A New Implicit Algorithm of Asymptotically Quasi-nonexpansive Maps in Uniformly Convex Banach Spaces

H. Fukhar-ud-din, A. R. Khan and M. A. A. Khan

Abstract—**In this paper, we introduce and study weak and strong convergence of a two-step implicit algorithm for a finite family of asymptotically quasi-nonexpansive maps in a uniformly convex Banach space. The results are proved for a more general implicit algorithm under weaker assumptions on the control sequences of parameters. Our results are generalizations of several well-known results in the current literature.**

Index Terms—**Asymptotically nonexpansive map, asymptotically quasi- nonexpansive map, common fixed point, condition (A), demi-compactness.**

I. Introduction

Throughout the paper, we assume that E is a uniformly convex Banach space, T a selfmap on a nonempty subset C of E, $F(T) = \{x \in C : T(x) = x\}$, the set of fixed points of T, $\{1, 2, 3, ...N\}$, the indexing set I and $F = \cap_{i\in I} F(T_i)$, where $T_i (i \in I)$ are selfmaps on C. The map T is : (i) asymptotically nonexpansive if there is a sequence $\{u_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} u_n = 1$ such that $\|T^n x - T^n y\| \leq u_n \|x - y\|$ for $x, y \in C$ and $n \geq 1$ (ii) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there is a sequence $\{u_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} u_n = 1$ such that $\|T^n x - p\| \leq u_n \|x - p\|$ for $x \in C$, $p \in F(T)$ and $n \geq 1$ (iii) uniformly L-Lipschitzian if for some $L \geq 1$, $\|T^n x - T^n y\| \leq L \|x - y\|$ for $x, y \in C$ and $n \geq 1$.

A Banach space E is uniformly convex if for each $r \in (0, 2]$, the modulus of convexity of E, given by

$$\delta(r) = \inf\left\{1 - \frac{1}{2}\|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq r\right\},$$

satisfies the inequality $\delta(r) > 0$. For a sequence, the symbol $\to$ (resp.$\rightharpoonup$) denotes strong (resp. weak) convergence. The space E is said to satisfy *Opial's property* [11] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. A map $T : C \to E$ is demiclosed at $y \in C$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

In 1972, Goebel and Kirk [8] proposed and analyzed a concept of asymptotically nonexpansive maps as an important generalization of nonexpansive maps. Since then, the study of approximation theory of fixed points of asymptotically (quasi-)nonexpansive maps has been developed through explicit and implicit algorithms in various settings (see, e.g. [1-8, 10, 12, 15, 18]). Implicit algorithms have an advantage over explict algorithms for nonlinear problems in Hilbert spaces and Banach spaces in view of their accuracy. For the existence of a common fixed point of a finite family of maps in Hilbert and Banach space, we have to assume that the maps are usually linear or at least to be weakly continuous and affine. In nonlinear case, a stronger geometric structure is required to obtain the desired existence result. So approximation results are of paramount importance in this field of investigations.

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In 2001, Xu and Ori [16] proved the following result.

Theorem 1.1 (**[16, Theorem 2]**). *Let $\{T_i : i \in I\}$ be a family of nonexpansive selfmaps on a closed convex subset C of a Hilbert space with $F \neq \emptyset$, let $x_0 \in C$ and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then the algorithm $x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n$, where $n \geq 1$ and $T_n = T_n \mathrm{mod} N$, converges weakly to a point in F.*

They posed an open question: What condition on the maps $\{T_i : i \in I\}$ and (or) parameters $\{\alpha_n\}$ are sufficient to guarantee strong convergence of the algorithm in Theorem 1.1. The answer is given in affirmative by Sun [12] as follows.

Theorem 1.2 (**[12, Theorem 3.1]**). *Let C be a nonempty closed convex subset of a Banach space. Let $\{T_i : i \in I\}$ be N asymptotically quasi-nonexpansive selfmaps of C with sequence $\{u_{in}\} \subset [1, \infty)$, such that $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in I$ and $F \neq \emptyset$. Suppose that $x_0 \in C$ and $\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, \frac{1}{2})$. Then the algorithm $\{x_n\}$ defined by the implicit iteration scheme*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_{i(n)}^{k(n)} x_n,\ n \geq 1 \qquad (1.1)$$

where $n = (k-1)N+i$, $i = i(n) \in I$ and $k = k(n) \geq 1$ is a positive integer such that $k(n) \to \infty$ as $n \to \infty$, *converges strongly to a common fixed point of $\{T_i : i \in I\}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{d(x, p) : p \in F\}$.*

Theorem 1.3 (**[12, Theorem 3.3]**). *Let C be a nonempty bounded closed convex subset of a Banach space. Let $\{T_i : i \in I\}$ be N uniformly L-Lipschitizian asymptotically quasi-nonexpansive selfmaps of C with sequence $\{u_{in}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$* for all $i \in I$ and *$F \neq \emptyset$. Suppose that there exists one semi-compact member T* in *$\{T_i : i \in I\}$* and that *$x_0 \in C$* and *$\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, \frac{1}{2})$. Then the algorithm $\{x_n\}$ defined implicitly in (1.1) converges strongly to a common fixed point of $\{T_i : i \in I\}$.*

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Recently, Guo and Cho [10] have established strong convergence of $\{x_n\}$ in (1.1) under new conditions on the control sequences to a common fixed point of the family $\{T_i : i \in I\}$ of asymptotically nonexpansive maps.

Inspired and motivated by these facts, we investigate convergence of a two-step implicit algorithm.

Let C be a nonempty subset of a real Banach space E and $T_i : C \to C$ be N asymptotically quasi- nonexpansive maps. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$ and x_0 a given point in C. We define the two-step implicit algorithm in a compact form as

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_{i(n)}^{k(n)} x_n. \end{aligned} \tag{1.2}$$

Recall that a family $\{T_i : i \in I\}$ of N asymptotically quasi-nonexpansive selfmaps of C with $F \neq \emptyset$ is said to satisfy condition (A) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that the inequality

$$\|x - Tx\| \geq f(d(x, F)) \text{ for all } \ x \in C,$$

holds for at least one $T \in \{T_i : i \in I\}$.

In the sequel, we shall need the following lemmas.

Lemma 1.4 ([3, Lemma 3]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of non-negative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$. If $a_{n+1} \leq (1 + b_n)a_n$, $n \geq 1$, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 1.5 ([17, Theorem 2]). *Let $r > 0$ be a fixed real number. Then a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\begin{aligned} \|\lambda x + (1-\lambda)y\|^2 \quad &\leq \quad \lambda \|x\|^2 + (1-\lambda) \|y\|^2 \\ &\quad -\lambda(1-\lambda)g(\|x - y\|), \end{aligned}$$

for all $x, y \in B_r[0] = \{x \in E : \|x\| \leq r\}$ and $0 \leq \lambda \leq 1$.

II. Convergence of implicit algoritm

We start this section with the following technical result.

Proposition 2.1. Let C be a nonempty subset of E and $T_i : C \to C$ be N asymptotically quasi-nonexpansive maps. Then, for all $n \geq 1$ and $x, y \in C, p \in F(T_i)$

(1) there exists a sequence $\{u_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} u_n = 1$ such that $\|T_i^n x - p\| \leq u_n \|x - p\|$,

(2) $\{T_i : i \in I\}$ is uniformly L-*Lipschitzian* with a *Lipschitz* constant $L \geq 1$, that is, there exists a constant $L \geq 1$ such that $\|T_i^n x - p\| \leq L \|x - p\|$.

Proof. (1) Since for each $i \in I$, $T_i : C \to C$ is an asymptotically quasi-nonexpansive map, therefore, there exists a sequence $\{u_n^i\} \subset [1,\infty)$ with $\lim_{n\to\infty} u_n^i = 1$, such that

$$\|T_i^n x - p\| \leq u_n^i \|x - p\|, \text{ for all } n \geq 1.$$

Let $u_n = \max_{1\leq i\leq N} u_n^i$. Then we have $\{u_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} u_n = 1$ and

$$\|T_i^n x - p\| \leq u_n^i \|x - p\| \leq u_n \|x - p\|, \text{ for all } n \geq 1.$$

(2) Taking $L = \sup_{n\geq1} u_n$, the conclusion (2) can be obtained immediately from the conclusion of (1). □

The following result extends Theorem 1.2 for the two-step implicit scheme (1.2).

Theorem 2.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $\{T_i : i \in I\}$ be N asymptotically quasi-nonexpansive selfmaps of C* and *$F \neq \emptyset$. Suppose that $x_0 \in C$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and satisfy the following conditions:*

$(C1)$: $\sum_{n=1}^{\infty}(u_{in} - 1) < \infty$

$(C2)$: *There exists constant $\epsilon \in (0, \frac{1}{2})$ such that* $0 < 1 - \alpha_n \leq \epsilon$ and $\epsilon \leq \beta_n \leq 1 - \epsilon$.

Then the implicit algorithm $\{x_n\}$ generated by (1.2), converges to a point in F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. The necessity of the conditions is obvious. Thus we only prove the sufficiency. Let $p \in F$, we have

$$\begin{aligned} \|y_n - p\| &= \left\|(1-\beta_n)x_n + \beta_n T_{i(n)}^{k(n)} x_n - p\right\| \\ &\leq (1-\beta_n) \|x_n - p\| + \beta_n u_{in} \|x_n - p\| \\ &\leq u_{in} \|x_n - p\|. \end{aligned}$$

So,

$$\begin{aligned} \|x_n - p\| &= \left\|\alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} y_n - p\right\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)u_{in} \|y_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)u_{in}^2 \|x_n - p\|. \end{aligned} \tag{2.1}$$

Denote $\mu_n = u_{in}^2 - 1$. From $\sum_{n=1}^{\infty}(u_{in} - 1) < \infty$, we get $\sum_{n=1}^{\infty} \mu_n < \infty$.

Therefore, (2.1) becomes

$$\begin{aligned} \|x_n - p\| &\leq \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)(1+\mu_n) \|x_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (1-\alpha_n+\mu_n) \|x_n - p\| \\ &\leq \|x_{n-1} - p\| + \frac{\mu_n}{\alpha_n} \|x_n - p\|. \end{aligned} \tag{2.2}$$

Note that $1 - \epsilon \leq \alpha_n$. Hence (2.2) reduces to

$$\|x_n - p\| \leq \|x_{n-1} - p\| + \frac{\mu_n}{1-\epsilon} \|x_n - p\|.$$

Simplifying, we have

$$\begin{aligned} \|x_n - p\| &\leq \frac{1-\epsilon}{1-\epsilon-\mu_n} \|x_{n-1} - p\| \\ &\leq (1 + \frac{\mu_n}{1-\epsilon-\mu_n}) \|x_{n-1} - p\|. \end{aligned} \tag{2.3}$$

Since $\mu_n \to 0$, there exists a positive integer n_0 such that $\mu_n \leq \frac{1-\epsilon}{2}$, for all $n \geq n_0$. It follows from (2.3) that

$$\|x_n - p\| \leq \left(1 + \frac{2\mu_n}{1-\epsilon}\right) \|x_{n-1} - p\|, \text{ for all } n \geq n_0, \tag{2.4}$$

which further implies that

$$\|x_n - F\| \leq \left(1 + \frac{2\mu_n}{1-\epsilon}\right) \|x_{n-1} - F\|, \text{ for all } n \geq n_0. \tag{2.5}$$

Now applying Lemma 1.4 to the inequalities (2.4) and (2.5), we conclude that both $\lim_{n\to\infty} d(x_n, p)$ and $\lim_{n\to\infty} d(x_n, F)$ exist. Since $\liminf_{n\to\infty} d(x_n, F) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$. Next we show that $\{x_n\}$

is a Cauchy sequence. Note that $1+x \le e^x$ for $x>0$ and hence by (2.4), we have

$$\begin{aligned}\|x_{n+m}-p\| &\le \left(1+\frac{2\mu_{n+m}}{1-\epsilon}\right)\|x_{n+m-1}-p\| \\ &\le \exp\left[\sum_{i=n}^{n+m}\frac{2\mu_i}{1-\epsilon}\right]\|x_n-p\| \\ &\le \exp\left[\frac{2}{1-\epsilon}\sum_{i=1}^{\infty}\mu_i\right]\|x_n-p\| \\ &< M\|x_n-p\|.\end{aligned}$$

where $M=\exp\left[\frac{2}{1-\epsilon}\sum_{i=1}^{\infty}\mu_i\right]$.
Since $\lim_{n\to\infty} d(x_n,F)=0$, so for any given $\epsilon>0$, there exists a positive integer n_1 such that $d(x_n,F)<\frac{\epsilon}{2M}$. That is, there exists $p_0\in F$ such that $\|x_{n_0}-p_0\|<\frac{\epsilon}{2M}$. Hence, for any $n\ge n_1$ and $m\ge 1$, we have

$$\begin{aligned}\|x_{n+m}-x_n\| &\le \|x_{n+m}-p_0\|+\|x_n-p_0\| \\ &\le \exp\left[\frac{2}{1-\epsilon}\sum_{i=n_0}^{n+m}\mu_i\right]\|x_{n_0}-p_0\| \\ &\quad +\exp\left[\frac{2}{1-\epsilon}\sum_{i=n_0}^{n}\mu_i\right]\|x_{n_0}-p_0\| \\ &\le \exp\left[\frac{2}{1-\epsilon}\sum_{i=1}^{\infty}\mu_i\right]\|x_{n_0}-p_0\| \\ &\quad +\exp\left[\frac{2}{1-\epsilon}\sum_{i=1}^{\infty}\mu_i\right]\|x_{n_0}-p_0\| \\ &< 2M\|x_{n_0}-p_0\| \\ &< \epsilon.\end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in E and so it must converge. Let $\lim_{n\to\infty} x_n = q$(say). As $\lim_{n\to\infty} d(x_n,F)=0$, therefore, $d(q,F)=0$. This implies that there exists $p\in F$ such that $d(q,p)=0$. That is $q=p$. Hence q is a common fixed point of T_i for all $i\in I$. □

Corollary 2.3. *Suppose that all the conditions of Theorem 2.2 hold. Then the implicit algorithm $\{x_n\}$ generated by (1.2), converges to a point $p\in F$ if and only if $\{x_n\}$ has an infinite subsequence $\{x_{n_j}\}$ with limit p.*

Now, we prove weak convergence of the implicit algorithm (1.2).

Theorem 2.4. *Suppose that all the conditions of Theorem 2.2 hold. Further, if J is the identity map and $(J-T_i)$ is demiclosed at 0 for every $i\in I$, then the implicit algorithm $\{x_n\}$ in (1.2) converges weakly to a common fixed point of $\{T_i : i\in I\}$.*

Proof. The inequality (2.4) in the proof of Theorem 2.2 implies with the help of Lemma 1.4 that $\lim_{n\to\infty}\|x_n-p\|$ exists. We may assume that $\lim_{n\to\infty}\|x_n-p\|=d\ge 0$. Since $\{\|x_n-p\|\}$ is convergent, so $\{x_n\}$ is bounded. Therefore, there exists $r>0$ such that $\{x_n\}\subset B_r[p]\cap C=D$ and so D is a nonempty closed bounded and convex subset of C.

From the scheme (1.2) and Lemma 1.5, we have

$$\begin{aligned}\|y_n-p\|^2 &= \left\|(1-\beta_n)x_n+\beta_n T_{i(n)}^{k(n)}x_n-p\right\|^2 \\ &\le (1-\beta_n)\|x_n-p\|^2+\beta_n u_{in}^2\|x_n-p\|^2 \\ &\quad -\beta_n(1-\beta_n)g\left(\left\|T_{i(n)}^{k(n)}x_n-x_n\right\|\right) \\ &\le u_{in}^2\|x_n-p\|^2 \\ &\quad -\beta_n(1-\beta_n)g\left(\left\|T_{i(n)}^{k(n)}x_n-x_n\right\|\right).\end{aligned}$$

So,

$$\begin{aligned}\|x_n-p\|^2 &\le \alpha_n\|x_{n-1}-p\|^2+(1-\alpha_n)u_{in}^2\|y_n-p\|^2 \\ &\quad -\alpha_n(1-\alpha_n)g\left(\left\|x_{n-1}-T_{i(n)}^{k(n)}y_n\right\|\right) \\ &\le \alpha_n\|x_{n-1}-p\|^2+(1-\alpha_n)u_{in}^4\|x_n-p\|^2 \\ &\quad -(1-\alpha_n)u_{in}^2\beta_n(1-\beta_n)g\left(\left\|T_{i(n)}^{k(n)}x_n-x_n\right\|\right) \\ &\quad -\alpha_n(1-\alpha_n)g\left(\left\|x_{n-1}-T_{i(n)}^{k(n)}y_n\right\|\right) \\ &\le \|x_{n-1}-p\|^2+\epsilon M u_{in}^4 \\ &\quad -\epsilon^3u_{in}^2 g\left(\left\|T_{i(n)}^{k(n)}x_n-x_n\right\|\right) \\ &\quad -\epsilon^2 g\left(\left\|x_{n-1}-T_{i(n)}^{k(n)}y_n\right\|\right).\end{aligned}$$

This implies that

$$\begin{aligned}\epsilon^3h_{k(n)}^2 g\left(\left\|T_{i(n)}^{k(n)}x_n-x_n\right\|\right) &\le \|x_{n-1}-p\|^2-\|x_n-p\|^2 \\ &\quad +\epsilon M u_{in}^4\end{aligned}$$

and

$$\epsilon^2 g\left(\left\|x_{n-1}-T_{i(n)}^{k(n)}y_n\right\|\right)\le \|x_{n-1}-p\|^2-\|x_n-p\|^2+\epsilon M u_{in}^4.$$

It follows from the above inequalities that, for any $m>0$,

$$\begin{aligned}\epsilon^3u_{in}^2\sum_{n=1}^{m} g\left(\left\|T_{i(n)}^{k(n)}x_n-x_n\right\|\right) &\le \|x_0-p\|^2-\|x_m-p\|^2 \\ &\quad +\epsilon M\sum_{n=1}^{m}u_{in}^4\end{aligned}$$

and

$$\begin{aligned}\epsilon^2\sum_{n=1}^{m} g\left(\left\|x_{n-1}-T_{i(n)}^{k(n)}y_n\right\|\right) &\le \|x_0-p\|^2-\|x_m-p\|^2 \\ &\quad +\epsilon M\sum_{n=1}^{m}u_{in}^4\end{aligned}$$

which imply on letting $m\to\infty$,

$$\sum_{n=1}^{\infty} g\left(\left\|T_{i(n)}^{k(n)}x_n-x_n\right\|\right)<\infty$$

$$\text{and}\qquad \sum_{n=1}^{\infty} g\left(\left\|x_{n-1}-T_{i(n)}^{k(n)}y_n\right\|\right)<\infty.$$

Hence

$$\lim_{n\to\infty}\left\|T_{i(n)}^{k(n)}x_n-x_n\right\|=0, \tag{2.6}$$

and

$$\lim_{n\to\infty}\left\|x_{n-1}-T_{i(n)}^{k(n)}y_n\right\|=0. \tag{2.7}$$

Set $\left\|T_{i(n)}^{k(n)}x_n - x_{n-1}\right\| = \sigma_n$. Now, we prove that $\sigma_n \to 0$ as $n \to \infty$.
Note that

$$\begin{aligned}\sigma_n &\leq \left\|T_{i(n)}^{k(n)}x_n - T_{i(n)}^{k(n)}y_n\right\| + \left\|x_{n-1} - T_{i(n)}^{k(n)}y_n\right\| \\ &\leq u_{in}\left\|x_n - y_n\right\| + \left\|x_{n-1} - T_{i(n)}^{k(n)}y_n\right\| \\ &\leq u_{in}\beta_n\left\|x_n - T_{i(n)}^{k(n)}x_n\right\| + \left\|x_{n-1} - T_{i(n)}^{k(n)}y_n\right\| \to 0 \\ &\quad \text{as } n \to \infty.\end{aligned} \tag{2.8}$$

Using (2.6) and (2.7) in (2.8), we get that

$$\lim_{n\to\infty}\sigma_n = 0. \tag{2.9}$$

Since $\|x_n - x_{n-1}\| \leq (1-\alpha_n)\left\|T_{i(n)}^{k(n)}y_n - x_{n-1}\right\|$, therefore

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0. \tag{2.10}$$

and

$$\lim_{n\to\infty}\|x_n - x_{n+j}\| = 0, \text{ for } j = 1, 2....N. \tag{2.11}$$

For any positive integer $n > N$, $n = (n-N)(\bmod N)$. Also $n = (k(n)-1)N + i(n)$. Hence $n - N = ((k(n)-1) - 1)N + i(n) = (k(n-N)N + i(n-N)$.
That is, $k(n-N) = k(n) - 1$ and $i(n-N) = i(n)$. Therefore, for $n > N$,

$$\begin{aligned}\|x_{n-1} - T_nx_n\| &\leq \left\|x_{n-1} - T_{i(n)}^{k(n)}x_n\right\| + \left\|T_{i(n)}^{k(n)}x_n - T_nx_n\right\| \\ &\leq \sigma_n + L\left\|T_{i(n)}^{k(n)-1}x_n - x_n\right\| \\ &= \sigma_n + L^2\|x_n - x_{n-N}\| \\ &\quad + L\left\|T_{i(n-N)}^{k(n-N)}x_{n-N} - x_{(n-N)-1}\right\| \\ &\quad + L\left\|x_{(n-N)-1} - x_n\right\| \\ &= \sigma_n + L^2\|x_n - x_{n-N}\| \\ &\quad + L\sigma_{n-N} + L\left\|x_{(n-N)-1} - x_n\right\|,\end{aligned}$$

which yields(on using (2.9) and (2.11):

$$\lim_{n\to\infty}\|x_{n-1} - T_nx_n\| = 0. \tag{2.12}$$

On using (2.9) and (2.12), it follows that

$$\begin{aligned}\lim_{n\to\infty}\|x_n - T_nx_n\| &\leq \lim_{n\to\infty}\|x_n - x_{n-1}\| \\ &\quad + \lim_{n\to\infty}\|x_{n-1} - T_nx_n\| = 0.\end{aligned} \tag{2.13}$$

Consequently, for any $j = 1, 2, ..., N$, from (2.11) and (2.13), we have

$$\begin{aligned}\|x_n - T_{n+j}x_n\| &\leq \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\| \\ &\quad + \|T_{n+j}x_{n+j} - T_{n+j}x_n\| \\ &\leq (1+L)\|x_n - x_{n+j}\| \\ &\quad + \|x_{n+j} - T_{n+j}x_{n+j}\|.\end{aligned}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty}\|x_n - T_{n+j}x_n\| = 0.$$

This implies that the sequence

$$\bigcup_{j=1}^{N}\{\|x_n - T_{n+j}x_n\|\}_{n=1}^{\infty} \to 0 \text{ as } n \to \infty.$$

Since for each $l = 1, 2, ..., N$, $\{\|x_n - T_lx_n\|\}_{n=1}^{\infty}$ is a subsequence of $\cup_{j=1}^{N}\{\|x_n - T_{n+j}x_n\|\}_{n=1}^{\infty}$, therefore we have

$$\lim_{n\to\infty}\|x_n - T_lx_n\| = 0 \text{ for } l = 1, 2, ...N. \tag{2.14}$$

Next, we prove that $\omega_\omega(x_n)$, the weak $\omega-$limits set of $\{x_n\}$, is nonempty and $\omega_\omega(x_n) \subset F$. Indeed, since E is uniformly convex and D is a nonempty closed bounded convex subset of C so D is weakly compact and weakly closed. This implies that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to a point $q \in \omega_\omega(x_n)$, which shows that $\omega_\omega(x_n)$ is nonempty. For any $q \in \omega_\omega(x_n)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q$. Again, by (2.14), we have $\lim_{n\to\infty}\|x_{n_k} - T_lx_{n_k}\| = 0$. Now for each $l \in I, (J - T_l)$ is demiclosed at 0, therefore $(J - T_l)q = 0$. Hence $\omega_\omega(x_n) \subset F$. In a similar fashion we can prove that, if $q_1 \in \omega_\omega(x_n)$, then $q_1 \in F$. For the uniqueness, assume that $q \neq q_1$, and $x_{n_k} \to q$, $x_{n_j} \to q_1$. By Opial's condition, we conclude that

$$\begin{aligned}\lim_{n\to\infty}\|x_n - q\| &= \lim_{n\to\infty}\|x_{n_k} - q\| < \lim_{n\to\infty}\|x_{n_k} - q_1\| \\ &= \lim_{n\to\infty}\|x_n - q_1\| = \lim_{n\to\infty}\|x_{n_j} - q_1\| \\ &< \lim_{n\to\infty}\|x_{nj} - q\| = \lim_{n\to\infty}\|x_n - q\|.\end{aligned}$$

This is a contradiction. Thus $\{x_n\}$ converges weakly to a common fixed point of $\{T_i : i \in I\}$. □

Remark 2.5. (1) It is remarked that Theorem 2.2 also holds for a two-step algoritm with bounded error terms given below

$$\begin{aligned}x_n &= \alpha_nx_{n-1} + \beta_nT_{i(n)}^{k(n)}y_n + \gamma_n\theta_n, \\ y_n &= \alpha_nx_n + \beta_nT_{i(n)}^{k(n)}x_n + \gamma_n\theta_n'.\end{aligned}$$

Furthermore, it is a matter of routine to investigate weak and strong convergence results of an algorithm with error terms under appropriate conditions on the control sequences of parameters. Therefore, Theorem 2.2 extends and improves Theorems 2.1-2.2 in [9] and Theorem 3.1 in [12-14].
(2) Note that weak convergence results: Theorems 1-2 in [4], Theorems 2.3-2.4 in [9], Theorem 2.2 in [15], Theorem 2 in [16] and Theorems 1-2 in [18] are direct consequences of Theorem 2.4.

As an application of our Theorem 2.2, we establish another strong convergence result as follows.

Theorem 2.6. *Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}$ and $\{x_n\}$ be as in Theorem 2.2. If $\{T_i\}$ satisfies condition (A), then $\{x_n\}$ converges strongly to a point in $F \neq \emptyset$.*

Proof. As in the proof of Theorem 2.2, we have that $\lim_{n\to\infty} d(x_n, F)$ exists. Furthermore, (2.14) implies that $\lim_{n\to\infty}\|x_n - T_lx_n\| = 0$ for $l = 1, 2, ...N$. So condition (A) guarantees that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a non-decreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, Theorem 2.2 implies that $\{x_n\}$ converges strongly to a point in F. □

Remark 2.7.
(1) Observe that the condition (A) is much weaker than the demi-compactness. Therefore, Theorem 2.6 is an

improvement and generalization of several well-known results in the current literature such as Theorem 1.3, Theorems 3.1-3.2 in [1], Theorems 4-5 in [4], Theorem 4 in [10], Theorem 2.3 in [15] and Theorem 3 in [18]. Furthermore, Theorem 2.2 and Theorem 2.6 answer in affirmative the question posed by Xu and Ori [16].

(2) Following the line of action of the results proved so far, we can prove these results with suitable changes for the following classes of functions :

(i) generalized asymptotically-quasi nonexpansive maps

$(i.e., \|T^n x - p\| \leq u_n \|x-p\| + \delta_n$, where $\lim_{n\to\infty} u_n = 1$ and $\lim_{n\to\infty} \delta_n = 0)$.

(ii) asymptotically nonexpansive maps in the intermediate sense [2]

$\{i.e., \limsup_{n\to\infty} \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \leq 0\}$.

We leave the details to the reader.

Open Question : Can Theorem 2.2 be proved without the fast rate of convergence condition $(C1)$.

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