A New Preconditioned AOR Iterative Method and Comparison Theorems for Linear Systems

Aijuan Li

***Abstract*—In this paper, a new preconditioned AOR iterative method is proposed with the preconditioner $I + S_\alpha$. Some comparison theorems are given when the coefficient matrix A of linear system is an irreducible $L-$ matrix. Numerical example shows that our methods are superior to the basic AOR iterative method.**

***Index Terms*—preconditioner, AOR iterative method, irreducible L-matrix**

I. Introduction

WE consider the linear system of n equations

$$Ax = b, \tag{1}$$

where $A = (a_{ij}) \in R^{n\times n}$ is an $n \times n$ nonsingular matrix, with $a_{ii} \neq 0, i = 1, 2, \cdots, n$ and x,b are n-dimensional vectors. If A is split into

$$A = M - N$$

where M is nonsingular.

Then the basic iterative method for solving (1) can be expressed in the form

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, k = 0, 1, \cdots$$

where $x^{(0)}$ is an initial vector. As it is well known, the above iterative process is convergent to the unique solution $x = A^{-1}b$ for each initial value $x^{(0)}$ if and only if the spectral radius of the iteration matrix $M^{-1}N$ satisfies $\rho(M^{-1}N) < 1$.

Throughout the paper, we assume that $A = I - L - U$, where I is the identity matrix, $-L$ and $-U$ are strictly lower triangular and strictly upper triangular matrices of A, respectively. Then the iteration matrix of the AOR iterative method [1] for solving the linear system (1) is

$$L_{\gamma,\omega} = (I - \gamma L)^{-1}[(1 - \omega)I + (\omega - \gamma)L + \omega U] \tag{2}$$

where ω and γ are real parameters with $\omega \neq 0$.

It is well known that for certain values of the parameters γ and ω. We can obtain the Successive Over-relaxation (SOR), JOR and Jacobi methods, whose iterative matrices are denoted by $L_{\omega,\omega}$, J_ω and J, respectively.

In order to accelerate the convergence of iterative method for solving the linear system (1), the original linear system (1) is transformed into the following preconditioned linear system

$$PAx = Pb, \tag{3}$$

where $P \in R^{n\times n}$ is nonsingular and called a preconditioner. Then the corresponding basic iterative method is given in general by

$$x^{(k+1)} = M_p^{-1}N_px^{(k)} + M_p^{-1}Pb, \quad k = 0, 1, 2, \cdots$$

where $PA = M_p - N_p$ is a splitting of PA and M_p is nonsingular. Similar to the original system (1), we call the basic iterative methods corresponding to the preconditioned system the preconditioned iterative methods, such as the preconditioned AOR method and the preconditioned SOR method.

In [2]-[9], some different preconditioners have been proposed by several authors. In [10], the author presented preconditioned AOR methods for consistent linear systems by using four types of preconditioners $P = P_k(k = 1, 2, 3, 4)$. The preconditioner P_1 is of the form $P_1 = I + S_1$, where

$$S_1 = \begin{pmatrix} 0 & -\alpha_1 a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -\alpha_2 a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1}a_{n-1n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The preconditioner P_2 is of the form $P_2 = I + S_2$, where

$$S_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\alpha_n a_{n1} & 0 & \cdots & 0 \end{pmatrix}$$

The preconditioner P_3 is of the form $P_3 = I + S_3$, where

$$S_3 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -\alpha_2 a_{21} & 0 & \cdots & 0 \\ -\alpha_3 a_{31} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\alpha_n a_{n1} & 0 & \cdots & 0 \end{pmatrix}$$

The preconditioner P_4 is of the form $P_4 = I + S_4$, where

$$S_4 = \begin{pmatrix} 0 & 0 & \cdots & -\alpha_1 a_{1n} \\ 0 & 0 & \cdots & -\alpha_2 a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\alpha_{n-1}a_{n-1n} \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

The author provided some comparison theorems which showed the improvement on convergence rates of the basic iterative methods.

In this paper, we propose a new preconditioned AOR iterative method with a preconditioner $P = I + S_\alpha$, where

$$S_\alpha = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ -\alpha_2 a_{21} & 0 & \cdots & 0 & 0 \\ 0 & -\alpha_3 a_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\alpha_n a_{nn-1} & 0 \end{pmatrix}$$

Aijuan Li is with the School of Science, Shandong University of Technology, Zi'bo 255049, People's Republic of China. Email:juanzi612@163.com

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For convenience, some notations, definitions, lemmas and theorems that will be used in the following parts are given below.

II. Preliminaries

In this paper, $\rho(A)$ denotes the spectral radius of the matrix A.

Definition 2.1([11]). A matrix A is a $L-$ matrix if $a_{ii} \geq 0, i = 1, 2, \cdots, n$ and $a_{ij} \leq 0$ for all $i, j = 1, 2, \cdots, n, i \neq j$.

Definition 2.2([12]). A matrix A is irreducible if the directed graph associated to A is strongly connected.

Lemma 2.1([11]). Let A be a nonnegative and irreducible $n \times n$ matrix. Then
(1) A has a positive real eigenvalue equal to its spectral radius $\rho(A)$.
(2) For $\rho(A)$, there corresponds an eigenvector $x > 0$.
(3) $\rho(A)$ is a simple eigenvalue of A.
(4) $\rho(A)$ increases when any entry of A increases.

Lemma 2.2([2]). Let A be a nonnegative matrix. Then
(1)If $\alpha x \leq Ax$ for some nonnegative vector $x, x \neq 0$, then $\alpha \leq \rho(A)$.
(2)If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$, $\alpha x \neq Ax, Ax \neq \beta x$ for some nonnegative vector x, then $\alpha < \rho(A) < \beta$ and x is a positive vector.

III. Preconditioned AOR iterative method and comparison theorems

For the linear system (1), we consider its preconditioned form

$$PAx = Pb$$

where $P = I + S_\alpha$.

Let $PA = A_\alpha$ and $Pb = b_\alpha$, we will obtain preconditioned system

$$A_\alpha x = b_\alpha, \tag{4}$$

where $A_\alpha = (I + S_\alpha)A$ and $b_\alpha = (I + S_\alpha)b$.

Now, we express the coefficient matrix of (4) as

$$\begin{aligned} A_\alpha &= (I + S_\alpha)(I - L - U) \\ &= I - L - U + S_\alpha - S_\alpha L - S_\alpha U \\ &= D_\alpha - L_\alpha - U_\alpha \end{aligned}$$

where $S_\alpha U = E_\alpha + F_\alpha$, E_α and F_α are diagonal matrix and strictly upper triangular matrix of the matrix $S_\alpha U$, $D_\alpha = I - E_\alpha$, $L_\alpha = L - S_\alpha + S_\alpha L$, $U_\alpha = U + F_\alpha$.

Then the preconditioned AOR iteration matrix

$$L_{\gamma,\omega,\alpha} = (D_\alpha - \gamma L_\alpha)^{-1}[(1-\omega)D_\alpha + (\omega-\gamma)L_\alpha + \omega U_\alpha] \tag{5}$$

where ω and γ are real parameters with $\omega \neq 0$.

Theorem 3.1 Let A and A_α be the coefficient matrices of linear system (1) and (4), respectively. $L_{\gamma,\omega}$ and $L_{\gamma,\omega,\alpha}$ are defined by (2) and (5), respectively. Let A be an irreducible $L-$ matrix. Suppose that $0 < \alpha_i a_{ii-1} a_{i-1i} < 1$, $\alpha_i a_{ii-1} \neq 0$ and $0 < \alpha_i \leq 1$ for $i = 2, 3, \cdots, n$. If $0 \leq \gamma \leq \omega \leq 1$ $(\omega \neq 0, \gamma \neq 1)$, then
(1)$\rho(L_{\gamma,\omega,\alpha}) < \rho(L_{\gamma,\omega})$ if $\rho(L_{\gamma,\omega}) < 1$.
(2)$\rho(L_{\gamma,\omega,\alpha}) > \rho(L_{\gamma,\omega})$ if $\rho(L_{\gamma,\omega}) > 1$.
(3)$\rho(L_{\gamma,\omega,\alpha}) = \rho(L_{\gamma,\omega})$ if $\rho(L_{\gamma,\omega}) = 1$.

Proof. Since A is an irreducible $L-$ matrix and $0 \leq \gamma \leq \omega \leq 1$, the proof which $L_{\gamma,\omega}$ is irreducible and nonnegative has been obtained in [13]. Now we obtain that $L_{\gamma,\omega,\alpha}$ is irreducible and nonnegative.
Since A is an irreducible $L-$ matrix and $\alpha_i a_{ii-1} \neq 0$, we get $\alpha_i a_{ii-1} > 0$. If $0 < \alpha_i \leq 1$ for $i = 2, 3, \cdots, n$, then

$$\begin{aligned} L_{\gamma,\omega,\alpha} &= (D_\alpha - \gamma L_\alpha)^{-1}[(1-\omega)D_\alpha + (\omega-\gamma)L_\alpha \\ &\quad + \omega U_\alpha] \\ &= (I - \gamma D_\alpha^{-1} L_\alpha)^{-1} D_\alpha^{-1}[(1-\omega)D_\alpha \\ &\quad + (\omega-\gamma)L_\alpha + \omega U_\alpha] \\ &= (I - \gamma D_\alpha^{-1} L_\alpha)^{-1}[(1-\omega)I \\ &\quad + (\omega-\gamma)D_\alpha^{-1}L_\alpha + \omega D_\alpha^{-1}U_\alpha] \\ &= [I + \gamma D_\alpha^{-1}L_\alpha + (\gamma D_\alpha^{-1}L_\alpha)^2 + \cdots][(1- \\ &\quad \omega)I + (\omega-\gamma)D_\alpha^{-1}L_\alpha + \omega D_\alpha^{-1}U_\alpha] \\ &= (1-\omega)I + \omega(1-\gamma)D_\alpha^{-1}L_\alpha + \omega D_\alpha^{-1}U_\alpha + \\ &\quad \gamma D_\alpha^{-1}L_\alpha[(\omega-\gamma)D_\alpha^{-1}L_\alpha + \omega D_\alpha^{-1}U_\alpha] + \\ &\quad [(\gamma D_\alpha^{-1}L_\alpha)^2 + (\gamma D_\alpha^{-1}L_\alpha)^3 + \cdots][(1-\omega)I + \\ &\quad (\omega-\gamma)D_\alpha^{-1}L_\alpha + \omega D_\alpha^{-1}U_\alpha] \\ &\geq 0 \end{aligned}$$

Since A is irreducible $L-$ matrix, then $(1-\omega)I + \omega(1-\gamma)D_\alpha^{-1}L_\alpha + \omega D_\alpha^{-1}U_\alpha$ is irreducible. Thus, $L_{\gamma,\omega,\alpha}$ is also irreducible and nonnegative.
From Lemma 2.1, there exists an $x > 0$ such $L_{\gamma,\omega}x = \lambda x$, where $\lambda = \rho(L_{\gamma,\omega})$, we obtain

$$(I - \gamma L)^{-1}[(1-\omega)I + (\omega-\gamma)L + \omega U]x = \lambda x$$

Therefore

$$[(\omega - \gamma + \lambda\gamma)L + \omega U]x = (\omega + \lambda - 1)x$$

$$S_\alpha L x = (\frac{\lambda - 1 + \omega}{\omega - \gamma + \lambda\gamma} S_\alpha - \frac{\omega S_\alpha U}{\omega - \gamma + \lambda\gamma})x$$

$$(S_\alpha L - S_\alpha)x = [\frac{(\lambda-1)(1-\gamma)S_\alpha}{\omega - \gamma + \lambda\gamma} - \frac{\omega S_\alpha U}{\omega - \gamma + \lambda\gamma}]x$$

Thus

$$\begin{aligned} L_{\gamma,\omega,\alpha}x - \lambda x &= (D_\alpha - \gamma L_\alpha)^{-1}[(1-\omega)D_\alpha + \\ &\quad (\omega-\gamma)L_\alpha + \omega U_\alpha]x - \lambda x \\ &= (D_\alpha - \gamma L_\alpha)^{-1}[(1-\omega)D_\alpha + \\ &\quad (\omega-\gamma)L_\alpha + \omega U_\alpha - \lambda(D_\alpha - \gamma L_\alpha)]x \\ &= (D_\alpha - \gamma L_\alpha)^{-1}[(1-\omega-\lambda)D_\alpha + \\ &\quad (\omega - \gamma + \lambda\gamma)L_\alpha + \omega U_\alpha]x \\ &= (D_\alpha - \gamma L_\alpha)^{-1}[(1-\omega-\lambda)(I - E_\alpha) \\ &\quad + (\omega - \gamma + \lambda\gamma)(L - S_\alpha + S_\alpha L) + \\ &\quad \omega(U + F_\alpha)]x \\ &= (D_\alpha - \gamma L_\alpha)^{-1}[(1-\omega-\lambda)I - \\ &\quad (1-\omega-\lambda)E_\alpha + (\omega-\gamma+\lambda\gamma)L + (\omega- \\ &\quad \gamma+\lambda\gamma)(S_\alpha L - S_\alpha) + \omega(U + F_\alpha)]x \\ &= (D_\alpha - \gamma L_\alpha)^{-1}[-(\omega - \gamma + \lambda\gamma)L - \\ &\quad \omega U - (1-\omega-\lambda)E_\alpha + (\omega-\gamma+\lambda\gamma)L \\ &\quad + (\omega-\gamma+\lambda\gamma)(S_\alpha L - S_\alpha) \\ &\quad + \omega U + \omega F_\alpha]x \\ &= (D_\alpha - \gamma L_\alpha)^{-1}[(\lambda+\omega-1)E_\alpha + \\ &\quad (\omega-\gamma+\lambda\gamma)(\tfrac{(\lambda-1)(1-\gamma)S_\alpha}{\omega-\gamma+\lambda\gamma} - \\ &\quad \tfrac{\omega S_\alpha U}{\omega-\gamma+\lambda\gamma}) + \omega F_\alpha]x \\ &= (D_\alpha - \gamma L_\alpha)^{-1}[(\lambda+\omega-1)E_\alpha + \\ &\quad (\lambda-1)(1-\gamma)S_\alpha - \omega S_\alpha U + \omega F_\alpha]x \\ &= (D_\alpha - \gamma L_\alpha)^{-1}[(\lambda+\omega-1)E_\alpha + \\ &\quad (\lambda-1)(1-\gamma)S_\alpha - \omega(E_\alpha + F_\alpha) + \\ &\quad \omega F_\alpha]x \\ &= (\lambda-1)(D_\alpha - \gamma L_\alpha)^{-1}[E_\alpha + \\ &\quad (1-\gamma)S_\alpha]x \end{aligned}$$

Since $\alpha_i a_{ii-1} \neq 0$, we obtain $S_\alpha \geq 0$. We know that the first component of $[E_\alpha + (1-\gamma)S_\alpha]x$ is zero and $(D_\alpha - \gamma L_\alpha)^{-1}[E_\alpha + (1-\gamma)S_\alpha]x \geq 0$. Since $L_{\gamma,\omega,\alpha}$ is irreducible and $L_{\gamma,\omega,\alpha}x \neq \lambda x$. If $\lambda > 1$, From Lemma 2.2, we obtain $\rho(L_{\gamma,\omega,\alpha}) > \rho(L_{\gamma,\omega})$. If $\lambda < 1$, we obtain $\rho(L_{\gamma,\omega,\alpha}) < \rho(L_{\gamma,\omega})$. If $\lambda = 1$, we obtain $\rho(L_{\gamma,\omega,\alpha}) = \rho(L_{\gamma,\omega})$. This completes the proof.

In Theorem 3.1, we take $\gamma = 0, \omega = 1$, then we obtain the following Corollary 3.1.

Corollary 3.1 Let A and A_α be the coefficient matrices of linear system (1) and (4), respectively. $L_{0,1}$ and $L_{0,1,\alpha}$ be the Gauss-Seidel iteration matrix of basic Gauss-Seidel iterative method and preconditioned Gauss-Seidel iterative method, respectively. Let A be an irreducible $L-$ matrix. Suppose that $0 < \alpha_i a_{ii-1} a_{i-1i} < 1$, $\alpha_i a_{ii-1} \neq 0$ and $0 < \alpha_i \leq 1$ for $i = 2, 3, \cdots, n$. Then
(1)$\rho(L_{0,1,\alpha}) < \rho(L_{0,1})$ if $\rho(L_{0,1}) < 1$.
(2)$\rho(L_{0,1,\alpha}) > \rho(L_{0,1})$ if $\rho(L_{0,1}) > 1$.
(3)$\rho(L_{0,1,\alpha}) = \rho(L_{0,1})$ if $\rho(L_{0,1}) = 1$.

In Theorem 3.1, we take $\gamma = \omega$, then we obtain the following Corollary 3.2.

Corollary 3.2 Let A and A_α be the coefficient matrices of linear system (1) and (4), respectively. $L_{\omega,\omega}$ and $L_{\omega,\omega,\alpha}$ be the SOR iteration matrix of basic SOR iterative method and preconditioned SOR iterative method, respectively. Let A be an irreducible $L-$ matrix. Suppose that $0 < \alpha_i a_{ii-1} a_{i-1i} < 1$, $\alpha_i a_{ii-1} \neq 0$ and $0 < \alpha_i \leq 1$ for $i = 2, 3, \cdots, n$. Then
(1)$\rho(L_{\omega,\omega,\alpha}) < \rho(L_{\omega,\omega})$ if $\rho(L_{\omega,\omega}) < 1$.
(2)$\rho(L_{\omega,\omega,\alpha}) > \rho(L_{\omega,\omega})$ if $\rho(L_{\omega,\omega}) > 1$.
(3)$\rho(L_{\omega,\omega,\alpha}) = \rho(L_{\omega,\omega})$ if $\rho(L_{\omega,\omega}) = 1$.

IV. Numerical Example

In this section, we give the following example to illustrate the results obtained in section 3.

Example The coefficient matrix A of (1) is given by

$$A = \begin{pmatrix} 1 & -0.2 & -0.3 & -0.1 & -0.2 \\ -0.1 & 1 & -0.1 & -0.3 & -0.1 \\ -0.2 & -0.1 & 1 & -0.1 & -0.2 \\ -0.2 & -0.1 & -0.1 & 1 & -0.3 \\ -0.1 & -0.2 & -0.2 & -0.1 & 1 \end{pmatrix}$$

We obtain the spectral radius of AOR iterative matrix under the different preconditioners with real parameters γ,ω and $\alpha_i (i = 2, 3, \cdots, n)$.
If we denote the spectral radius of the preconditioned AOR iterative matrix by $\rho(L_{\gamma,\omega,\alpha_1})$ when $\alpha_2 = 0.1$, $\alpha_3 = 0.2$, $\alpha_4 = 0.1$, $\alpha_5 = 0.3$.
We denote the spectral radius of the preconditioned AOR iterative matrix by $\rho(L_{\gamma,\omega,\alpha_2})$ when $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, $\alpha_5 = 1$.
We denote the spectral radius of the preconditioned AOR iterative matrix by $\rho(L_{\gamma,\omega,\alpha_3})$ when $\alpha_2 = 0.4$, $\alpha_3 = 0.5$, $\alpha_4 = 0.3$, $\alpha_5 = 0.8$.
We denote the spectral radius of the preconditioned AOR iterative matrix by $\rho(L_{\gamma,\omega,\alpha_4})$ when $\alpha_2 = 0.8$, $\alpha_3 = 0.5$, $\alpha_4 = 0.9$, $\alpha_5 = 1$.
Then we obtain the Table I and Table II.

Table I. The comparison of the spectral radius of AOR iterative matrix(1)

γ, ω	$\rho(L_{\gamma,\omega})$	$\rho(L_{\gamma,\omega,\alpha_1})$	$\rho(L_{\gamma,\omega,\alpha_2})$
$\gamma = 0, \omega = 1$	0.6551	0.6480	0.6205
$\gamma = 0.1, \omega = 0.2$	0.9288	0.9274	0.9221
$\gamma = 0.2, \omega = 0.3$	0.8896	0.8876	0.8798
$\gamma = 0.5, \omega = 0.6$	0.7531	0.7495	0.7359
$\gamma = 0.6, \omega = 0.6$	0.7423	0.7389	0.7262
$\gamma = 0.7, \omega = 0.8$	0.6400	0.6358	0.6203
$\gamma = 0.5, \omega = 0.9$	0.6297	0.6242	0.6038

Table II. The comparison of the spectral radius of AOR iterative matrix(2)

γ, ω	$\rho(L_{\gamma,\omega})$	$\rho(L_{\gamma,\omega,\alpha_3})$	$\rho(L_{\gamma,\omega,\alpha_4})$
$\gamma = 0, \omega = 1$	0.6551	0.6367	0.6388
$\gamma = 0.1, \omega = 0.2$	0.9288	0.9252	0.9247
$\gamma = 0.2, \omega = 0.3$	0.8896	0.8843	0.8838
$\gamma = 0.5, \omega = 0.6$	0.7531	0.7437	0.7438
$\gamma = 0.6, \omega = 0.6$	0.7423	0.7335	0.7341
$\gamma = 0.7, \omega = 0.8$	0.6400	0.6291	0.6308
$\gamma = 0.5, \omega = 0.9$	0.6297	0.6156	0.6157

From Table I and Table II, we can see that if $\rho(L_{\gamma,\omega}) < 1$, $\rho(L_{\gamma,\omega,\alpha}) < \rho(L_{\gamma,\omega})$, which shows that our methods are superior to the basic AOR iterative method.

References

[1] A.Hadjidimos, "Accelerated overrelation method," *Math Comput.*, V32, pp. 149-157, 1978.
[2] A.D.Gunawardena, S.K.Jain, L.Snyder, "Modified iterative methods for consistent linear systems," *Linear Algebra Appl.*, V154-156, pp. 123-143, 1991.
[3] D.J.Evans, M.M.Martins, M.E.Trigo, "The AOR iterative method for new preconditioned linear systems," *J.Comput.Appl.Math.*, V132, pp. 461-466, 2001.
[4] H.Kotakemori, K.Harada, M.Morimoto, H.Niki, " A comparison theorem for the iterative method with the preconditioner$(I + S_{\max})$," *J.Comput.Appl.Math.*, V145, pp. 373-378, 2002.
[5] A.Hadjidioms, D.Noutsos, M.Tzoumas, "More on modifications and improvements of classical iterative schemes for M-matrices," *Linear Algebra Appl.*, V364, pp. 253-279, 2003.
[6] H.Niki, K.Harada, M.Morimoto, M.Sakakihara, "The survey of preconditioners used for accelerating the rate of convergence in the Gauss-Seidel method," *J.Comput.Appl.Math.*, V164-165, pp. 587-600, 2004.
[7] L.y.Sun, "A comparison theorem for the SOR iterative method," *J.Comput.Appl.Math.*, V181, pp. 336-341, 2005.
[8] C.Y.Zhang, C.X.Xu, S.H.Luo, "Covergence of block iterative methods for linear systems with generalized H-matrices," *J.Comput.Appl.Math.*, V229, pp. 70-84, 2009.
[9] A.J.Li, "Improving AOR iterative methods for irreducible L-matrices," *Engineering Letters*, V19, Issue 1, pp.46-49, 2011.
[10] J.H.Yun, "Comparison results of the preconditioned AOR methods for L-matrices," *Appl.Math.Comput.*, V218, pp. 3399-3413, 2011.
[11] D.M.Young, Iterative solution of large sinear systems, New York,Academic.1971.
[12] R.S.Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood cliffs.NJ,1962.
[13] Y.T.Li, S.F.Yang, "A multi-parameters preconditioned AOR iterative method for linear systems, " *App.Math.Comput.*, V206, pp. 465-473, 2008.