General Parametric Solution of the Panlevé-Ince Nonlinear ODE and of Some Relative Equations in Mathematical Physics

D.E. Panayotounakos, Th. I. Zarmpoutis and G. Kostogiannis[1]

Abstract —**In this paper, we prove that the modified Painlevé-Ince (PI) equation, the force-free Duffing nonlinear type oscillator (DO) and the Lotka-Volterra (LV) nonlinear ordinary differential equation ODEs can be solved in general parametric form. The developed mathematical methodology and the extracted results, being expressed by way of new theorems including admissible functional transformations and substitutions, generalize the corresponding ones given by V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan.**

Index Terms — **Force-free Duffing Oscillator, modified, Lotka-Volterra equation, parametric solution Painlevé-Ince equations.**

I. Introduction

A well known nonlinear ordinary differential equation (ODE) in nonlinear dynamics and applied mathematics is the modified Painlevé- Ince (PI) second order ODE which is $y''_{xx} + ay\,y'_x + \beta y^3 = 0$, where α and β are arbitrary parameters. Mathematicians and physicians have an attracted interest in this equation [4-6], [9], [14]. For example, Painlevé studied this equation and indentified general solution for $\beta = \alpha^2/9$ or $\beta = -\alpha^2$, while V. Chandrasekar, M. Senthilvelan, and M. Lakshmanan [4] gave the time independent integrals and the corresponding Hamiltonians for the above PI equation. Moreover, the PI equation is intimately connected with the two other well known nonlinear models, the free-force Duffing Oscillator (DO) type equation and the two dimensional Lotka-Volterra (LV) equations. Many investigators studied and solved these two equations, but only under various restrictions [3-6], [9], [12-14] and [19],[21]. H. Duler [5] and W. Kapteyn [12] four years later provided the first two investigations about LV equation. The equivalent system to LV equation was examined in 1952 by N. Bautin [3] and by J.Petrovskii and E.Landis [17] in 1955. A detailed reference to the Duffing problem was included in [5]. Interest in the six Painlevé equations was reignited by Ablowitz et al [1-2]. Recently, T. Hasuike [20] proposed a solution algorithm based on a parametric solution approach, P.R. Gordoa and A. Pickering [8] gave a new derivation of two Painlevé hierarchies, Yi Zhang et al [22] provided an exact solution and A. Pichering [18] presented a Hamiltonian approach.

In this paper we prove that the modified PI equation $y''_{xx} + ay\,y'_x + \beta y^3 = 0$, where α and β are free parameters, is integrable in general parametric form for any value of α and β. The PI nonlinear, second order ODE is intimately connected with two other well known nonlinear models, the force-free Duffing type Oscillator (DO) and the two dimensional LV equation. The general parametric solutions of these two nonlinear ODEs are also constructed by means of new proposed theorems inserting admissible functional transformations and substitutions. The developed theory includes no restrictions, generalizes the successful results given V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan [4], using the mathematical methodology by D. Panayotounakos, Th. I. Zarmpoutis and C. Siettos [15-16].

II. Some Basic Results

The symbols $(\ )'_x = d/dx, (\ )''_{xx} = d^2/dx^2, \ldots$ are used to denote the total derivatives. The modified PI equation [4], [9] is not possible to be integrated straightforwardly. Consequently, we make use of the following theorem,

Theorem 1

Applying admissible functional transformation on the modified PI nonlinear ODE $y''_{xx} + \alpha y\,y'_x + \beta y^3 = 0$, where $a,b =$ suitable constants, exact parametric solutions, including two arbitrary constants of integration, can be extracted

Proof

The transformation

$$y'_x = p(y),\ p(y) \neq 0 \Leftrightarrow y''_{xx} = \frac{dp}{dy}\frac{dy}{dx} = p\,p'_y, \qquad (1)$$

reduces the modified PI equation

$$y''_{xx} + \alpha y\,y'_x + \beta y^3 = 0;\ \ a,b = \text{suitable constants} \neq 0,$$

to the Abel nonlinear ODE of the second kind

$$p\,p'_y + ayp + \beta y^3 = 0, \qquad (2)$$

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D.E. Panayotounakos is with the School of Applied Mathematics and Physical Sciences, National Technical University of Athens, Zografou Street 5, Hellas, GR 15773.

Th. I. Zarmpoutis is with the School of Applied Mathematics and Physical Sciences, National Technical Unive rsity of Athens, Zografou Street 5, Hellas, GR 15773 and he is the corresponding author. (Email zarmpoutis@tee.gr)

G. Kostogiannis is with the School of Applied Mathematics and Physical Sciences, National Technical University of Athens, Zografou Street 5, Hellas, GR 15773

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where p is a subsidiary differentiable function which has to be determined.

In addition the substitution

$$z=-a\int y dy=-\frac{ay^2}{2}, \qquad (3)$$

reduces the Abel equation (2) to the canonical (normal) form

$$p\,p'_z-p=-\frac{2\beta}{\alpha}z. \qquad (4)$$

The last equation (4) admits the parametric solution [19; p.29]

$$\begin{aligned} z&=C\exp\left(-\int\frac{\tau dt}{\tau^2-\tau+\frac{2\beta}{\alpha}}\right),\\ p&=-\frac{C\alpha}{2\beta}(\tau-1)\exp\left(-\int\frac{\tau dt}{\tau^2-\tau+\frac{2\beta}{\alpha}}\right), \end{aligned} \qquad (5)$$

$C=$ integration constant, $-\infty<z=\text{parameter}<+\infty$.

The above parametric solution (5) can be summarized as

$$\frac{p}{z}=-\frac{\alpha}{2\beta}(\tau-1),$$

and thus, if we estimate

$$\ln\left|\frac{z}{C}\right|=-\int\frac{\tau dt}{\tau^2-\tau+\frac{2\beta}{\alpha}},$$

and according to [7; p.68, 2.175.1], one obtains

$$\ln\left|\frac{z}{C}\right|=-\frac{1}{2}\ln\left|\tau^2-\tau+\frac{2\beta}{\alpha}\right|-\frac{1}{2}\int\frac{dt}{\tau^2-\tau+\frac{2\beta}{\alpha}},$$

or equivalently the result

$$\ln\left|\frac{z\sqrt{\tau^2-\tau+\frac{2\beta}{\alpha}}}{C}\right|=-\frac{1}{2}\int\frac{dt}{\tau^2-\tau+\frac{2\beta}{\alpha}}. \qquad (6)$$

The integral in the right hand side of (6) is evaluated as (elliptic, parabolic hyperbolic cases)

$$-\frac{1}{2}\int\frac{d\tau}{\tau^2-\tau+\frac{2\beta}{\alpha}}=$$

$$=\begin{cases} -\frac{1}{2}\frac{1}{\sqrt{-\Delta}}\ln\left|\frac{-1+2\tau-\sqrt{-\Delta}}{-1+2\tau+\sqrt{-\Delta}}\right|, & \text{if } \Delta=\frac{2\beta}{\alpha}-1<0;\\ -\frac{1}{-1+2\tau}, & \text{if } \Delta=\frac{2\beta}{\alpha}-1=0;\\ -\frac{1}{\sqrt{\Delta}}\arctan\left(\frac{-1+2\tau}{\sqrt{\Delta}}\right), & \text{if } \Delta=\frac{2\beta}{\alpha}-1>0,\\ -\infty<\tau=\text{parameter}<\infty. \end{cases} \qquad (7)$$

Solving equation (6) in terms of z, one extracts expressions with parameter τ, that is

$$z=\begin{cases} \dfrac{-\frac{1}{2}\left(\frac{-1+2\tau-\sqrt{-\Delta}}{-1+2\tau+\sqrt{-\Delta}}\right)^{\frac{1}{\sqrt{-\Delta}}}C}{\left(\tau^2-\tau+\frac{2\beta}{\alpha}\right)^{\frac{1}{2}}}, & \text{if } \Delta=\frac{2\beta}{\alpha}-1<0;\\ \dfrac{\left[\exp\left(-\frac{1}{-1+2\tau}\right)\right]C}{\left(\tau^2-\tau+\frac{2\beta}{\alpha}\right)^{\frac{1}{2}}}, & \text{if } \Delta=\frac{2\beta}{\alpha}-1=0;\\ \dfrac{-\frac{1}{2}\exp\left[\frac{1}{\sqrt{\Delta}}\tan^{-1}\left(\frac{-1+2\tau}{\sqrt{\Delta}}\right)\right]C}{\left(\tau^2-\tau+\frac{2\beta}{\alpha}\right)^{\frac{1}{2}}}, & \text{if } \Delta=\frac{2\beta}{\alpha}-1>0,\\ C=\text{suitable constant of integration};\\ \Delta=\text{discriminat of the } \tau^2-\tau+\frac{2\beta}{\alpha},\\ -\infty<\tau=\text{parameter}<\infty. \end{cases} \qquad (8)$$

Thus, using (3), one finds the following solution for y^2 (elliptic, parabolic hyperbolic types)

$$y^2=\begin{cases} \dfrac{\frac{1}{a}\left(\frac{-1+2\tau-\sqrt{-\Delta}}{-1+2\tau+\sqrt{-\Delta}}\right)^{\frac{1}{\sqrt{-\Delta}}}C}{\left(\tau^2-\tau+\frac{2\beta}{\alpha}\right)^{\frac{1}{2}}}, & \text{if } \Delta=\frac{2\beta}{\alpha}-1<0;\\ \dfrac{-\frac{2}{a}\left[\exp\left(-\frac{1}{-1+2\tau}\right)\right]C}{\left(\tau^2-\tau+\frac{2\beta}{\alpha}\right)^{\frac{1}{2}}}, & \text{if } \Delta=\frac{2\beta}{\alpha}-1=0;\\ \dfrac{\frac{1}{a}\exp\left[\frac{1}{\sqrt{\Delta}}\tan^{-1}\left(\frac{-1+2\tau}{\sqrt{\Delta}}\right)\right]C}{\left(\tau^2-\tau+\frac{2\beta}{\alpha}\right)^{\frac{1}{2}}}, & \text{if } \Delta=\frac{2\beta}{\alpha}-1>0,\\ C=\text{suitable constant of integration};\\ \Delta=\text{discriminat of the } \tau^2-\tau+\frac{2\beta}{\alpha},\\ -\infty<\tau=\text{parameter}<\infty. \end{cases} \qquad (9)$$

or equivalently

$$y=\begin{cases} \pm\left(\frac{C}{a}\right)^{1/2}F_1(z), & \text{if } \Delta=\frac{2\beta}{\alpha}-1<0\\ \mp\left(\frac{2C}{a}\right)^{1/2}F_2(z), & \text{if } \Delta=\frac{2\beta}{\alpha}-1=0;\\ \pm\left(\frac{C}{a}\right)^{1/2}F_3(z), & \text{if } \Delta=\frac{2\beta}{\alpha}-1>0\\ C=\text{constant of integration};\\ \alpha\cdot C>0,\ -\infty<\tau=\text{parameter}<\infty \end{cases} \qquad (10)$$

where

$$F_1(z)=\frac{\left(\dfrac{-1+2\tau-\sqrt{-\Delta}}{-1+2\tau+\sqrt{-\Delta}}\right)^{\frac{1}{2\sqrt{-\Delta}}}}{F(\tau)^{\frac{1}{4}}},$$

$$F_2(z)=\frac{\left[\exp\left(-\dfrac{1}{2(2\tau-1)}\right)\right]}{F(\tau)^{\frac{1}{4}}}, \tag{11}$$

$$F_3(z)=\frac{\exp\left[\dfrac{1}{2\sqrt{\Delta}}\arctan\left(\dfrac{-1+2\tau}{\sqrt{\Delta}}\right)\right]}{F(\tau)^{\frac{1}{4}}},$$

$$F(\tau)=\exp\left(-\int\frac{\tau d\tau}{\tau^2-\tau+\dfrac{2\beta}{\alpha}}\right).$$

Combining (1) and (5), the parametric function $x=x(\tau)$ is also obtained,

$$y'_x=\frac{dy}{dx}=p(\tau)\Leftrightarrow dx=\frac{dy}{p(\tau)}, \tag{12}$$

where $y=y(\tau)$ as in (10). Consequently, making use of (10) and the second of (5), one extracts

$$x=\begin{cases}\pm\dfrac{\beta}{\alpha\sqrt{\alpha C}}\displaystyle\int\frac{F'_{1_\tau}d\tau}{(\tau-1)F(\tau)}+C^*, & \text{if }\Delta<0,\ \alpha,C<0;\\ \mp\dfrac{2\beta}{\alpha\sqrt{\alpha C}}\displaystyle\int\frac{F'_{2_\tau}d\tau}{(\tau-1)F(\tau)}+C^*, & \text{if }\Delta=0,\ \alpha,C<0;\\ \pm\dfrac{\beta}{\alpha\sqrt{\alpha C}}\displaystyle\int\frac{F'_{3_\tau}d\tau}{(\tau-1)F(\tau)}+C^*, & \text{if }\Delta>0,\ \alpha,C<0,\\ C^*=\text{new integration constant.}\end{cases} \tag{13}$$

where $F_i(\tau),\ (\iota=1,2,3)$ as in (11).

This result completes the proof of the Theorem 1. We underline, that C and C^* are suitable constants of integration and $-\infty<\tau<+\infty$ is the parameter. Also, without loss of generality, it is supposed that $\alpha>0$ and $C>0$. A first conclusion of the above analysis is that the general solution of the PI nonlinear ODE may be not unique inside a main interval of the parameter $\tau\in(\tau_1,\tau_2)$. According to the sign of the discriminant Δ, the exact solutions, may be divided into three parametric solutions given by formulae (10) and (13) which are valid into consecutive subintervals of the main interval (τ_1,τ_2).

We continue with the following corollary and theorems concerning nonlinear oscillator equations

Corollary 1

The already examined modified PI nonlinear ODE [9] , [21] $y''_{xx}+ay\,y'_x+\beta y^3=0;\quad a,\beta=$ suitable constants can be transformed to a nonlinear oscillator equation

$$w''_{tt}(aw+\gamma)w'_t+\beta w^3+\frac{a\gamma}{3}w^2+\frac{2\gamma}{9}w=0,$$

through the invertible point admissible functional transformation

$$y=w\exp\left(\frac{\gamma}{3}\tau\right),\ x=-\frac{3}{7}\exp\left(-\frac{\gamma}{3}\tau\right).$$

Here γ is an arbitrary parameter. In case of $\alpha=0$, the previous nonlinear ODE in terms of $w(\tau)$ also describes the force-free Duffing oscillators.

The proof of this corollary is presented by V. Chandrasekar et al [4] and consequently the following theorem holds.

Theorem 2

The nonlinear oscillator equation

$$w''_{\tau\tau}+(\alpha w+\gamma)w'_\tau+\beta w^3+\frac{\alpha\gamma}{3}w^2+\frac{2\gamma^2}{9}w=0,$$

by way of suitable admissible functional transformations can be reduced to an Abel equation of the second kind; in the sequel it can be transformed to a first order linear ODE performing general solutions.

Proof

The admissible substitution

$$w'_\tau=p(w),\ p\neq0\Leftrightarrow w''_{\tau\tau}=p'_w w'_\tau=pp'_w, \tag{14}$$

transforms the considered nonlinear oscillator equation to the second kind Abel nonlinear ODE of the second kind [11], [19]

$$p\,p'_w=-(aw+\gamma)p-\left(\beta w^3+\frac{a\gamma}{3}w^2+\frac{2\gamma}{9}w\right)=0. \tag{15}$$

Setting $-\alpha=\alpha^*;\ \ -\beta=c^*;\ \ \alpha\gamma/3=\alpha^*b^*;\ \ 2\gamma^2/9=$ $=2b^{*2}\left(\gamma^2/9=b^{*2}\right)$, one estimates $\gamma=-3b^*;\ \ \beta=-c^*$; and the above Abel equation of the second kind (15) becomes in the form

$$p\,p'_w=-\left(\alpha^*w+3b^*\right)p+c^*w^3-\alpha^*b^*w^2-2b^{*2}w=0, \tag{16}$$

the type of which is included in [19; p. 47]. Then, the new substitution $p=w^2t+b^*w$ reduces the above Abel equation to the following linear ODE with respect to $w=w(t)$

$$\left(-2t^2+\alpha^*t+c^*\right)w'_t=t_w+b^*,$$

or equivalently

$$\left(2t^2+at+\beta\right)w'_t=-t\,w+\frac{\gamma}{3}, \tag{17}$$

where $\gamma=-3b^*$ is an arbitrary parameter. The general integral of this equation is given by the functional relation

$$w(t)=C\exp\left(-\int\frac{t}{R}dt\right)+\frac{\gamma}{3}\exp\left(-\frac{t}{R}dt\right)\int\frac{\exp\left(\dfrac{t}{R}\right)}{R}dt, \tag{18}$$

in which $R = 2t^2 - \alpha^* t - C^* \equiv 2t^2 + at + \beta,$ and $\pm\int \frac{tdt}{R}$ as in [7; p. 68, equation number 2.175], namely

$$\begin{aligned} \pm\int \frac{tdt}{R} &= \pm\int \frac{tdt}{2t^2+at+\beta} = \\ &= \pm\ln\left|2t^2+at+\beta\right| \mp \frac{\alpha}{4}\int \frac{dt}{2t^2+at+\beta} = \\ &\pm\ln\left|2t^2+at+\beta\right| \mp \frac{\alpha}{4} J. \end{aligned} \tag{19}$$

Here J is the integral

$$J = \int \frac{dt}{R} = \int \frac{dt}{2t^2+at+\beta} =$$
$$= \begin{cases} \frac{1}{\sqrt{-\Delta}} \ln\left|\frac{\alpha+2t-\sqrt{-\Delta}}{\alpha+2t+\sqrt{-\Delta}}\right|, & \text{if } \Delta = 2\beta-\alpha^2 < 0; \\ -\frac{2}{-a+4t}, & \text{if } \Delta = 2\beta-\alpha^2 = 0; \\ \frac{2}{\sqrt{\Delta}} \tan^{-1}\left(\frac{\beta+4t}{\sqrt{\Delta}}\right), & \text{if } \Delta = 2\beta-\alpha^2 > 0. \end{cases} \tag{20}$$

A first observation is that the solution of the given equation (17) is not unique inside a main interval of the variable t, but according to (20), it may be divided into three solutions which are valid separately inside three consecutive subintervals. From now on, following the inverse course, one constructs the general solution of the nonlinear oscillator equation. This completes the proof of the Theorem 2.

III. The nonlinear Lotka-Volterra equation

Let us consider the general two dimensional Lotka-Volterra (LV) equation given by the first order nonlinear system [5,13,21]

$$x'_t = x\left(a_1 + a_2 x + a_3 y\right),\ y'_t = y\left(b_1 + b_2 x + b_3 y\right), \tag{21}$$

where α_ι and b_i $(\iota = 1,2,3)$ are six real parameters. Such types of systems have been investigated since 1908. Useful results were developed by H. Dulac [6], W. Kapteyn [12] and M. Frommer [5]. These investigations were reexamined in 1955 by J. Petrovski and E. Landis [17] and also reaffirmed the correctness of Bautin's [3] investigation. A convenient decoupling methodology [4] of equations (21) sets the following general second order nonlinear ODE

$$\begin{aligned} &x''_{tt} - \left(1+\frac{b_3}{a_3}\right)\frac{x'_t}{x} + \left[\left(2a_2\frac{b_3}{a_3} - a_2 - b_2\right)x + \left(2a_1\frac{b_3}{a_3} - b_1\right)\right]x'_t + \\ &+\left(b_2 a_2 - \frac{b_3}{a_3}a_1^2\right)x^3 + \left(a_2 b_1 + b_2 a_1 - 2a_1 a_2\frac{b_3}{a_3}\right)x^2 + \\ &+\left(a_1 b_1 - \frac{b_3}{a_3}a_1^2\right)x = 0. \end{aligned} \tag{22}$$

Let us choose two of the parameters in (22) in the form $b_3 = -a_3$. So (22) results in the simpler form

$$\begin{aligned} &x''_{tt} - \left[\left(3a_2+b_2\right)x + 2a_1 + b_1\right]x'_t + \left(b_2 a_2 + a_1^2\right)x^3 \\ &+\left(a_2 b_1 + b_2 a_1 + 2a_1 a_2\right)x^2 + \left(a_1 b_1 + a_1^2\right)x = 0, \end{aligned} \tag{23}$$

with associated LV equations (including five parameters) given by

$$x'_t = x\left(a_1 + a_2 x + a_3 y\right),\ y'_t = y\left(b_1 + b_2 x - a_3 y\right). \tag{24}$$

A thorough investigation of these kinds of systems is developed by H. Davis [5]. More detailed investigations for the restricted form were presented by V. Chandrasekar, M. Senthilvelan, and M. Lakshmanan [4].

Setting

$$x'_t = w(x),\ x''_{tt} = w'_x x'_t = ww'_x;\ w(x) \neq 0, \tag{25}$$

equation (23) becomes to the following Abel equation of the second kind

$$\begin{aligned} &w w'_x - F_1(x) w = -F_2(x); \\ &F_1(x) = A_1 x + A_2,\ F_2(x) = A_3 x^3 + A_4 x^2 + A_5 x; \\ &A_1 = 2a_2 + b_2,\ A_2 = 2a_1 + b_1,\ A_3 = b_2 a_2 + a_1^2, \\ &A_4 = a_2 b_1 + b_2 a_1 + 2a_1 a_2,\ A_5 = a_1 b_1 + a_1^2, \end{aligned} \tag{26}$$

which further by the admissible substitution

$$\begin{aligned} &\xi = \int F_1(x)\,dx = \frac{A_1 x^2 + 2A_2 x}{2} \Leftrightarrow \\ &\Leftrightarrow x = \frac{-A_2 \pm \sqrt{A_2^2 + 2A_1\xi}}{\mathrm{A}_1}, \end{aligned} \tag{27}$$

is reduced to the canonical form

$$\begin{aligned} &w w'_\xi - w = G(\xi); \\ &G(\xi) = -\frac{F_2(x)}{F_1(x)} = -\frac{x\left(A_3 x^2 + A_4 x + A_5\right)}{A_1 + A_2}, \\ &\xi = \frac{x\left(A_1 x + 2A_2\right)}{2}, \end{aligned} \tag{28}$$

where A_i $(i = 1,\ldots,5)$ as in (26). In addition, another type of admissible substitution, that is

$$w(\xi) = \frac{1}{u(\xi)} \Leftrightarrow w'_\xi = -\frac{u'_\xi}{u^2};\ u(\xi) \neq 0, \tag{29}$$

reduces (28) to the Abel nonlinear ODE of the first kind [11]

$$u'_\xi = -G(\xi)u^3 + u^2. \tag{30}$$

The solution of (30), by way of (29) and (26), is equivalent with the solutions $w(\xi)$ or $w(x)$ ($x(t)$ and $y(t)$ become through (24)). For this purpose we introduce the following proposition.

Proposition 1

Using the admissible functional transformation

$$u(\xi) = -\frac{1}{t\xi'_t},\ t\xi'_t \neq 0,$$

the nonlinear ODE

$$u'_\xi(\xi) = -G(\xi)u^3 + u^2,$$

is transformed to a new nonlinear ODE of the second kind of the Emden-Fowler type

$$\xi_{tt}''=-t^2G(\xi);\ \ t\neq 0,$$

where $G(\xi)$ is given continuous function.

We postpone the proof of this proposition, because it is extensively developed in [15] and we set

$$u=t^{-2}\Rightarrow du=-2t^{-3}dt=-u\sqrt{u}dt,$$

$$z=t^{-2}\frac{\xi_t'}{\xi}\Rightarrow dt=\left[\frac{t^{-2}}{\xi}\xi_{tt}''+\left(\frac{t^{-2}}{\xi}\right)_t'\xi_t'\right]dt, \qquad (31)$$

so that, by way of the total differentials (31), du; dz and the expressions for ξ_{tt}''; ξ_t' which are given in Proposition 1, we derive the equation

$$u_z'=\frac{du}{dz}=\frac{-u\sqrt{u}}{\left(\frac{G(\xi)}{\xi}u^2-2z\sqrt{u}-z^2u\right)}, \qquad (32)$$
$$\left(G(\xi)\ \text{as in}\ (28)\right)$$

Furthermore, putting

$$\sqrt{u(t)}=p(t), \qquad (33)$$

and supposing that

$$\Omega(z)=\frac{G(\xi)p^2}{\xi}-2zp, \qquad (34)$$

one gets the nonlinear ODE

$$p_z'=-\frac{p}{\Omega(z)-z^2p}, \qquad (35)$$

where $\Omega(z)$ is a function which must be determined. Equation (35) is reduced in the first Abel nonlinear ODE

$$\left[z^2p-\Omega(z)\right]p_z'=0, \qquad (36)$$

fact that permits us to use the Julia construction [10; 11, p.27] being expressed as

Proposition 2

If the variable coefficients of the general Abel equation of the second kind

$$\left[g_1(x)y+g_0(x)\right]y_x'=f_2(x)y^2+f_1(x)y+f_0(x),$$

satisfy the functional relation

$$g_0\left(2f_2+g_{1_x}'\right)=g_1\left(f_1+g_{0_x}'\right);\quad g_1\neq 0,$$

then, its general solution is given by

$$\frac{g_1y^2+2g_0y}{g_1J}=2\int\frac{f_0}{g_1J}dx+C,$$

where $J(x)$ is the integral

$$J(x)=\exp\int\frac{2f_2}{g_1}dx,\quad C=\text{integration constant}.$$

Applying the functional relation given by the above Proposition 2 to (36), one defines $\Omega(z)$ through the first order linear ODE $2\Omega(z)=z(1-\Omega_z')$, that is $\Omega(z)=z/3+C_1/z^2$, ($C_1=$ integration constant). Since the integration of the original nonlinear system (25) must include two constants of integration we can set $C_1=0$ and thus $\Omega(z)=z/3$. On the other hand, the Abel equation (36) according to the Proposition 2 admits the general solution because of $J=1$, $z^2p^2-2zp/3-z^2C=0$, $f_0(z)=0$, $g(z)=z/3$. Thus, one estimates

$$p(z)=\frac{1}{3z}\left[1\pm\sqrt{\left(1-36z^2C\right)}\right]; \qquad (37)$$
$C=$ first constant of integration

where without loss of generality, we suppose $36z^2G\leq 1$, $C\leq 1/\left(36z^2\right)$; $-\infty<z=$parameter$<+\infty$.

By now, following the inverse proceeding we are able to define function $x=x(t)$ and thus $y=y(t)$ using the first of (24). Also one observes that the solution of the problem under consideration may not be unique inside a main interval of the parameter being introduced, because of the sign concerning square roots and the sign of the subsquare quantity (C is the first integration constant), but it can be divided to several solutions valid inside consecutive subintervals of the main interval. Then, matching of the corresponding solutions must be done in each point that solution changes in order to ensure the appropriate smoothness.

Summarizing one extracts the following results

$$\begin{aligned}
&\Omega(z)=z/3;\\
&p(z)=F_1(z)=\frac{1}{3z}\left(1\pm\sqrt{1-36z^2C}\right);\\
&\frac{G(\xi)}{\xi}=\frac{\Omega(\xi)}{p^2}+\frac{2z}{p}\Leftrightarrow\frac{G(\xi)}{\xi}=\frac{z+6zF_1^2(z)}{F_1^2(z)};\\
&u=p^2=F_1^2(z);\\
&t^2=\frac{1}{u}\Leftrightarrow t=\pm\frac{1}{F_1(z)};\\
&-\infty<z=\text{parameter}<+\infty;\\
&C=\text{first constant of integration.}
\end{aligned} \qquad (38)$$

Moreover, by the last of (38) one derives

$$dt=\mp\frac{F_{1_z}'}{F_1^2(z)}dz,$$

while by the transformation $u=-1/\left(t\xi_t'\right)$ one extracts

$$\xi_t'=\frac{d\xi}{dt}=\frac{d\xi}{dz}\frac{dz}{dt}=-\frac{1}{tu}=\mp\frac{1}{F_1^3(z)}\Leftrightarrow\frac{d\xi}{dz}=\mp\frac{1}{F_1^3(z)}\frac{dt}{dz}.$$

Thus, though the fourth and the last of (38), one also estimates

$$\frac{d\xi}{dz}=-\frac{F_{1_z}'(z)}{F_1^5(z)},\ \text{or}\ \xi=\frac{1}{4}F_1^{-4}(z).$$

In other words, we have already derived

$$u = F_1^2(z),$$
$$F_1(z) = \frac{1}{3z}\left(1 \pm \sqrt{1-36z^2C}\right);$$
$$\xi = \frac{1}{4}F_1^{-4}(z), \quad (39)$$
$$-\infty < z = \text{parameter} < +\infty;$$
$$C = \text{first integration constant} \le \frac{1}{36z^2},$$

and based on the substitution $w = 1/u$, we evaluate

$$w = \frac{1}{F_1^2(z)},$$
$$F_1(z) = \frac{1}{3z}\left(1 \pm \sqrt{1-36z^2C}\right);$$
$$\xi = \frac{1}{4}F_1^{-4}(z), \quad (40)$$
$$-\infty < z = \text{parameter} < +\infty;$$
$$C = \text{first integration constant} \le \frac{1}{36z^2}.$$

The possible elimination of the parameter z among w and ξ leads to an explicit $w = w(\xi)$ or equivalently to an implicit solution $h(w,\xi) = 0$. From the third of (28) and without lost of generality for $x(x+2A_2) < 0$, we are able to write $\xi = x(A_1x + 2A_2)/2,$ so that the parametric solution (40) becomes

$$w = \frac{1}{F_1^2(z)},\ F_1(z) = \frac{1}{3z}\left(1 \pm \sqrt{1-36z^2C}\right);$$
$$x(A_1x + 2A_2) = \frac{1}{2}F_1^{-4}(z);$$
$$-\infty < z = \text{parameter} < +\infty; \quad (41)$$
$$C = \text{first integration constant} \le \frac{1}{36z^2};$$
$$x(A_1x + 2A_2) >;\ A_1 = 3a_1 + b_2, A_2 = 2a_1 + b_1.$$

The above results, as previously prescribed, complete the solution of the problem under consideration. Remark: The second constant of integration C^* will be introduced through the already prescribed inverse mathematical procedure leading to the evaluation of the functions $x(t), y(t)$. This procedure demands the integration by parts of first of equation (25).

IV. Conclusions

In this paper, we have considered a modified Painlevé-Ince (PI), a force-free Duffing nonlinear type oscillator (DO) and a Lotka-Volterra (LV) ODE. Using admissible functional transformation, theorems, propositions and corollaries we obtained the exact solutions in parametric form. The solution of each equation is not unique inside a main interval of the independent variable, but it may be divided into several solutions which are valid inside consecutive subintervals. Therefore, it is necessary to match the corresponding solutions in each point that solution changes in order to ensure the appropriate smoothness.

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