On the Nature of Chaos

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Abstract—**Based on a very special property of the shift map (Theorem 1), we believe that chaos should involve not only nearby points can diverge apart but also far apart points can get close to each other and these happen infinitely often. Therefore, we propose to call a continuous map $f(x)$ from an infinite compact metric space (G, d) (with metric d) to itself chaotic if there exists a positive number δ such that for any point x and any nonempty open set V (not necessarily an open neighborhood of x) in G there is a point y in V such that $\limsup_{n\to\infty} d(f^n(x), f^n(y)) \ge \delta$ and $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.**

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1 Introduction

Let (G, d) be an infinite compact metric space with metric d and let f be a continuous map from G into itself. For any positive integer n, let the n^{th} iterate of f be defined by putting $f^1 = f$ and $f^n = f \circ f^{n-1}$. Let p be a point in G and let $m > 1$ be an integer. We say that p is a fixed point of f if $f(p) = p$. We say that p is a periodic point of f with least period m (or simply a period-m point of f) if $f^m(p) = p$ and $f^i(p) \ne p$ for all $1 \le i < m$. We say that f is (topologically) transitive if, for any two nonempty open sets U and V in G, there exists a positive integer n such that $f^n(U) \cap V \ne \emptyset$.

Now let $\Sigma_2 = \{\beta : \beta = \beta_0\beta_1\cdots$, where $\beta_i = 0$ or $1\}$ be the metric space with metric d defined by

$$d(\beta_0\beta_1\cdots, \gamma_0\gamma_1\cdots) = \sum_{i=0}^{\infty} \frac{|\beta_i - \gamma_i|}{2^{i+1}}$$

and let σ be the shift map defined by $\sigma(\beta_0\beta_1\beta_2\cdots) = \beta_1\beta_2\cdots$. The shift map σ is often used **[5, 14]** to model the chaoticity of a dynamical system. For example, it is well-known that, for the logistic map $f_\mu(x) = \mu x(1 - x)$ with any $\mu > 4$, the dynamics of f_μ on the nonwandering set where the most interesting phenomena occur is topologically conjugate to the shift map σ on Σ_2. But, what is the shift map chaotic about?

It is well known **[5]** that the shift map has a point α with dense orbit (and hence is topologically transitive), i.e., the orbit $O_\sigma(\alpha) = \{\sigma^i(\alpha) : i = 0, 1, 2, \cdots\}$ is dense in Σ_2, has dense periodic points and has sensitive dependence on initial conditions (i.e., there exists a positive number δ such that for any point x in G and any open neighborhood V of x there exist a point y in V and a positive integer n such that $d(f^n(x), f^n(y)) \ge \delta$). Sensitive dependence on initial conditions, which is easily understood intuitively as nearby points, however close, will eventually separate a distance, is generally believed to be the central ingredient of chaos. However, does it really reveal the true nature of chaos? In **[3]**, it is shown that sensitive dependence on initial conditions for a contious self-map is a consequence of topological transitivity and dense periodic points and hence is a topological property. On the other hand, if we let W denote the dense invariant (but non-compact) subset of Σ_2 which consists of all elements with finitely many 1's in its expansion, then it is easy to see that, on W, the shift map is topologically transitive and has sensitive dependence on initial conditions. Yet, every point of W is eventually fixed, i.e., for every β in W, there is a positive integer n such that $\sigma^n(\beta) = \bar{0} = 000\cdots$. So, W is a system no one would like to call it chaotic. These seem to suggest that sensitive dependence on initial conditions tells only part, but not the whole, of the chaos story. But then what is the other part?

We know that the shift map has a property called extreme sensitive dependence on initial conditions (it is called Li-Yorke sensitivity in **[1, 12]**) which is stronger than sensitive dependence on initial conditions, i.e., there exists $\varepsilon > 0$ (for the shift map, we can choose $\varepsilon = 1$) such that for any point α in Σ_2 and any open neighborhood V of α there is a point β in V such that $\limsup_{n\to\infty} d(\sigma^n(\alpha), \sigma^n(\beta)) \ge \varepsilon$ and $\liminf_{n\to\infty} d(\sigma^n(\alpha), \sigma^n(\beta)) = 0$. We also know **[6]** that the shift map has a dense uncountable invariant 1-scrambled set (S is a δ-scrambled set **[13]** for some $\delta > 0$ if and only if, for any $x \ne y$ in S, $\limsup_{n\to\infty} d(\sigma^n(x), \sigma^n(y)) \ge \delta$ and $\liminf_{n\to\infty} d(\sigma^n(x), \sigma^n(y)) = 0$). However, are extreme sensitive dependence on initial conditions and the existence of a dense uncountable invariant 1-scrambled set all that the shift map is chaotic about? This motivates us to investigate the chaoticity of the shift map even further. Surprisingly, we find that the shift map is more chaotic than we previously thought.

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2 The chaoticity of the shift map

First we introduce some terminology. For any two finite strings $D = \beta_0\beta_1\cdots\beta_m$ and $E = \gamma_0\gamma_1\cdots\gamma_n$ of 0's and 1's, let $B\cdot E = \beta_0\beta_1\cdots\beta_m\gamma_0\gamma_1\cdots\gamma_n$ denote the concatenation of D and E. If $D_1, D_2, \cdots, D_k$ are k finite strings of 0's and 1's, the concatenation $D_1\cdot D_2\cdot\ldots\cdot D_k$ of $D_1, D_2, \cdots, D_k$ are defined similarly. For simplicity, we shall write $D_1D_2\cdots D_k$ for $D_1\cdot D_2\cdot\ldots\cdot D_k$. For any $\gamma_k = 0$ or 1, let $\gamma'_k = 0$ if $\gamma_k = 1$ and $\gamma'_k = 1$ if $\gamma_k = 0$. For any finite string $E = \gamma_k\gamma_{k+1}\gamma_{k+2}\cdots\gamma_{k+n}$ of 0's and 1's, let E' denote the finite string $\gamma'_k\gamma'_{k+1}\gamma'_{k+2}\cdots\gamma'_{k+n}$.

It is evident that any element of Σ_2 whose expansion contains every finite sequence of 0's and 1's is a transitive point of σ, i.e., a point with dense orbit, and there are uncountably many of such points. Let $m \geq 5$ be a *fixed* integer and let $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ be a *fixed* transitive point in Σ_2. Let $X = \{x_1 = x_{1,0}x_{1,1}x_{1,2}\cdots, x_2 = x_{2,0}x_{2,1}x_{2,2}\cdots, x_n = x_{n,0}x_{n,1}x_{n,2}\cdots, \cdots\}$ be any countably infinite subset of Σ_2. For any integers $0 \leq i < j$, let

$$C(x_m, i:j) = x_{m,i}x_{m,i+1}x_{m,i+2}\cdots x_{m,j}$$

and

$$C'(x_m, i:j) = x'_{m,i}x'_{m,i+1}x'_{m,i+2}\cdots x'_{m,j}.$$

For simplicity, let $0^1 = 0, 0^2 = 00, 0^3 = 000, (01)^2 = 0101, (0011)^3 = 0011\,0011\,0011$, and so on.

For any $\gamma = \gamma_0\gamma_1\gamma_2\cdots$ in Σ_2, define a new point $\tau_\gamma = \tau_\gamma(\gamma, X)$ in Σ_2 as follows and let

$$Y = \{\sigma^n(\tau_\gamma) : n \geq 0, \gamma \in \Sigma_2\},$$

where $\tau_\gamma = (\tau_\gamma)_0(\tau_\gamma)_1(\tau_\gamma)_2\cdots = \alpha_0\alpha_1\alpha_2\cdots\alpha_{m!-1}$ $A_\gamma(m!)\ A_\gamma((m+1)!)\ A_\gamma((m+2)!)\cdots$,

and, for any $k \geq m$, $A_\gamma(k!) = (\tau_\gamma)_{k!}(\tau_\gamma)_{k!+1}(\tau_\gamma)_{k!+2}\cdots(\tau_\gamma)_{(k+1)!-1}$ is the concatenation of the following k strings of 0's and 1's, each of length $k!$,

$$\alpha_0\alpha_1\alpha_2\cdots\alpha_{k!-1}$$

$$(\gamma_0)^{(k-1)!}(\gamma_1)^{(k-1)!}(\gamma_2)^{(k-1)!}\cdots(\gamma_{k-1})^{(k-1)!}$$

$$(01)^{(k-2)!}(0011)^{(k-2)!}\cdots(0^{k-1}1^{k-1})^{(k-2)!}$$

$$B(x_1, 4k!)B(x_2, 5k!)B(x_3, 6k!)\cdots B(x_{k-3}, k\cdot k!),$$

where, for $1 \leq i \leq k-3$, $B(x_i, (3+i)k!)$ is the concatenation of the following $2k$ strings of 0's and 1's, each of length $\frac{1}{2}(k-1)!$,

$C(x_i, (3+i)k! : (3+i)k! + [\frac{1}{2}(k-1)! - 1])$

$\cdots$

$C(x_i, (3+i)k! + (j-1)[\frac{1}{2}(k-1)! - 1] : (3+i)k! + j[\frac{1}{2}(k-1)! - 1])$

$\cdots$

$C(x_i, (3+i)k! + (k-1)[\frac{1}{2}(k-1)! - 1] : (3+i)k! + k[\frac{1}{2}(k-1)! - 1])$

$C'(x_i, (3+i)k! + \frac{1}{2}k! : (3+i)k! + \frac{1}{2}k! + [\frac{1}{2}(k-1)! - 1])$

$\cdots$

$C'(x_i, (3+i)k! + \frac{1}{2}k! + (j-1)[\frac{1}{2}(k-1)! - 1] : (3+i)k! + \frac{1}{2}k! + j[\frac{1}{2}(k-1)! - 1])$

$\cdots$

$C'(x_i, (3+i)k! + \frac{1}{2}k! + (k-1)[\frac{1}{2}(k-1)! - 1] : (3+i)k! + \frac{1}{2}k! + k[\frac{1}{2}(k-1)! - 1])$.

We note that, in the expansion of τ_γ,

(1) there are infinitely many strings $\alpha_0\alpha_1\alpha_2\cdots\alpha_{k!-1}$, $k \geq m$, which imply, for each $i \geq 0$, the denseness of the point $\sigma^i(\tau_\gamma)$,

(2) there are infinitely many strings

$$(\gamma_0)^{(k-1)!}(\gamma_1)^{(k-1)!}(\gamma_2)^{(k-1)!}\cdots(\gamma_{k-1})^{(k-1)!},\ k \geq m,$$

which imply that

$$\limsup_{n\to\infty} d(\sigma^n(\tau_\beta), \sigma^n(\tau_\gamma)) = 1 \text{ for } \beta \neq \gamma,$$

(3) for any integer $j \geq 1$, there are infinitely many strings containing $(0^j1^j)^{(k-2)!}$:

$$(01)^{(k-2)!}(0011)^{(k-2)!}\cdots(0^{k-1}1^{k-1})^{(k-2)!},\ k > j,$$

which imply that, for any integer $j \geq 1$,

$$\limsup_{n\to\infty} d(\sigma^n(\tau_\beta), \sigma^n(\sigma^j(\tau_\gamma))) = 1 \text{ for any } \beta \text{ and } \gamma,$$

and since this string also contains long string of 0's, i.e., 0^{k-1}, we obtain, for any $j \geq 1$,

$$\liminf_{n\to\infty} d(\sigma^n(\tau_\beta), \sigma^n(\sigma^j(\tau_\gamma))) = 0 \text{ for any } \beta \text{ and } \gamma,$$

(4) for any positive integers i and j, there are infinitely many strings

$$C(x_i, (3+i)k! + (j-1)[\tfrac{1}{2}(k-1)! - 1] : (3+i)k! + j[\tfrac{1}{2}(k-1)! - 1]),\ k > j,$$

which imply that

$$\liminf_{n\to\infty} d(\sigma^n(x_i), \sigma^n(\sigma^{j-1}(\tau_\gamma))) = 0,$$

(5) for any positive integers i and j, there are infinitely many strings

$C'(x_i, (3+i)k! + \frac{1}{2}k! + (j-1)[\frac{1}{2}(k-1)! - 1] : (3+i)k! + \frac{1}{2}k! + j[\frac{1}{2}(k-1)! - 1]),\ k > j,$

which imply that

$$\limsup_{n\to\infty} d(\sigma^n(x_i), \sigma^n(\sigma^{j-1}(\tau_\gamma))) = 1.$$

Therefore, with this set $Y = \{\sigma^n(\tau_\gamma) : n \geq 0, \gamma \in \Sigma_2\}$, we have obtained the following result.

Theorem 1. *For any given countably infinite subset X of Σ_2, there exists a dense uncountable invariant 1-scrambled set Y of transitive points in Σ_2 such that, for any x in X and any y in Y, we have*

$$\limsup_{n\to\infty} d(\sigma^n(x), \sigma^n(y)) = 1$$

and

$$\liminf_{n\to\infty} d(\sigma^n(x), \sigma^n(y)) = 0.$$

In the following, we present some examples of continous maps on the interval which also have similar chaotic property as the shift map. These examples include the tent map **[9]** (in section 3), a topologically conjugate class of tent-like maps **[9]** (in section 4) and a non-full map (in section 5). The proofs of the chaotic properties of these examples are by symbolic dynamics which is similar to that of the shift map as described in section 2 except that the respective scrambled sets are chosen differently.

3 The chaoticity of the tent map $T(x)$

Let $T(x) = 1 - |2x - 1|$ be the tent map defined on $[0,1]$. For any point a in $[0,1)$, let $a = .a_0a_1a_2\cdots$ be the usual binary representation of a with $a_i = 0$ for *infinitely many* i's (for $a = 1$, let $a_k = 1$ for all $k \geq 0$). For $a_k = 0$ or 1, let $a_k' = 1 - a_k$ for all $k \geq 0$. Then, we have (see **[17]**), for all $n \geq 1$,

$$T^n(.a_0a_1a_2\cdots) = \begin{cases} .a_na_{n+1}a_{n+2}\cdots, & \text{if } a_{n-1} = 0, \\ .a_n'a_{n+1}'a_{n+2}'\cdots, & \text{if } a_{n-1} = 1. \end{cases}$$

Therefore, any number whose binary representation contains every finite strings of 0's and 1's of the form $0c_1c_2\cdots c_n$, for any $n \geq 1$, has a dense orbit in $[0,1]$ and so T is transitive.

Let $I(0) = [0, \frac{1}{2}]$ and $I(1) = [\frac{1}{2}, 1]$. For $\alpha_i = 0$ or 1, let $I(\alpha_0\alpha_1\cdots\alpha_n)$ denote a closed subinterval of $I(\alpha_0\alpha_1\cdots\alpha_{n-1})$ of minimum length (**[4, 11]**) such that $T(I(\alpha_0\alpha_1\cdots\alpha_n)) = I(\alpha_1\alpha_2\cdots\alpha_n)$. Then, T maps the endpoints of $I(\alpha_0\alpha_1\cdots\alpha_n)$ onto those of $I(\alpha_1\alpha_2\cdots\alpha_n)$ and maps the interior of $I(\alpha_0\alpha_1\cdots\alpha_n)$ onto the interior of $I(\alpha_1\alpha_2\cdots\alpha_n)$ and the length of each $I(\alpha_0\alpha_1\cdots\alpha_{n-1})$ is $1/2^n$. Let $\Sigma_2 = \{\alpha : \alpha = \alpha_0\alpha_1\alpha_2\cdots$, where $\alpha_i = 0$ or $1\}$ be the compact metric space with metric d defined by $d(\alpha_0\alpha_1\cdots, \beta_0\beta_1\cdots) = \sum_{i=0}^{\infty} |\alpha_i - \beta_i|/2^{i+1}$ and let σ be the shift map on Σ_2 defined by $\sigma(\alpha_0\alpha_1\alpha_2\cdots) = \alpha_1\alpha_2\cdots$. For any $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ in Σ_2, let

$$I(\alpha) = \bigcap_{n=0}^{\infty} I(\alpha_0\alpha_1\cdots\alpha_n).$$

Then it is easy to see that each $I(\alpha)\,(\subset I(\alpha_0))$ consists of one point, say $I(\alpha) = \{x_\alpha\}$, and

$$T(I(\alpha)) = I(\sigma\alpha).$$

Furthermore, it is also easy to see that if $< \alpha(n) >$ is a sequence of points in Σ_2 which converges to α, then $< T(x_{\alpha(n)}) >$ converges to $T(x_\alpha)$ in I. Now let $\bar{0} \in \Sigma_2$ denote the sequence consisting of all 0's. Then it is clear that $I(\bar{0}) = \{0\}$. Since $T(I(1\bar{0})) = I(\bar{0}) = \{\,0\,\}$ and since the point 1 is the only point in $I(1) = [\frac{1}{2}, 1]$ mapping to 0, we obtain that $I(1\bar{0}) = \{\,1\,\}$ (see [**11**, Proposition 20] for a more general case). These facts will be needed in the proof of Theorem 2 below.

Let $m \geq 5$ be a *fixed* integer and let $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ be a *fixed* transitive point in Σ_2. Then it is clear that the unique point x_α in $I(\alpha)$ is a transitive point in I. Let

$$X = \{x_1, x_2, \cdots, x_n, \cdots\}$$

be any given countably infinite subset of I. For each integer $n \geq 1$, there is a (not necessarily unique) element $\beta_{n,0}\beta_{n,1}\beta_{n.2}\cdots$ in Σ_2 such that $\{x_n\} = I(\beta_{n,0}\beta_{n,1}\beta_{n.2}\cdots)$.

For simplicity, let $0^1 = 0, 0^2 = 00, 0^3 = 000, (01)^2 = 0101, (0011)^3 = 0011\,0011\,0011$, and so on.

For any integers $0 \leq i < j$ and $n \geq 1$, let

$$C(x_n, i:j) \ = \beta_{n,i}\beta_{n,i+1}\cdots\beta_{n,j-1}\,0$$

and

$$C^*(x_n, i:j) = \begin{cases} 10^{j-i}, & \text{if } \beta_{n,i} = 0, \\ 0^{j-i+1}, & \text{if } \beta_{n,i} = 1. \end{cases}$$

For any element $\gamma = \gamma_0\gamma_1\gamma_2\cdots$ in Σ_2, we define a new element in Σ_2 by putting $\tau_\gamma = (\tau_\gamma)_0(\tau_\gamma)_1(\tau_\gamma)_2\cdots = \alpha_0\alpha_1\cdots\alpha_{m!-2}0\ A_\gamma((m+1)!)\ A_\gamma((m+2)!)\ A_\gamma((m+3)!)\cdots$, where $A_\gamma(k!) = (\tau_\gamma)_{k!}(\tau_\gamma)_{k!+1}(\tau_\gamma)_{k!+2}\cdots(\tau_\gamma)_{(k+1)!-1} =$

$$\alpha_0\alpha_1\alpha_2\cdots\alpha_{k!-2}\,0$$
$$\gamma_0(0)^{(k-1)!-1}\,\gamma_1(0)^{(k-1)!-1}\,\cdots\,\gamma_{k-1}(0)^{(k-1)!-1}$$
$$1(0)^{k!-1}$$
$$B(x_1, 4k!)\quad B(x_2, 5k!)\quad \cdots\quad B(x_{k-3}, k\cdot k!),$$

where $B(x_i, (3+i)k!)$ is a finite sequence of 0's and 1's of length $k!$ such that $B(x_i, (3+i)k!) =$

$C(x_i, (3+i)k! : (3+i)k! + [\frac{1}{2}(k-1)! - 1])$

$\cdots$

$C(x_i, (3+i)k! + (j-1)[\frac{1}{2}(k-1)! - 1] : (3+i)k! + j[\frac{1}{2}(k-1)! - 1])$

$\cdots$

$C(x_i, (3+i)k! + (k-1)[\frac{1}{2}(k-1)! - 1] : (3+i)k! + k[\frac{1}{2}(k-1)! - 1])$

$C^*(x_i, (3+i)k! + \frac{1}{2}k! : (3+i)k! + \frac{1}{2}k! + [\frac{1}{2}(k-1)! - 1)]$

$\cdots$

$C^*(x_i, (3+i)k! + \frac{1}{2}k! + (j-1)[\frac{1}{2}(k-1)! - 1] : (3+i)k! + \frac{1}{2}k! + j[\frac{1}{2}(k-1)! - 1])$

$\cdots$

$C^*(x_i, (3+i)k! + \frac{1}{2}k! + (k-1)[\frac{1}{2}(k-1)! - 1] : (3+i)k! + \frac{1}{2}k! + k[\frac{1}{2}(k-1)! - 1])$.

Let $Y = \{T^n(x_{\tau_\gamma}) : \{x_{\tau_\gamma}\} = I(\tau_\gamma), \gamma \in \Sigma_2, n = 0, 1, 2, \cdots\}$. Then it is easy to check that the following result holds.

Theorem 2. *Let $T(x) = 1 - |2x-1|$ be the tent map defined on $[0,1]$. Then for any given countably infinite subset X of $[0,1]$, there exists a dense invariant uncountable 1-scrambled set Y of transitive points in $[0,1]$ such that, for any $x \in X$ and any $y \in Y$, we have*

$$\limsup_{n\to\infty} |T^n(x) - T^n(y)| \geq \frac{1}{2}$$

and

$$\liminf_{n\to\infty} |T^n(x) - T^n(y)| = 0.$$

4 A topologically conjugate class of tent-like maps

Now assume that f is a *full* map on $[0,1]$, i.e., for some point $0 < a < 1$, $f\,(= f_a)$ is a continuous map from $[0,1]$ onto itself such that (i) $f(0) = 0 = f(1)$ and $f(a) = 1$ and (ii) f is strictly increasing on $[0,a]$ and strictly decreasing on $[a,1]$. Note that the tent map T defined in section 2 is just a special case of f. We first show that if f satisfies (a) f has a dense orbit; or (b) f has dense periodic points; or (c) f has sensitive dependence on initial conditions, then $x < f(x) < 1$ for all $0 < x < a$.

Suppose there is a fixed point $0 < v < a$. If $f(x) > x$ for some $0 < x < v$, let u be the smallest fixed point of f in $[x,v]$. Since f is strictly increasing on $[x,u]$, every point in $[x,u]$ is attracted to the fixed point u. So, f cannot satisfy any one of (a), (b) and (c). If $f(x) < x$ for some $0 < x < v$, let w be the largest fixed point of f in $[0,x]$. Since f is strictly increasing on $[w,x]$, every point in $[w,x]$ is attracted to the fixed point w. So, f cannot satisfy any one of (a), (b) and (c). Therefore, if f satisfies any one of (a), (b) and (c), then f is strictly increasing and $x < f(x) < 1$ on $(0,a)$, and f is strictly decreasing and $0 < f(x) < 1$ on $(a,1)$.

Let $I(0) = [0,a]$ and $I(1) = [a,1]$. For $\alpha_i = 0$ or 1, let $I(\alpha_0\alpha_1\cdots\alpha_n)$ be any closed subinterval of $I(\alpha_0\alpha_1\cdots\alpha_{n-1})$ of minimum length **[4, 11]** such that $f(I(\alpha_0\alpha_1\cdots\alpha_n)) = I(\alpha_1\alpha_2\cdots\alpha_n)$. Hence, f maps the endpoints of $I(\alpha_0\alpha_1\cdots\alpha_n)$ onto those of $I(\alpha_1\alpha_2\cdots\alpha_n)$ and maps the interior of $I(\alpha_0\alpha_1\cdots\alpha_n)$ onto the interior of $I(\alpha_1\alpha_2\cdots\alpha_n)$. Consequently,

$$\begin{aligned} a &\notin \cup_{i=0}^n \operatorname{int}(f^i(I(\alpha_0\alpha_1\cdots\alpha_n))) \\ &= \cup_{i=0}^n \operatorname{int}(I(\sigma^i(\alpha_0\alpha_1\cdots\alpha_n))), \end{aligned}$$

where int(J) denotes the interior of the interval J. For any $\alpha = \alpha_0\alpha_1\cdots$ in Σ_2, let $I(\alpha) = \bigcap_{n=0}^\infty I(\alpha_0\alpha_1\cdots\alpha_n)$. Then $f(I(\alpha)) = I(\sigma\alpha)$ and each $I(\alpha)\,(\subset I(\alpha_0))$ is either a nondegenerate compact interval or consists of one point **[11]**. Furthermore, if $I(\alpha)$ is a nondegenerate compact interval then $a \notin \cup_{i\geq 0} \operatorname{int}(f^i(I(\alpha)))$, f maps the endpoints of $I(\alpha)$ onto the endpoints of $f(I(\alpha))$ and the interior of $I(\alpha)$ onto the interior of $f(I(\alpha))$. Note that it is shown in [**11**, Propositions 20 & 21] that $I(\bar{0}) = \{0\}$ and $I(1\bar{0}) = \{1\}$ and, if $I(\alpha) \cap I(\beta) \neq \emptyset$ for some $\alpha \neq \beta$ in Σ_2, then for some point p in $[0,1]$ and some $k \geq 0$ and $\gamma_i = 0$ or 1, $0 \leq i \leq k-1$, we have

$$\{\alpha,\beta\} = \{\gamma_0\gamma_1\cdots\gamma_{k-1}01\bar{0},\ \gamma_0\gamma_1\cdots\gamma_{k-1}11\bar{0}\},$$

$$I(\alpha) = I(\beta) = \{p\} \quad \text{and} \quad f^k(p) = a.$$

Conversely, if

$$\{\alpha,\beta\} = \{\gamma_0\gamma_1\cdots\gamma_{k-1}01\bar{0},\ \gamma_0\gamma_1\cdots\gamma_{k-1}11\bar{0}\},$$

then $I(\alpha) = I(\beta) = \{p\}$ for some point p and $f^k(p) = a$. These facts will be needed later.

Assume that $I(\alpha) = \bigcap_{n=0}^\infty I(\alpha_0\alpha_1\cdots\alpha_n)$ is a nondegenerate interval. Then so is $f^i(I(\alpha))$ for every $i \geq 0$ since f is not constant on any interval. Since $a \notin \cup_{i\geq 0} \operatorname{int}(f^i(I(\alpha)))$, f^i is strictly monotonic on $I(\alpha)$ for every $i \geq 1$. Assume that f satisfies any one of (a), (b) and (c), and assume that, for some integer $m \geq 1$, $f^m(I(\alpha))$ is a nondegenerate interval such that $\operatorname{int}(f^m(I(\alpha))) \cap \operatorname{int}(I(\alpha)) \neq \emptyset$ (this happens when f satisfies (a) or (b)). Then $f^m(I(\alpha)) = I(\alpha)$ and f^m maps the endpoints of $I(\alpha)$ onto itself and f^m is monotonic on $I(\alpha)$. By resorting to f^{2m} if necessary, we may assume that f^m is increasing on $I(\alpha)$ and fixes both endpoints of $I(\alpha)$. But then, one endpoint of $I(\alpha)$ which is a fixed point of f^m attracts all points of $\operatorname{int}(I(\alpha))$ (under f^m) which clearly contradicts the assumption that f satisfies any one of (a), (b) and (c). If f satisfies (c) and $f^i(I(\alpha))$ and $f^j(I(\alpha))$ have disjoint interiors whenever $i \neq j$, then since the interval $[0,1]$ has finite length, we must have $\lim_{n\to\infty}$ diameter$(f^n(I(\alpha))) = 0$ which contradicts the assumption that f has sensitivity. This shows that if f satisfies any one of (a), (b) and (c), then $I(\alpha)$ consists of exactly one point for every α in Σ_2 and every point of $[0,1]$ belongs to $I(\alpha)$ for some (not necessarily unique) α in Σ_2.

In the following, assume that f satisfies any one of (a), (b) and (c). For the sake of clarity, we write $I_f(\alpha)$ instead of $I(\alpha)$ to emphasize the role of f. For every x in $[0,1]$, there is an α in Σ_2 such that $I_f(\alpha) = \{x\}$. If there is another $\beta \ne \alpha$ in Σ_2 such that $I_f(\beta) = \{x\}$, then it follows from the above that, for some $k \ge 0$, $\{\alpha, \beta\} = \{\gamma_0\gamma_1\cdots\gamma_k 01\bar{0}, \gamma_0\gamma_1\cdots\gamma_k 11\bar{0}\}$. But then $I_T(\alpha) = I_T(\beta) = \{w\}$ for some w with $T^k(w) = \frac{1}{2}$. So, the map $\psi : [0,1] \to [0,1]$ defined by letting $\psi(I_f(\alpha)) = I_T(\alpha)$ is well-defined (cf. [**11**, Theorem 22]). It is easy to see that ψ is a homeomorphism such that $\psi(0) = 0, \psi(a) = \frac{1}{2}, \psi(1) = 1$ and $(\psi f)(I_f(\alpha)) = \psi(f(I_f(\alpha))) =$

$$\psi(I_f(\sigma\alpha)) = I_T(\sigma\alpha) = TI_T(\alpha) = (T\psi)(I_f(\alpha)).$$

Therefore, f is topologically conjugate to T through ψ. This, together with Theorem 2 above, implies the following result.

Theorem 3. *Let $0 < a < 1$ and let f be a continuous map from $[0,1]$ onto itself such that (i) $f(0) = 0 = f(1)$ and $f(a) = 1$ and (ii) f is strictly increasing on $[0,a]$ and strictly decreasing on $[a,1]$. Then the following statements are equivalent:*

(a) *f has a dense orbit.*

(b) *f has dense periodic points.*

(c) *f has sensitive dependence on initial conditions.*

Furthermore, if f has a dense orbit, then f is topologically conjugate to the tent map $T(x) = 1 - |2x-1|$ on $[0,1]$ and, for any countably infinite subset X of $[0,1]$, f has a dense uncountable invariant 1-scrambled set Y of transitive points in $[0,1]$ such that, for any $x \in X$ and $y \in Y$, we have

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| \ge \min\{a, 1-a\}$$

and

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0.$$

5 A non-full map

Now let $g(x)$ be the continuous map from $[0,1]$ onto itself defined by

$$g(x) = \begin{cases} 1-2x, & 0 \le x \le \frac{1}{2}, \\ x - \frac{1}{2}, & \frac{1}{2} \le x \le 1. \end{cases}$$

This map $g(x)$ is a classical example in the *Theory of Discrete Dynamical Systems on the Interval* that has exactly one period-3 orbit.

On the other hand, let $\phi(x)$ be the continuous map [**15, 16**] from the one-point compactification space $[0,\infty]$ of $[0,\infty)$ onto itself defined by

$$\phi(x) = \begin{cases} \infty, & \text{if} \quad x = 0, \\ |1 - \frac{1}{x}|, & \text{if} \quad 0 < x, \\ 1, & \text{if} \quad x = \infty. \end{cases}$$

In [**8**], we show that $g(x)$ and $\phi(x)$ are topologically conjugate and $\phi(x)$ has similar chaotic property as the shift map. Consequently, we have the following result.

Theorem 4. *Let $g(x)$ be the continuous map from $[0,1]$ onto itself defined by*

$$g(x) = \begin{cases} 1-2x, & 0 \le x \le \frac{1}{2}, \\ x - \frac{1}{2}, & \frac{1}{2} \le x \le 1. \end{cases}$$

Then, for any countably infinite subset X of $[0,1]$, g has a dense uncountable invariant $(\frac{1}{2})$-scrambled set Y of transitive points in $[0,1]$ such that, for any $x \in X$ and $y \in Y$, we have

$$\limsup_{n\to\infty} |g^n(x) - g^n(y)| \ge \frac{1}{2}$$

and

$$\liminf_{n\to\infty} |g^n(x) - g^n(y)| = 0.$$

6 The true nature of chaos

Theorem 1 has a very important consequence. That is, given any point x in Σ_2, then at just about everywhere in Σ_2, whether it is close to x or far away from it we can always find a point y (in the dense set Y) whose iterates satisfy $\limsup_{n\to\infty} d(\sigma^n(x), \sigma^n(y)) = 1$ and $\liminf_{n\to\infty} d(\sigma^n(x), \sigma^n(y)) = 0$. This seems to suggest that in a chaotic system the iterates of not only nearby points can diverge apart but also far apart points can get close to each other and these happen infinitely often. After all, two present far apart points may be very close to each other some time earlier.

Theorem 1 also reveals a very striking property for the shift map. That is, when we let $X = \{x_0, \sigma(x_0), \sigma^2(x_0), \cdots\}$ denote the orbit of any given point x_0 in Σ_2, then Theorem 1 implies the existence of a dense uncountable invariant 1-scrambled set Y of transitive points in Σ_2 such that, for every positive integer m and every y in Y, $\limsup_{n\to\infty} d(\sigma^n(\sigma^m(x_0)), \sigma^n(y)) = 1$ and $\liminf_{n\to\infty} d(\sigma^n(\sigma^m(x_0)), \sigma^n(y)) = 0$. In particular, this says that, for any point x_0 and any time m earlier, at about everywhere (the corresponding dense set Y) in Σ_2, we can find a point y (in Y) whose trajectory eventually catches up with that of the point x_0 to within any prescribed distance (since $\liminf_{n\to\infty} d(\sigma^n(\sigma^m(x_0)), \sigma^n(y)) = 0$) even though x_0 starts out time m earlier than y.

7 A definition of chaos

Let (G, d) be an infinite compact metric space with metric d and let f be a continuous map from G into itself. We say that f is chaotic (see also **[18, 19]**) if there exists a positive number δ such that for any point x and any nonempty open set V (not necessarily an open neighborhood of x) in G there is a point y in V such that $\limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq \delta$ and $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$. By **[2, 20]** and Theorems 1, 2 and 4, we have the following result.

Theorem 5. *The following statements hold.*

(a) *The shift map σ is chaotic on Σ_2.*

(b) *The tent map $T(x) = 1 - |2x - 1|$ is chaotic on $[0, 1]$.*

(c) *The map f defined on $[0, 1]$ by putting $f(x) = 1 - 2x$ for $0 \leq x \leq \frac{1}{2}$ and $f(x) = x - \frac{1}{2}$ for $\frac{1}{2} \leq x \leq 1$ is chaotic on $[0, 1]$.*

(d) *F_μ is chaotic on Λ_μ for any $\mu \geq 4$, where $F_\mu(x) = \mu x(1 - x)$ and $\Lambda_\mu = \cap_{n=0}^{\infty} F_\mu^{-n}([0, 1])$ for $\mu > 4$ and $\Lambda_\mu(x) = [0, 1]$ for $\mu = 4$.*

The chaotic maps in Theorem 5 are all transitive and Li-Yorke sensitive. However, not all transitive maps or Li-Yorke sensitive maps are chaotic. The following is such an example.

Example 1. Let $g(x)$ be the continuous map from $[-1, 1]$ onto itself defined by putting $g(x) = 2x + 2$ for $-1 \leq x \leq -\frac{1}{2}$; $g(x) = -2x$ for $-\frac{1}{2} \leq x \leq 0$; and $g(x) = -x$ for $0 \leq x \leq 1$. Then g is transitive and Li-Yorke sensitive but not chaotic because the period-2 point $-\frac{2}{3}$ and the interval $[0, 1]$ are jumping alternately and never get close to each other.

Chaotic maps are clearly Li-Yorke sensitive **[12]**, but not necessarily transitive. The following is such an example.

Example 2. Let $T(x) = 1 - |2x - 1|$ for $0 \leq x \leq 1$ and let h be the continuous map from $[-\frac{1}{2}, 1]$ to itself defined by putting $h(x) = -x$ for $-\frac{1}{2} \leq x \leq 0$ and $h(x) = T(x)$ for $0 \leq x \leq 1$. Then, on $[-\frac{1}{2}, 1]$, h is chaotic but not transitive.

The following result gives a necessary condition for an interval map to be chaotic.

Theorem 6. *Let I denote a compact interval in the real line and let f be a continuous map from I into itself. If f is chaotic, then f has periodic points of all even periods and, f^2 is turbulent, i.e., there exist closed subintervals I_0 and I_1 of I with at most one point in common such that $f^2(I_0) \cap f^2(I_1) \supset I_0 \cup I_1$.*

Proof. Let z be a fixed point of f and let W be a nonempty open set in I. Assume that f is chaotic. Then there exist a positive number δ and a point c in W such that $\limsup_{n\to\infty} |f^n(c) - f^n(z)| \geq \delta$ and $\liminf_{n\to\infty} |f^n(c) - f^n(z)| = 0$. In particular, the ω-limit set $\omega(f, c)$ of c with respect to f (x is in $\omega(f, c)$ if and only if $\lim_{k\to\infty} f^{n_k}(c) = x$ for some sequence of positive integers $n_k \to \infty$) contains the fixed point z of f and a point different from z. It follows from **[10]** that f has periodic points of all even periods and f^2 is turbulent. $\square$

The converse of Theorem 6 is false as shown by Example 1.

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