(x) - u_\varepsilon^*(0,0,\log(x)) + \mathcal{O}(\varepsilon).$$

More precisely, we localize the variational inequality in an open bounded set, and we discretize it with a finite-difference method, to obtain a variational inequality in finite dimension, the unique solution of which is an approximation of $u_\varepsilon^*(t,x_1,x_2)$. Finally, in section 5 we give the numerical results.

II. Transformation and resolution of the problem (P)

Set $\hat{S}_t^x := \log(S_t^x)$, then the process $\hat{S}_t^x$ is a continuous version of the flow of the stochastic differential equation:

$$\text{(E)} \qquad d\hat{S}_t^x = \beta dt + \sigma dW_t,$$

with $\hat{S}_0^x = \log(x)$. It is shown in [18] that the problem (P) may be transformed into an optimal stopping one whose criterion depends only on τ and S_τ^x. For any given $\tau \in \mathcal{T}_{0,T_\nu}$, the expression of the optimal hedging ratio is given by the following

Proposition 2: **([18], Proposition 12)** Let $\tau \in \mathcal{T}_{0,T_\nu}$. The optimal ratio $\xi^*(\tau)$ solution of the problem

$$(\mathrm{P}^\tau) \qquad \inf_{\xi \in \mathcal{V}_\tau} \mathbb{E}^r\left[(m - \xi(A_T^x - A_\tau^x))^2\right],$$

is given by

$$\xi^*(\tau) = \frac{-F\left(\tau, S_\tau^x, r\right) + e^{r\tau} e^{\sigma^2(T-\tau)} F\left(\tau, S_\tau^x, r+\sigma^2\right)}{S_\tau^x\left\{e^{\sigma^2(T-\tau)} - 1\right\}}.$$

We replace $\xi^*(\tau)$ by its expression, we will get

Proposition 3: The problem of control (P) is transformed into the following optimal stopping problem, the criterion of which depends only of τ and S_τ^x:

$$\text{(P)} \qquad V(x) = v(x) - \sup_{\tau \in \mathcal{T}_{0,T_\nu}} \mathbb{E}^r\left[\varphi(\tau, S_\tau^x)\right],$$

where

$$\varphi(t,x) = \frac{\left(-e^{-rt}F\left(t,x,r\right) + e^{\sigma^2(T-t)}F\left(t,x,r+\sigma^2\right)\right)^2}{e^{\sigma^2(T-t)} - 1}.$$

Proof: See in [18], Theorem 5 and the last remark of Subsection 6.2. ■

Remark 2: Since the function φ is not defined at the point T and $\lim_{t \to T} \varphi(t,x) = \infty$, we shall restrict ourselves to the interval $[0, T-\nu]$, where ν is a positive real small enough. Since $F(.,.,r) \in C^{1,2}([0,T_\nu] \times \mathbb{R})$, the function φ is continuous on $[0,T_\nu] \times \mathbb{R}_+$. Let $\overline{F}$ be a continuous extension of F from $[0,T_\nu] \times \mathbb{R}_+$ to $\mathbb{R}^2$ such that $\forall t \le 0, \overline{F}(t,.,r) = \overline{F}(0,.,r)$ and $\forall x \le 0, \overline{F}(.,x,r) = \overline{F}(.,0,r)$. We have then for all $(t,x) \in \mathbb{R}^2 - \{[0,T_\nu] \times \mathbb{R}_+\}$:

$$\frac{\partial \overline{F}}{\partial t}(t,x,r) = \frac{\partial \overline{F}}{\partial x}(t,x,r) = 0.$$

Let us define the functions:

$$\begin{aligned} \delta(t) &= \frac{1}{e^{\sigma^2 T} - 1}\mathbf{1}_{]-\infty,0]}(t) + \frac{e^{-2rt}}{e^{\sigma^2(T-t)} - 1}\mathbf{1}_{[0,T_\nu]}(t) \\ &+ \frac{e^{-2rT_\nu}}{e^{\sigma^2\nu} - 1}\mathbf{1}_{[T_\nu,+\infty[}(t) \end{aligned} \tag{9}$$

$$\begin{aligned} \kappa(t) &= e^{\sigma^2 T}\mathbf{1}_{]-\infty,0]}(t) + e^{\sigma^2(T-t)+rt}\mathbf{1}_{[0,T_\nu]}(t) \\ &+ e^{\sigma^2\nu + rT_\nu}\mathbf{1}_{[T_\nu,+\infty[}(t). \end{aligned} \tag{10}$$

Then the function:

$$\overline{\varphi}(t,x) = \delta(t)\left(-\overline{F}\left(t,x,r\right) + \kappa(t)\overline{F}\left(t,x,r+\sigma^2\right)\right)^2,$$

is a continuous extension of φ from $[0,T_\nu] \times \mathbb{R}$ to $\mathbb{R}^2$. Set

$$\begin{aligned} \psi(t,x) &= \overline{\varphi}(t,e^x) = \delta(t)\left(-\overline{F}\left(t,e^x,r\right)\right. \\ &+ \left.\kappa(t)\overline{F}\left(t,e^x,r+\sigma^2\right)\right)^2. \end{aligned} \tag{11}$$

The problem (P) is equivalently rewritten as:

$$\text{(P)} \qquad V(x) = v(x) - \sup_{\tau \in \mathcal{T}_{0,T_\nu}} \mathbb{E}^r\left[\psi(\tau, \hat{S}_\tau^x)\right].$$

Remark 3: Notice that $V(x) < v(x)$. This means that the market-maker must hedge at least once before the option expires.

A. Continuity of the value function $V(x)$

In order to show the continuity of the value function of the problem (P), we need the following proposition:

Proposition 4: The function $\psi(.,.)$ defined in Equation (11) satisfies the following hypothesis

$$(H^1) \qquad \psi \text{ is continuous, } \psi(t,x) \le Me^{M|x|},$$

for some constant $M > 0$ uniformly in $t \in \mathbb{R}$

Proof: We have the following lemma

Lemma 1: We have:

$$\forall r > 0, \forall (t,x) \in \mathbb{R}^2, |\overline{F}(t,e^x,r)| \le L_0(1 + e^{\alpha|x|}),$$

where L_0 is a constant which depends only on α, r, T and L.

Proof: It is enough to show the inequality on $[0,T_\nu] \times \mathbb{R}$. For all $r > 0$ and $(t,x) \in [0,T_\nu] \times \mathbb{R}$, we have by hypothesis (H^0),

$$|\overline{F}(t,e^x,r)| \le L\left(1 + \mathbb{E}^r[(S_{T-t}^{e^x})^\alpha]\right),$$

then we obtain:

$$|\overline{F}(t,e^x,r)| \le L\left(1 + e^{r\alpha T}e^{\alpha x}\mathbb{E}^r\left[\sup_{0 \le z \le T} \zeta_{z,\sigma}^\alpha\right]\right), \tag{12}$$

where $\zeta_{.,\sigma}$ is the $\mathbb{Q}^r$-exponential martingale given by

$$\zeta_{z,\sigma} := \exp\left\{-\sigma B_z - \frac{\sigma^2}{2}z\right\}. \tag{13}$$

■

Since $\delta(t)$ and $\kappa(t)$ given in Equations (9) and (10) are bounded on $\mathbb{R}$ and $\mathbb{E}^r\left[\sup_{0 \le z \le T} \zeta_{z,\sigma}^\alpha\right] < \infty$ by Doob's inequality in $L^p, p > 1$, it follows that

$$0 \le \psi(t,x) \le L_1(1 + e^{2\alpha|x|}) \le 2L_1 e^{2\alpha|x|} \le Me^{M|x|},$$

where L_1 is a positive constant which depends of α, r, T, ν, σ and L and $M = 2(L_1 + \alpha)$. ■

In light of this result, we have:

Proposition 5: **([10], Proposition 2.2)**
The value function $V(x)$ of the optimal stopping problem (P) is continuous and

$$V(\hat{S}_t^x) = v(x) - \mathrm{esssup}_{\tau \in \mathcal{T}_{0,T_\nu}} \mathbb{E}^r\left[\psi\left(\tau, \hat{S}_\tau^x\right)/\mathcal{F}_t\right].$$

Remark 4: Note that we have $v(x) - V(x) \ge \psi(t,x), \forall (t,x) \in [0,T_\nu] \times \mathbb{R}_+^*$. Then, $v(x) - V(\hat{S}_t^x)$ is the smallest supermartingale above the process $\psi(t, \hat{S}_t^x)$ at each time t; it is therefore the Snell envelope of the process $\psi(t, \hat{S}_t^x)$.

B. Resolution of the problem (P)

Proposition 6: The stopping time:

$$\tau^x := \inf\left\{0 \le s \le T_\nu; v(x) - V(\hat{S}^x_s) = \psi(t, \hat{S}^x_s)\right\},$$

is a solution of the problem (P).

Proof: The process $v(x) - V(\hat{S}^x_t)$ is the Snell envelope of the process

$$\left\{\psi(t, \hat{S}^x_t),\ \mathcal{F}_t, 0 \le t \le T_\nu\right\}.$$

The latter which is an adapted continuous process, is uniformly integrable; indeed, by the equality

$$\hat{S}^x_t = \log(x) + \int_0^t \beta dv + \int_0^t \sigma dW_v,$$

it follows that

$$\mathbb{E}^r\left(\sup_{0\le s\le T_\nu} e^{M\left|\hat{S}^x_t\right|}\right) \le Ce^{M|\log(x)|}. \tag{14}$$

where C depends only on T_ν, M, r and σ. The inequality (14) and the hypothesis (H^1) imply

$$\mathbb{E}^r\left[\sup_{0\le s\le T_\nu} \psi(t, \hat{S}^x_t)\right] \le MCe^{M|\log(x)|}.$$

This shows the uniform integrability of the process $\psi(t, \hat{S}^x_t)$. Consequently, the stopping time τ^x is a solution of the problem (P). ■

III. Disrupted problem

A. Definition of the disrupted problem

Let a fictitious market composed of two assets, a riskless one of process price $S^1_t = e^t$ and a risky one of process price $S^2_t = S_t$ which follows the stochastic differential equation (2). Set

$$\begin{aligned} X^1_s &:= \log(S^1_s) = s, \\ X^2_s &:= \log(S^2_s) = \log(S_s), \qquad a.s. \end{aligned}$$

Then we have:

$$\text{(E)} \qquad \begin{cases} dX^1_s = ds \\ dX^2_s = \beta ds + \sigma dW_s \qquad a.s. \end{cases}$$

Set:

$$\begin{aligned} X &= {}^t(X^1, X^2) \\ \Delta &= \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \beta \end{pmatrix} \\ \Gamma &= \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix} \end{aligned} \tag{15}$$

Equation (E) becomes:

$$\text{(E)} \qquad dX_t = \Delta(X_t)dt + \Gamma(X_t)dM_t,$$

where $M = \begin{pmatrix} B^0 \\ W \end{pmatrix}$, with B^0 a standard Brownian motion independent of W.

Let $y = {}^t(x_1, x_2)$ and let $X^{t,y} = {}^t(X^{1,t,x_1}, X^{2,t,x_2})$ be a continuous version of the flow of the stochastic differential equation

$$\text{(E)} \qquad dX^{t,y}_s = \Delta(X^{t,y}_s)ds + \Gamma(X^{t,y}_s)dM_s,$$

such that $X^{t,y}_t = {}^t(x_1, x_2)$. The problem (P) is equivalently rewritten as:

$$\text{(P)} \qquad V(x) = v(x) - \sup_{\tau\in\mathcal{T}_{0,T_\nu}} \mathbb{E}^r\left[\psi(X^{0,0,\log(x)}_\tau)\right].$$

Since the operator $\frac{1}{2}\sigma\,{}^t\sigma$ is not elliptic, we can not use classical methods in order to compute the value $\sup_{\tau\in\mathcal{T}_{0,T_\nu}} \mathbb{E}^r\left[\psi(X^{0,0,\log(x)}_\tau)\right] = v(x) - V(x)$ of the American option with payoff ψ, maturity T_ν and underlying process $\{X^{0,0,\log(x)}_t, 0 \le t \le T_\nu\}$ (e.g. [10] or [19]). To overcome this difficulty, we disrupt the modelling of S^1 and S^2 by a parameter $\varepsilon \in \mathbb{R}$, to get the following fictitious assets $S^{1,\varepsilon}$ and $S^{2,\varepsilon}$ such that the dynamic of the couple $(S^{1,\varepsilon}, S^{2,\varepsilon})$ is a 2-dimensional Black-Scholes model. For all $s \ge t$:

$$\begin{aligned} S^{1,\varepsilon}_s &:= (1-\varepsilon)e^{\alpha_\varepsilon s + \varepsilon B^0_s} \qquad a.s. \\ S^{2,\varepsilon}_s &:= S^2_s = S_s \qquad a.s. \end{aligned}$$

where

$$\alpha_\varepsilon = 1 - \frac{\varepsilon^2}{2}. \tag{16}$$

Set:

$$\begin{aligned} X^{1,\varepsilon}_s &:= \log(S^{1,\varepsilon}_s) = log(1-\varepsilon) + \alpha_\varepsilon s + \varepsilon B^0_s,\ a.s. \\ X^{2,\varepsilon}_s &:= \log(S^2_s) = \log(S_s)\ a.s. \\ X^\varepsilon &:= {}^t(X^{1,\varepsilon}, X^{2,\varepsilon}) \\ \Gamma_\varepsilon &:= \begin{pmatrix} \varepsilon & 0 \\ 0 & \sigma \end{pmatrix} \text{ and } \qquad \Delta_\varepsilon = \begin{pmatrix} \alpha_\varepsilon \\ \beta \end{pmatrix}. \end{aligned}$$

We have then:

$$dX^\varepsilon_s = \Delta_\varepsilon(X^\varepsilon_s)ds + \Gamma_\varepsilon(X^\varepsilon_s)dM_s.$$

Fix $y = {}^t(x_1, x_2)$ and let $X^{\varepsilon,t,y} = {}^t(X^{1,\varepsilon,t,x_1}, X^{2,\varepsilon,t,x_2})$ be a continuous version of the flow of the stochastic differential equation

$$\text{(E}_\varepsilon\text{)} \qquad dX^{\varepsilon,t,x}_s = \Delta_\varepsilon(X^{\varepsilon,t,x}_s)ds + \Gamma_\varepsilon(X^{\varepsilon,t,x}_s)dM_s,$$

such that $X^{\varepsilon,t,y}_t = {}^t(x_1, x_2)$. The dynamics of the components of the couple $X^{\varepsilon,t,y}$ are given by:

$$\begin{aligned} X^{1,\varepsilon,t,x_1}_s &= \alpha_\varepsilon(s-t) + \varepsilon(B^0_s - B^0_t) + x_1 \\ X^{2,\varepsilon,t,x_2}_s &= \log(S^x_s) - \log(S^x_t) + x_2 \\ &= \log(S^x_s) - \log(x) + x_2. \end{aligned}$$

We have :

$$\lim_{\varepsilon\longrightarrow 0}(X^{1,\varepsilon,0,0}_s, X^{2,\varepsilon,0,\log(x)}_s) = (s, \log(S^x_s)).$$

Set:

$$a^\varepsilon = \frac{1}{2}\Gamma_\varepsilon\,{}^t\Gamma_\varepsilon = \begin{pmatrix} \frac{1}{2}\varepsilon^2 & 0 \\ 0 & \frac{1}{2}\sigma^2 \end{pmatrix}. \tag{17}$$

The operator a^ε is elliptic and all conditions $(H_1), (H_2), (H_3)$ and (H_4) of [10] are now satisfied. Let us then consider the disrupted problem. Let $\varepsilon \in \mathbb{R}$ and define the function u^*_ε on $[0, T_\nu] \times \mathbb{R}^2$, as the value function of the following optimal stopping problem:

$$\text{(P}^\varepsilon\text{)} \quad u^*_\varepsilon(t, x_1, x_2) = \sup_{\tau\in\mathcal{T}_{t,T_\nu}} \mathbb{E}^r\left[\psi\left(X^{1,\varepsilon,t,x_1}_\tau, X^{2,\varepsilon,t,x_2}_\tau\right)\right],$$

where $\mathcal{T}_{t,T_\nu}$ is the set of all stopping times with values in $[t, T_\nu]$ almost surely.

Remark 5: Note that $u^*_\varepsilon(t, X^{1,\varepsilon,t,x_1}_t, X^{2,\varepsilon,t,x_2}_t)$ is the value at time t of the American option with payoff ψ and maturity T_ν, in the market with underlying $(X^{1,\varepsilon,t,x_1}_s, X^{2,\varepsilon,t,x_2}_s)$ and null interest rate.

B. Continuity of the value function of the disrupted problem

We have shown in Proposition 4, that the function $\psi(.,.)$ defined in Equation (11) satisfies the hypothesis (H^1). In light of this result, we get:

Proposition 7: **([10], Proposition 2.2)**
The value function of the optimal stopping problem (P^ε) is continuous and

$$u_\varepsilon^*(t, X_t^{1,\varepsilon,t,x_1}, X_t^{1,\varepsilon,t,x_2}) = \mathrm{esssup}_{\tau\in\mathcal{T}_{t,T_\nu}} \mathbb{E}^r\left[\psi\left(X_\tau^{1,\varepsilon,t,x_1}, X_\tau^{2,\varepsilon,t,x_2}\right)/\mathcal{F}_t\right].$$

Remark 6: Note that we have $u_\varepsilon^*(t,x_1,x_2) \geq \psi(x_1,x_2), \forall t\in[0,T_\nu]$ and $\forall x=(x_1,x_2)\in\mathbb{R}^2$. Moreover, for $t=T_\nu$, we have $u_\varepsilon^*(T_\nu,x_1,x_2)=\psi(x_1,x_2)$. On the other hand, $u_\varepsilon^*(s, X_s^{1,\varepsilon,t,x_1}, X_s^{2,\varepsilon,t,x_2})$ is the smallest supermartingale above the process $\psi(X_s^{1,\varepsilon,t,x_1}, X_s^{2,\varepsilon,t,x_2})$ at each time s; it is therefore the Snell envelope of the process $\psi(X_s^{1,\varepsilon,t,x_1}, X_s^{2,\varepsilon,t,x_2})$.

C. Resolution of the disrupted problem

Proposition 8: The stopping time:

$$\begin{aligned}\tau^{\varepsilon,x} &:= \inf\left\{t\leq s\leq T_\nu; u_\varepsilon^*(s, X_s^{1,\varepsilon,t,x_1}, X_s^{2,\varepsilon,t,x_2})\right.\\ &= \left.\psi(X_s^{1,\varepsilon,t,x_1}, X_s^{1,\varepsilon,t,x_2})\right\},\end{aligned}$$

is a solution of the problem (P^ε).

Proof: The process $u_\varepsilon^*(s, X_s^{1,\varepsilon,t,x_1}, X_s^{1,\varepsilon,t,x_2})$ is the Snell envelope of the process

$$\psi(X_s^{\varepsilon,t,y}) = \left\{\psi(X_s^{1,\varepsilon,t,x_1}, X_s^{2,\varepsilon,t,x_2}), t\leq s\leq T_\nu\right\}.$$

This last which is an adapted continuous process, is uniformly integrable; indeed, by the equality

$$X_s^{\varepsilon,t,y} = y + \int_t^s \Delta_\varepsilon(X_v^{\varepsilon,t,y})dv + \int_t^s \Gamma_\varepsilon(X_v^{\varepsilon,t,y})dM_v,$$

it follows that

$$\mathbb{E}^r\left(\sup_{t\leq s\leq T_\nu} e^{M(\left|X_s^{1,\varepsilon,t,x_1}\right|+\left|X_s^{2,\varepsilon,t,x_2}\right|)}\right) \leq Ce^{M(|x_1|+|x_2|)}, \tag{18}$$

where C depends only of T_ν, M, r, σ and ε. The inequality (18) and the hypothesis (H^1) imply

$$\mathbb{E}^r\left[\sup_{t\leq s\leq T_\nu}\psi(X_s^{\varepsilon,t,y})\right] \leq Ce^{M(|x_1|+|x_2|)}$$

This shows the uniform integrability of the process $\psi(X_\cdot^{\varepsilon,t,y})$. Consequently, the stopping time $\tau^{\varepsilon,x}$ is a solution of the disrupted problem (P^ε). ■

D. Continuity with respect to the volatility ε

Now, we shall focus our attention on the dependence of u_ε^* on ε, and we shall show that the function u_ε^* is continuous with respect to ε. Since the function ψ verifies hypothesis (H^1) as shows proposition 4, we will get the continuity of the function u with respect to ε on $\mathbb{R}$, as shows the following

Proposition 9: The function u_ε^* is continuous with respect to ε.

Proof: We have for all $\varepsilon_1,\varepsilon_2\in\mathbb{R}$:

$$\begin{aligned}&\left|u_{\varepsilon_1}^*(t,x_1,x_2) - u_{\varepsilon_2}^*\right|(t,x_1,x_2) \leq\\ &\mathbb{E}^r\left(\sup_{t\leq s\leq T_\nu}\left|\psi\left(X_s^{1,\varepsilon_1,t,x_1}, X_s^{2,\varepsilon_1,t,x_2}\right)\right.\right.\\ -\quad &\left.\left.\psi\left(X_s^{1,\varepsilon_2,t,x_1}, X_s^{2,\varepsilon_2,t,x_2}\right)\right|\right).\end{aligned}$$

Taking account of relation (18) and hypothesis (H^1), we get the following uniform integrability property:

$$\begin{aligned}&\mathbb{E}^r\left(\sup_{t\leq s\leq T_\nu}\left|\psi\left(X_s^{1,\varepsilon_1,t,x_1}, X_s^{2,\varepsilon_1,t,x_2}\right)\right.\right.\\ -\quad &\left.\left.\psi\left(X_s^{1,\varepsilon_2,t,x_1}, X_s^{2,\varepsilon_2,t,x_2}\right)\right|\right) \leq 2Ce^{M(|x_1|+|x_2|)}.\end{aligned}$$

By this last property, as well as the continuity of ψ and of the flow of (E_ε), the dominated convergence theorem provides the continuity of u_ε^* with respect to ε. ■

As a corollary, we have the following:

Corollary 1: The function $u_\varepsilon^*(t,x_1,x_2)$ converges as ε goes to zero, to the function $u_0^*(t,x_1,x_2)$. In particular, for $x_1=t=0$, $x_2=\log(x)$ we obtain :

$$\begin{aligned}V(x) &= v(x) - u_0^*(0,0,\log(x))\\ &= v(x) - \sup_{\tau\in\mathcal{T}_{0,T_\nu}}\mathbb{E}^r\left[\psi\left(\tau,\log(S_\tau^x)\right)\right]\\ &= v(x) - \lim_{\varepsilon\longrightarrow 0}\sup_{\tau\in\mathcal{T}_{0,T_\nu}}\\ &\qquad \mathbb{E}^r\left[\psi\left(\log(1-\varepsilon)+\alpha_\varepsilon\tau+\varepsilon B_\tau^0, \log(S_\tau^x)\right)\right].\end{aligned}$$

Now we are interested by the rate of convergence to zero of $|u_\varepsilon^*(t,x_1,x_2) - u_0^*(t,x_1,x_2)|$ as ε goes to zero. This is the object of the following

Proposition 10: Suppose that the function ψ satisfies a Lipschitz condition on $\mathbb{R}^2$, i.e. fulfills the hypothesis:

$$(H^3)\qquad \exists K>0/\forall(x_1,x_2),(x_1',x_2')\in\mathbb{R}^2,$$
$$|\psi(x_1',x_2') - \psi(x_1,x_2)| \leq K\left(|x_1'-x_1| + |x_2'-x_2|\right).$$

Then u_ε^* satisfies the following Lipschitz condition, with respect to ε on $\mathbb{R}$ uniformly in t and (x_1,x_2):

$$\begin{aligned}&\forall\varepsilon_1,\varepsilon_2\in\mathbb{R}, \forall(t,x_1,x_2)\in\mathbb{R}^3,\\ &\left|u_{\varepsilon_1}^*(t,x_1,x_2) - u_{\varepsilon_2}^*(t,x_1,x_2)\right| \leq 2KT_\nu^{\frac{1}{2}}\left|\varepsilon_1-\varepsilon_2\right|.\end{aligned}$$

In particular, it is continuous with respect to ε on $\mathbb{R}$.

Proof: Let $\varepsilon_1,\varepsilon_2\in\mathbb{R}, t\in\mathbb{R}_+$ and $(x_1,x_2)\in\mathbb{R}^2$. We have:

$$\begin{aligned}&\left|u_{\varepsilon_1}^*(t,x_1,x_2) - u_{\varepsilon_2}^*(t,x_1,x_2)\right|\\ \leq\quad &\sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r[\left|\psi\left(X_\tau^{1,\varepsilon_1,t,x_1}, X_\tau^{2,\varepsilon_1,t,x_2}\right)\right.\\ -\quad &\left.\psi\left(X_\tau^{1,\varepsilon_2,t,x_1}, X_\tau^{2,\varepsilon_2,t,x_2}\right)\right|]\\ \leq\quad &K\sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r\left[\left|X_\tau^{1,\varepsilon_2,t,x_1} - X_\tau^{1,\varepsilon_1,t,x_1}\right|\right.\\ +\quad &\left.\left|X_\tau^{2,\varepsilon_2,t,x_2} - X_\tau^{2,\varepsilon_1,t,x_2}\right|\right]\\ \leq\quad &K\sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r\left[|\varepsilon_2-\varepsilon_1|\left|B_\tau^0 - B_t^0\right|\right].\end{aligned}$$

Since for all $\tau\in\mathcal{T}_{t,T_\nu}$, τ is bounded, and $\{(B_t^0)^2-t, t\geq 0\}$ is a martingale, it comes by the optional sampling theorem, that $\mathbb{E}^r[(B_\tau^0)^2] = \mathbb{E}^r[\tau] < \infty$. Therefore for all $\tau\in\mathcal{T}_{t,T_\nu}$, $\mathbb{E}^r\left[\left|B_\tau^0\right|\right] \leq \sqrt{\mathbb{E}^r\left[(B_\tau^0)^2\right]} = \sqrt{\mathbb{E}^r(\tau)} \leq T_\nu^{\frac{1}{2}}$. Finally, we have:

$$\begin{aligned}&\forall\varepsilon_1,\varepsilon_2\in\mathbb{R}, \forall(x_1,x_2)\in\mathbb{R}^2,\\ &\left|u_{\varepsilon_1}^*(t,x_1,x_2) - u_{\varepsilon_2}^*(t,x_1,x_2)\right| \leq 2KT_\nu^{\frac{1}{2}}\left|\varepsilon_2-\varepsilon_1\right|.\end{aligned}$$

■

In fact, the function ψ does not in general verify a Lipchitz condition on $\mathbb{R}^2$. In the following proposition we are going

to show that the function ψ, for the case of a call or a put, satisfies the following hypothesis
(H^4)
There exists some constants γ and $\eta \in \mathbb{R}_+^*$ such that

$$\begin{aligned}&\forall (t,x),(t',x') \in \mathbb{R}^2,\\ &e^{-\eta(|x|+|x'|)}|\psi(t,x)-\psi(t',x')| \leq \gamma\left[|t-t'|+|x-x'|\right].\end{aligned}$$

We are going to show that, under the hypothesis (H^4), the function $u_\varepsilon(t,x_1,x_2)$ satisfies a Lipchitz condition with respect to ε on $\mathbb{R}$, uniformly in t and x_1.

Proposition 11: The function $\psi(.,.)$ given in Equation (11) corresponding to the case of a call $f(x) = (x-K)_+, K>0$ or a put $f(x) = (K-x)_+, K>0$, satisfies the hypothesis (H^4).

Proof: We treat only the case of Calls, the case of Puts is done in the same way. We have

$$\begin{aligned}&\forall \quad (t,x) \in [0,T_\nu]\times\mathbb{R}:\\ \overline{F}(t,e^x,r) &= e^x N(d_1(t,e^x,r)) - Ke^{-r(T-t)}\\ &\times \quad N(d_1(t,e^x,r)-\sigma\sqrt{T-t}),\end{aligned}$$

with $N(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{\frac{-z^2}{2}}dz$ and $d_1(t,e^x,r) := \frac{x-\log K+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$. On the other hand, since for all $\forall (t,x) \in [t,T_\nu]\times\mathbb{R}$, $\frac{\partial F}{\partial x}(t,e^x,r) = N(d_1(t,e^x,r))$, we get:

$$\forall (t,x) \in \mathbb{R}\times\mathbb{R}, \left|\frac{\partial \overline{F}}{\partial x}(t,e^x,r)\right| \leq 1. \tag{19}$$

Now, we have:

$$\forall (t,x) \in \mathbb{R}^2,\ |\overline{F}(t,e^x,r)| \leq e^x. \tag{20}$$

For all $(t,x) \in \mathbb{R}^2$, set:

$$q(t,x) := -\overline{F}\left(t,e^x,r\right) + \kappa(t)\overline{F}\left(t,e^x,r+\sigma^2\right).$$

Since $\forall t \in \mathbb{R}, 0 \leq \kappa(t) \leq e^{\sigma^2 T + rT_\nu}$, we obtain by (20):

$$\forall (t,x) \in \mathbb{R}^2,\ |q(t,x)| \leq (1+e^{\sigma^2 T+rT_\nu})e^x. \tag{21}$$

Next, we have:

$$\begin{aligned}&\psi(t,x)-\psi(t',x') = \delta(t)(q(t,x)+q(t',x')) \quad (22)\\ &[(q(t,x)-q(t',x))+(q(t',x)-q(t',x'))]\\ &+q^2(t',x')(\delta(t)-\delta(t')).\end{aligned}$$

Taking account of (19), we obtain:

$$\forall (t',x) \in \mathbb{R}\times\mathbb{R}, \left|\frac{\partial q}{\partial x}(t',x)\right| \leq \left(1+e^{\sigma^2 T+rT_\nu}\right)e^x. \tag{23}$$

Using the mean value theorem it follows that

$$\begin{aligned}&\forall \quad (t',x,x') \in \mathbb{R}^3, \qquad (24)\\ &|q(t',x')-q(t',x)| \leq (1+e^{\sigma^2 T+rT_\nu})|x-x'|e^{|x|+|x'|}.\end{aligned}$$

On the other hand, we have for all $(t,x) \in [0,T_\nu]\times\mathbb{R}$

$$\begin{aligned}\frac{\partial q}{\partial t}(t,x) &= -\frac{\partial \overline{F}}{\partial t}(t,e^x,r) + e^{\sigma^2(T-t)}\frac{\partial \overline{F}}{\partial t}(t,e^x,r+\sigma^2)\\ &- \sigma^2 e^{\sigma^2(T-t)}\overline{F}(t,e^x,r).\end{aligned}$$

After differentiation of the function $\overline{F}(t,e^x,r)$ with respect to t and a suitable majoration, it is easy to show that $\left|\frac{\partial \overline{F}}{\partial t}(t,e^x,r)\right| \leq C_0 e^{2x}$. This with relation (20), leads to:

$$\forall (t,x) \in [0,T_\nu]\times\mathbb{R}, \quad \left|\frac{\partial q}{\partial t}(t,x)\right| \leq \gamma_0 e^{2|x|}, \tag{25}$$

with γ_0 a positive constant. It follows, again by application of the mean value theorem, the following result:

$$\forall (t,t',x) \in \mathbb{R}^3, \quad |q(t,x)-q(t',x)| \leq \gamma_0 e^{2|x|}|t-t'|. \tag{26}$$

Finally, set $\gamma_1 := \sup_{0\leq s\leq T-\nu}|\delta'(s)| < \infty$, we have:

$$\forall (t,t') \in \mathbb{R}^2, |\delta(t)-\delta(t')| \leq \gamma_1|t-t'|. \tag{27}$$

Taking account of (21), (22), (24), (26) and (27) it comes that:

$$|\psi(t,x)-\psi(t',x')| \quad \leq \quad \gamma e^{3(|x|+|x'|)}\left[|t-t'|+|x-x'|\right],$$

where γ is a positive constant. ■

Thanks to hypothesis (H^4), which is realized by the function ψ, we shall now show that for a put or call u_ε^* satisfies a Lipchitz condition with respect to ε.

Proposition 12: The function u_ε^* corresponding to the case of a call or a put, satisfies the following Lipschitz condition with respect to ε on $\mathbb{R}$ uniformly in t and x_1:

$$\forall \varepsilon_1,\varepsilon_2 \in \mathbb{R}, \quad |u_{\varepsilon_1}^*(t,x_1,x_2)-u_{\varepsilon_2}^*(t,x_1,x_2)| \leq C(x_2)|\varepsilon_1-\varepsilon_2|$$

where $C(x_2)$ is a constant that depends only of x_2,r,T and σ. In particular, it is continuous with respect to ε on $\mathbb{R}$.

Proof: Let $\varepsilon_1,\varepsilon_2 \in \mathbb{R}, t \in [0,T_\nu]$ and $(x_1,x_2) \in \mathbb{R}^2$, We have taking account of hypothesis (H^4):

$$\begin{aligned}&\left|u_{\varepsilon_1}^*(t,x_1,x_2)-u_{\varepsilon_2}^*(t,x_1,x_2)\right|\\ \leq\ &\sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r[\left|\psi\left(X_\tau^{1,\varepsilon_1,t,x_1},X_\tau^{2,\varepsilon_1,t,x_2}\right)\right.\\ -\ &\left.\psi\left(X_\tau^{1,\varepsilon_2,t,x_1},X_\tau^{2,\varepsilon_2,t,x_2}\right)\right|]\\ \leq\ &\gamma\sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r\left[e^{3(|X_\tau^{2,\varepsilon_1,t,x_2}|+|X_\tau^{2,\varepsilon_2,t,x_2}|)}\right.\\ \times\ &\left.\left\{|X_\tau^{1,\varepsilon_1,t,x_1}-X_\tau^{1,\varepsilon_2,t,x_1}|+|X_\tau^{2,\varepsilon_1,t,x_2}-X_\tau^{2,\varepsilon_2,t,x_2}|\right\}\right]\\ \leq\ &\gamma\sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r\left[e^{6|\log(S_{\tau-t}^x)-\log(x)+x_2|}\left|\varepsilon_2-\varepsilon_1\right|\left|B_\tau^0-B_t^0\right|\right]\\ \leq\ &\gamma|\varepsilon_2-\varepsilon_1|\sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r\left[e^{6|\sigma W_{\tau-t}+(\beta)(\tau-t)+x_2|}|B_\tau^0-B_t^0|\right]\\ \leq\ &\gamma|\varepsilon_2-\varepsilon_1|A_0A_1,\end{aligned}$$

with

$$\begin{aligned}A_0 &:= \left(\sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r\left[(B_\tau^0-B_t^0)^2\right]\right)^{\frac{1}{2}},\\ A_1 &= \left(\sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r\left[e^{12|\sigma W_{\tau-t}+(\beta)(\tau-t)+x_2|}\right]\right)^{\frac{1}{2}}.\end{aligned}$$

We have

$$\begin{aligned}A_0 &= \left(\sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r\left[(B_{\tau-t}^0)^2\right]\right)^{\frac{1}{2}}\\ &= \left(\sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r\left[\tau-t\right]\right)^{\frac{1}{2}} \leq \sqrt{T_\nu}.\end{aligned}$$

On the other hand, we have :

$$\begin{aligned} A_1 &\leq e^{6(|x_2|+rT)}\left(\sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r\left[e^{12(\sigma W_{\tau-t}-\frac{\sigma^2}{2}(\tau-t))}\right.\right.\\ &+ \left.\left. e^{12(-\sigma W_{\tau-t}+\frac{\sigma^2}{2}(\tau-t))}\right]\right)^{\frac{1}{2}}\\ &\leq e^{6(|x_2|+rT)}\left(\sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r\left[\zeta_{\tau-t,12\sigma}e^{66(\tau-t)\sigma^2}\right.\right.\\ &+ \left.\left.\zeta_{\tau-t,-12\sigma}e^{78(\tau-t)\sigma^2}\right]\right)^{\frac{1}{2}}\\ &\leq e^{6(|x_2|+rT+39\sigma^2T)}\left(\sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r\left[\zeta_{\tau-t,12\sigma}\right.\right.\\ &+ \left.\left.\zeta_{\tau-t,-12\sigma}\right]\right)^{\frac{1}{2}}, \end{aligned}$$

where $\zeta_{.,\sigma}$ is the $\mathbb{Q}^r$ exponential martingale given by (13). Since we have $\mathbb{E}^r\left[\zeta_{\tau-t,12\sigma}\right] = \mathbb{E}^r\left[\zeta_{\tau-t,-12\sigma}\right] = 1$, we obtain:

$$A_1 \leq \sqrt{2}e^{6(|x_2|+rT+39\sigma^2T)}.$$

Consequently, we obtain:

$$\begin{aligned} &|u^*_{\varepsilon_1}(,t,x_1,x_2) - u^*_{\varepsilon_2}(,t,x_1,x_2)| \leq\\ &\gamma\sqrt{2T}e^{6(|x_2|+rT+39\sigma^2T)}|\varepsilon_2-\varepsilon_1|. \end{aligned}$$

■

Remark 7: We have:

$$u^*_\varepsilon(t,x_1,x_2) = u^*_0(t,x_1,x_2) + \mathcal{O}(\varepsilon),$$

this means that $u^*_\varepsilon(t,x_1,x_2)$ converges to $u^*_0(t,x_1,x_2)$ with a rate of order $\mathcal{O}(\varepsilon)$.
In the following, we shall establish an algorithm of computation of the risk value $V(x)$, value function of the optimal stopping problem (P).

IV. Variational inequality and Numerical Scheme

A. Variational inequality

In this section, we shall state the method of computing the tow-colours Rainbow American options, in the two-dimensional Black-Sholes model. After the logarithm change of variables, the price at time t of the American option on two stocks is given by:

$$(\mathbf{P}^\varepsilon)\quad u^*_\varepsilon(t,x_1,x_2) = \sup_{\tau\in\mathcal{T}_{t,T_\nu}}\mathbb{E}^r\left[\psi\left(X^{1,\varepsilon,t,x_1}_\tau, X^{2,\varepsilon,t,x_2}_\tau\right)\right].$$

The price at time 0 of such option $u^*_\varepsilon(0,x_1,x_2)$ can be formulated in terms of the solution $u(t,x_1,x_2)$ to the following variational inequality (see e.g. [10] or [19]):
(I^ε)

$$\begin{cases} \max(\psi-u, \frac{\partial u}{\partial t} + A^\varepsilon u) &= 0, \text{ on } [0,T_\nu[\times\mathbb{R}^2\\ u(T_\nu,x_1,x_2) = \psi(x_1,x_2) \end{cases}$$

where A^ε is the following differential operator:

$$A^\varepsilon f = \frac{\varepsilon^2}{2}\frac{\partial^2 f}{\partial x_1^2} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x_2^2} + \alpha_\varepsilon\frac{\partial f}{\partial x_1} + \beta\frac{\partial f}{\partial x_2}, \qquad (28)$$

where β and α_ε are given respectively by Equations (5) and (16).

B. Localization and discretization of the variational inequality (I^ε)

In this section, we deal with numerical computation of the solution u^*_ε of the variational inequality (I^ε), where ε is chosen such that u^*_ε and u^*_0 are close enough, for numerical purposes. First, we localize the problem by restriction of the domain space. We should then solve the problem in the finite interval

$$B_\rho :=]-\rho,\rho[^2.$$

We have the following

In order to solve numerically the V.I. (I^ε), we introduce the grid of mesh points (nk, ih, jh) where the parameters h and k are respectively the space and the time discretization step, which will go to zero. Let m the great integer such that $m+\frac{1}{2}\leq\rho$. Each point $M_{ij} = (ih, jh)$, is associated to the square of center M_{ij}

$$\begin{aligned} C^{(h)}_{ij} &:= \left]\left(i-\frac{1}{2}\right)h, \left(i+\frac{1}{2}\right)h\right[\\ &\times \left]\left(j-\frac{1}{2}\right)h, \left(j+\frac{1}{2}\right)h\right[. \end{aligned}$$

We set

$$\Omega_h := \left\{M_{ij}; C^{(h)}_{ij}\subset\Omega_\rho\right\} = \{M_{ij}; -m\leq i,j\leq m\},$$

and $E_h := Vect\left(\mathbf{1}_{C^{(h)}_{ij}}, -m\leq i,j\leq m\right)$ where $\mathbf{1}_{C^{(h)}_{ij}}$ is the indicatrice of the set $C^{(h)}_{ij}$. Each element u_h of E_h should have the following form:

$$u_h(x_1,x_2) = \sum_{i,j=-m}^{m} u_{i,j}\mathbf{1}_{C^{(h)}_{ij}}(x_1,x_2)$$

where $u_{ij} = u(ih, jh)$.

Let $N := \left[\frac{T}{k}\right]$ and denote by $\psi_{h,k}$ the approximation of the payoff function in the grid defined on Ω_ρ by

$$\begin{aligned} \psi_{h,k}(x_1,x_2) &:= \sum_{n=1}^{N}\psi_h\mathbf{1}_{[(n-1)k,nk[}(t)\\ &= \sum_{n=1}^{N}\left(\sum_{i,j=-m}^{m}\psi(M_{ij})\mathbf{1}_{C^{(h)}_{ij}}(x_1,x_2)\right)\mathbf{1}_{[(n-1)k,nk[}(t). \end{aligned}$$

0ne searches recursively for an approximating solution of the form:

$$u_{h,k}(t,x_1,x_2) := \sum_{n=1}^{N} u^n_h\mathbf{1}_{[(n-1)k,nk[}(t),$$

starting from $u^N_h = \psi$ with $u^n_h\in E_h$ for $0\leq n\leq N$. We approximate the differential operator A^ε defined in Equation (28) by the following discrete operator A^ε_h, which operates on the functions u^n_h defined on E_h by: :

$$A^\varepsilon_h u^n_h(x_1,x_2) = \sum_{i,j=-m}^{m}(A^\varepsilon_h u_h)^n_{i,j}\mathbf{1}_{C^{(h)}_{ij}}(x_1,x_2),$$

where

$$(A^\varepsilon_h u_h)^n_{i,j} = \frac{\varepsilon^2}{2}\frac{\delta^2 u^n_{i,j}}{\delta x_1^2} + \frac{\sigma^2}{2}\frac{\delta^2 u^n_{i,j}}{\delta x_2^2} + \alpha_\varepsilon\frac{\delta u^n_{i,j}}{\delta x_1} + \beta\frac{\delta u^n_{i,j}}{\delta x_2},$$

with central space derivative

$$\begin{aligned}
\frac{\delta^2 u^n_{i,j}}{\delta x_1^2} &= \frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{h^2}, \\
\frac{\delta^2 u^n_{i,j}}{\delta x_2^2} &= \frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{h^2}, \\
\frac{\delta u^n_{i,j}}{\delta x_1} &= \frac{u^n_{i+1,j} - u^n_{i-1,j}}{2h}, \\
\frac{\delta u^n_{i,j}}{\delta x_2} &= \frac{u^n_{i,j+1} - u^n_{i,j-1}}{2h}.
\end{aligned}$$

C. Dynamic Programming Approach

We should solve the variational inequality (I^ε) using the dynamic programming approach as in [19]. Let us consider the obstacle problem on $Q_\rho = [0, T_\nu] \times \Omega_\rho$.

$$\begin{aligned}
\max\left(\psi - u, \frac{\partial u}{\partial t} + A^\varepsilon u\right) &= 0 \qquad (29) \\
u(T_\nu, x_1, x_2) = \psi(x_1, x_2)
\end{aligned}$$

with the Dirichlet boundary condition $u = \psi$ on $[0, T[\times\partial\Omega_\rho$. We consider the following approximation of the obstacle problem (29)

$$\begin{aligned}
& u^N = \psi \\
& u^n \geq \psi \\
& \frac{u^{n+1} - u^n}{k} + A^\varepsilon_h u^n \leq 0 \qquad (30) \\
& \left(\frac{u^{n+1} - u^n}{k} + A^\varepsilon_h u^n, u^n - \psi\right) = 0
\end{aligned}$$

The V.I. in finite dimension (30) can be expressed in a linear complementarity problem. In this paper, to solve (30) we rather use the splitting methods considered as an analytic version of dynamic programming. The method consists in splitting the American problem in two steps: the approximating solution is recursively constructed starting from $u^N = \psi$ and computing u^n for $n = 0, \ldots, N-1$ as follows: firstly define the function v on $\partial\Omega_\rho$ by:

$$\begin{aligned}
v_{-m,j} &= \psi_{-(m+1),j}, v_{m,j} = \psi_{m+1,j}, v_{i,-m} = \psi_{i,-(m+1)} \\
v_{i,m} &= \psi_{i,m+1}
\end{aligned}$$

- **Step 1 :** We solve the following Cauchy problem on $[nk, (n+1)k[\times\Omega_\rho$, with Dirichlet boundary conditions

$$\begin{aligned}
& \frac{\partial v}{\partial t} + A^\varepsilon v = 0, \quad \text{in} \quad [nk, (n+1)k[\times\Omega_\rho \\
& v((n+1)k, .) = u((n+1)k, .)
\end{aligned}$$

Denote by $v_k[u((n+1)k, .)]$ be the solution.

- **Step 2** $u(nk, .) = \max(\psi(.), v_k[u((n+1)k, .)])$.

The convergence of this scheme is proved in [2]. The explicit scheme is conditionally stable and thus is convergent if $\frac{k}{h^2}$ goes to zero. Therefore, in order to solve the first step, we would rather use the following fully implicit scheme in time, where we have at each time step to solve:

$$\begin{aligned}
& \frac{v^{n+1} - v^n}{k} + A^\varepsilon_h(v^n + v) = 0 \qquad (31) \\
& v^N = \psi
\end{aligned}$$

Let U^n the vector of $\mathbb{R}^{(2m+1)^2}$:

$$\begin{aligned}
U^n &= {}^t\left[v^n_{-m,-m}, \ldots, v^n_{-m,m}, v^n_{-m+1,-m}, \ldots, v^n_{-m+1,m}, \right. \\
& \left. \ldots, v^n_{m,-m}, \ldots, v^n_{m,m}\right]
\end{aligned}$$

From (31), we obtain the linear system

$$U^{n+1} = A^\varepsilon U^n + Z^\varepsilon,$$

where $Z^\varepsilon = kA^\varepsilon_h v$ and A^ε is the block tridiagonal square matrix of order $(2m+1)^2$ with each block a square matrix of order $2m+1$:

$$A^\varepsilon = \begin{pmatrix}
B & C & 0 & \cdots & 0 & 0 \\
D & B & C & 0 & \cdots & 0 \\
0 & D & B & C & \cdots & 0 \\
0 & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & D & B & C \\
0 & 0 & 0 & \ldots & D & B
\end{pmatrix}$$

where, $C = eI_{2m+1}$, $D = dI_{2m+1}$ and

$$B = \begin{pmatrix}
a & b & 0 & \cdots & 0 & 0 \\
c & a & b & 0 & \cdots & 0 \\
0 & c & a & b & \cdots & 0 \\
0 & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & c & a & b \\
0 & 0 & 0 & \cdots & c & a
\end{pmatrix}$$

with I_{2m+1} is the identity matrix of order $2m+1$ and:

$$\begin{aligned}
a &= 1 + \frac{k}{h^2}(\varepsilon^2 + \sigma^2), \\
b &= -k\left(\frac{\varepsilon^2}{2h^2} + \frac{\alpha}{2h}\right), \\
\alpha &= 1 - \frac{\varepsilon^2}{2} \\
c &= -k\left(\frac{\sigma^2}{2h^2} - \frac{\alpha}{2h}\right), \\
d &= -k\left(\frac{\sigma^2}{2h^2} + \frac{\beta}{2h}\right), \\
e &= -k\left(\frac{\sigma^2}{2h^2} - \frac{\beta}{2h}\right).
\end{aligned}$$

V. The Numerical results

In this section, we derive the numerical results for the case of European call, with maturity $T = 1$, $\nu = 10^{-3}$ and strike $K = 100$. We take the following values of the Black-Scholes parameter's model : $S_0 = 100$, $r = 0.5$ and $\sigma = 0.2$.

Set $V^\varepsilon = v(x) - u^*_\varepsilon(0, \log(1-\varepsilon), \log(x))$ and $S^x_t = S_t$.

In order to approximate the variance $v(x)$ defined in Equation (8), given the current value $S_0 = x$ of the underlying asset, it is enough to approximate the expectation $\mathbb{E}^r[g(S^x_T)]$ for the functions $g(z) = z$ and $g(z) = z^2$. Since the process S_t is governed by the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

it is possible to valuate the payoff function $g(S_T)$ by simulation of the future stock price S_T. The last equation can be written on discrete form:

$$\Delta S^{k+1} = rS^k \Delta t + \sigma S^k \sqrt{\Delta t}\phi,$$

where $S^k = S_{t_k}$ and ϕ is a sample form a standardized normal distribution. Since $S^{k+1} = S^k + \Delta S^{k+1}$, if S^k is given, we can easily determine the stock price at time t_{k+1}. The generated values $S^k, k = 1, \ldots, N$ are called a realization of the discrete random walk, and represents a possible path of the future asset prices up to the expiry date. At expiry, the payoff function $g(S_T)$ is evaluated for the asset value S^N at time $t_N = T$. For a considered sequence of M realizations of the discrete random walk, we denote the corresponding generated asset values at expiry by $S^{N,j}, j = 1, \ldots, M$. The mean value of the payoff $g(S_T)$ is given by

$$\hat{g}(S_T) = \frac{1}{M} \sum_{j=1}^{M} g(S^{N,j}).$$

This mean value approximates the expectation of the payoff at expiry $\mathbb{E}^r[g(S_T)]$. By the large numbers law, this method known as the Monte-Carlo simulation one, converges to $\mathbb{E}^r[g(S_T)]$ with the rate $\mathcal{O}(1/\sqrt{M})$.

The implementation of this algorithm is done using a computer simulator generating the discrete random walks, i.e., random numbers generator. We obtain the following results:

x	$v(x)$
57	0.734319
58	0.962133
59	1.246217
60	1.597136

For each spot value, we compute the value of $u^*_\varepsilon(0, \log(1-\varepsilon), \log(x))$ for several values of the parameter ε.

Now we shall give the graphical representation of the curve $\varepsilon \longrightarrow V^\varepsilon$ for the spot values $57, 58, 59$ and 60. For each spot value x the following curves shows, that the function V^ε decreases to the risk value $V(x)$ as ε decreases to zero. It must be outlined here that we approximate a one dimensional model by a two-dimensional one. Hence, for each spot value the parameter ε must not reach zero but has to be taken greater than a very small level ε_0. The risk value V should be approximated by V^{ε_0}.

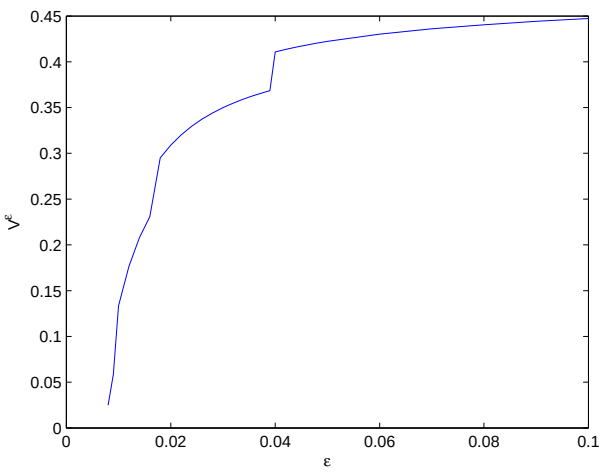

Fig. 1. The fonction $\varepsilon \longrightarrow V^\varepsilon$ for the spot $x = 57$.

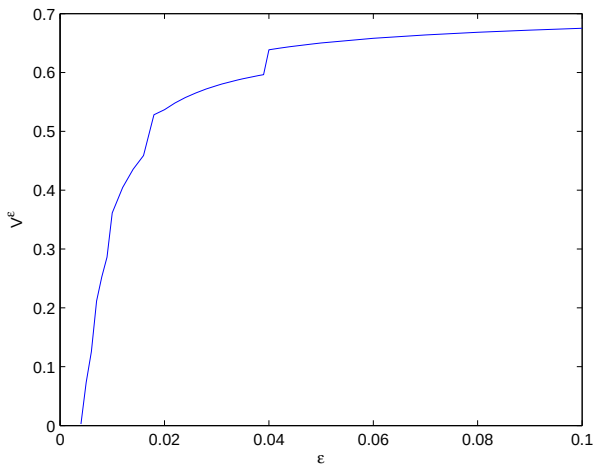

Fig. 2. The fonction $\varepsilon \longrightarrow V^\varepsilon$ for the spot $x = 58$.

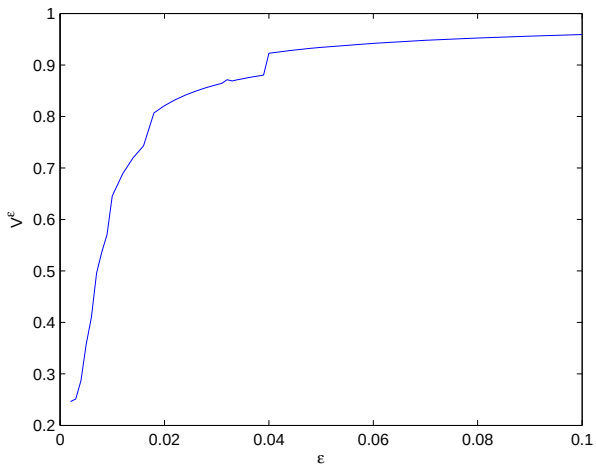

Fig. 3. The fonction $\varepsilon \longrightarrow V^\varepsilon$ for the spot $x = 59$.

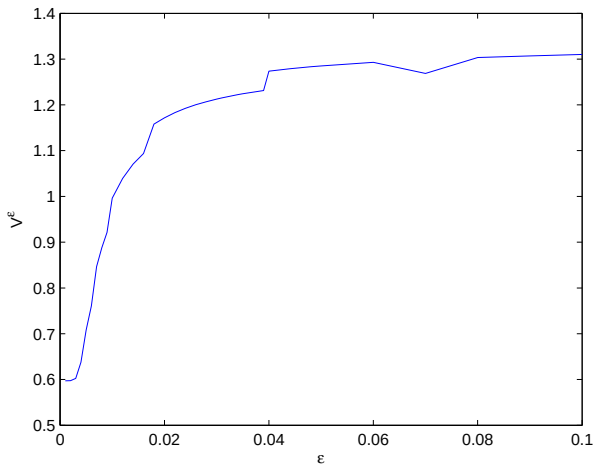

Fig. 4. The fonction $\varepsilon \longrightarrow V^\varepsilon$ for the spot $x = 60$.

The risk value $V(x)$ for the different chosen values of x, is given in the following table:

x	$V(x)$	ε_0
57	0.024852	0.008
58	0.002647	0.004
59	0.246359	0.002
60	0.597236	0.001

VI. Conclusions

In the setting of the Black-Scholes model, we study the problem (P) of selecting the best hedging (stopping) time, initial investment and allocation in the risky asset, for a quadratic criterion. We explicit the optimal ratio for each rebabancing (stopping) time. The problem (P) is therefore transformed into an optimal stopping one, which criterion depends only of the hedging (stopping) time and the stock price stopped at this time. This problem is a standard optimal stopping one, with a time dependent payoff. It is natural to try to apply the approach of [1], to optimal stopping, via one-dimensional evolution variationnal inequalities. Unfortunately, the payoff of the problem (P) does not satisfy the smoothness conditions required by [1] (subsection 4.10). To overcome this difficulty, we approximate the risk value by a fictitious two-colours Rainbow American option, unique solutions of a two-dimensional stationary variational inequality. By use of the dynamic programming approach for pricing two-colours Rainbow American options in the 2-dimensionnal Black-Scholes model, as in [19], we compute an approximation of the risk value $V(x)$. Using the technical of localization and discretization with the finite difference method, we establish an algorithm converging to an affine transformation of the risk value $V(x)$. We finally provide the numerical results.

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