Analytical Method of Average Run Length for Trend Exponential AR(1) Processes in EWMA Procedure

Wannaporn Suriyakat, Yupaporn Areepong, Saowanit Sukparungsee, and Gabriel Mititelu

Abstract—The Exponentially Weighted Moving Average (EWMA) procedure are used for monitoring and detecting small shifts in the process mean which performs quicker than the Shewhart control chart. Usually, the common assumption of the Statistical Process Control (SPC) is the observations are independent and identically distributed (IID). In practice, however, the observed data are from industry and finance is serially correlated with trend. In this paper, we extend to use CUSUM procedure to compare with EWMA procedure. The performance of latter is superior to the former when the magnitudes of shift are small to moderate. It is shown that EWMA procedure performs better than the CUSUM procedure for the case of trend exponential AR(1) processes.

Index Terms—Trend AR(1), Exponentially Weighted Moving Average, Average Run Length, Exponential White Noise

I. INTRODUCTION

The observations are usually independent and identically distributed (IID), but in reality they might be serially correlations with trend. Some researchers have considered the problem of data correlation as it is related to SPC (see [1]). The Exponentially Weighted Moving Average (EWMA) procedures are used to monitor and detect small shifts in the process mean which is quicker than the Shewhart control chart. The control limits and performance measures for EWMA control chart of correlated processes is based on variables or attributes (see [2] and [3]). Recently, several researchers have shown an increasing interest in the formulation and analytical of non-Gaussian models for serially correlated data, e.g., [4] and [5]. Exponential white noise has been studied in the connection with pollution problem (see [6]), and some paper has studied with exponential white noise by [6], [7], [8], [9] and [10]. In our study, an explicit formula for the EWMA chart for trend stationary exponential AR(1) processes is presented. An overview of the EWMA procedure for serially dependent data is given in Section 2. Later, Section 3 reviews the performance method for serially dependent data in EWMA procedure. Next, Section 4 discusses briefly the explicit formula for the average run length (ARL) for trend exponential AR(1) processes in EWMA procedure. In Section 5, a comparison of the performance of the EWMA procedure and CUSUM procedure is made. Finally, Section 6 concludes the discussion in the paper.

II. A REVIEW OF THE TREND EXPONENTIAL AR(1) PROCESSES IN EWMA PROCEDURE

In [11], [12], [13], [14] and [15] give detailed explanations EWMA procedure for serially dependent data. In the monitoring of the trend exponential AR(1) process in EWMA procedure, assume that we have the observations \(Y_t\) \((t = 0, 1, 2, \ldots)\) taken over time. The EWMA statistic \(Z_t\) is given by:

\[
Z_t = (1 - \lambda)Z_{t-1} + Y_t
\]

where \(\lambda\) is a smoothing constant \((0 < \lambda < 1)\), the sequence \(\{Y_t, t = 1, 2, 3, \ldots\}\) consists of the trend AR(1) processes and the initial value of \(Z_0\) is usually selected to be the process target of \(Y_0\) or the average of random data. The trend AR(1) processes is assumed to be as follow

\[
Y_t = \gamma + \beta t + \rho Y_{t-1} + X_t, \quad t \geq 1
\]

where \(\gamma\) is a constant, \(\beta\) is the trend slope in term of \(t\), and \(\rho\) is the autoregressive coefficient \((0 < \rho < 1)\). Let \(X_t\) is the independent random error term at time \(t\) following \(Exp(\alpha)\). The variance of \(Z_t\) for the large \(t\) will be

\[
\sigma^2_{Z_t} = \left(\frac{\lambda}{2-\lambda}\right)\left(\frac{1 + \rho(1-\lambda)}{1-\rho^2(1-\lambda)}\right)\sigma^2_X.
\]

Therefore the upper control limit (UCL) and lower control limit (LCL) for monitoring the process when plotting \(Z_t\) versus the time \(t\) are

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where $L$ is a constant to be chosen, and $\sigma_x$ is the standard deviation of a known underlying probability distribution. The process will be declared to be in an out-of-control state when $Z_t > B$. The alarm time for the EWMA in then given by

$$t = \inf\{t > 0 : Z_t > B\}. \tag{5}$$

Assume $E_\theta(.)$ denote the expectation at time $\theta$, where $\theta \leq \infty$. The ARLs of the EWMA control chart for the given process are that:

$$ARL_{\alpha} = E_\alpha(T) = T, \tag{6}$$

where $T$ is given (usually large) and

$$ARL_1 = E_1(\tau). \tag{7}$$

III. A REVIEWS OF THE PERFORMANCE METHOD FOR SERIALLY DEPENDENT DATA IN EWMA PROCEDURE

Usually, the performance of the control chart is measured by the average run length (ARL). The $ARL_{\alpha}$ is defined as the expected of false alarm time ($t$) before an in-control process is taken to signal to be out of control. For practical purposes, a sufficient large in-control $ARL_{\alpha}$ is desired. When the process is out-of-control, the performance of a control chart is usually used as $ARL_1$. The $ARL_1$ is defined as the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control. Ideally, $ARL_1$ should be small.

A control chart based on the Exponentially Weighted Moving Average (EWMA) model was first proposed by [16]. The methods to evaluate the performance of EWMA control charts for serially correlated have been studied by [17]. They used simulation method based on the presence of autocorrelation for EWMA control chart. The ARL, and steady state ARL of EWMA were estimated numerically by [18] using an integral equation approach and a Markov chain approach to investigate EWMA and CUSUM procedures for the process mean when data was described by an AR(1) process with additional random error. The EWMA control charts based on the observations which follow an AR(1) process, plus a random error, and to detect changes in the process mean, or in the process variance, the authors using a simulation approach is discussed by [14]. In [19] presented the ARL of the EWMA control chart for monitoring the mean of an AR(1) process, plus a random error by using an integral equation method. In [20] compared the ARL for the EWMAST chart, the CUSUM residual chart, the EWMA residual chart, the $X$ residual chart, and $X$ chart using simulation. In [21] calculated the ARL of $\bar{X}$ and EWMA charts using analytical and simulation techniques. In [22] studied the EWMA chart with residual-based approaches for detecting process shifts by using simulation. In [23] studied EWMA chart for an AR model and calculated ARL by Markov chain approach. In [24] evaluated the ARL of EWMA charts with heavy tailed distribution for monitoring the mean of the stationary processes by simulation method. In [25] computed exactly ARL with the Markov chain approach for a Poisson INAR(1) model of EWMA chart. In [26] designed the ARL performance of autocorrelated process control chart using a Monte Carlo simulation. In [27] used finite Markov Chain imbedding technique to investigate the run length properties for control charts when the process observations were autocorrelated. Recently, [28] have derived explicit formula of performance for EWMA control charts for AR(1) process observations with exponential white noise, [29] have derived explicit formula of $ARL_1$ for EWMA control chart for trend stationary exponential AR(1) processes.

IV. EXPLICIT FORMULA FOR TREND EXPONENTIAL AR(1) PROCESSES IN EWMA PROCEDURE

The performance of a control chart is measured by the average run length (ARL). The $ARL_1$ is defined as the expected of false alarm time ($\tau$) before an in-control process is taken to signal to be out of control. A sufficient large in-control $ARL_1$ is desired. When the process is out-of-control, the performance of a control chart is usually used as $ARL_{\alpha}$. It is the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control. Ideally, $ARL_{\alpha}$ should be small. The values of $ARL_{\alpha}$ and $ARL_1$ for an EWMA control chart with exponential white noise observations are derived by [28]. The authors used an integral equation approach and derived a Fredholm integral equation of second type for the $ARL_1$. The explicit formulas obtained by solving the integral equations are:

$$ARL = 1 - \frac{1 - \gamma}{\lambda e^{\lambda u + \rho v} + e^{\lambda u + \rho v} - 1} \tag{8}$$

where $\gamma$ is a constant, $\beta$ is the trend slope in term of $t$, $\rho$ is the autoregressive coefficient $0 < \rho < 1$, $\alpha$ is a parameter of the exponential distribution, $\lambda$ is a smoothing parameter, $u,v$ are initial values, and $B$ is boundary value.

V. NUMERICAL COMPARISONS OF PERFORMANCE

We present an explicit formula for trend exponential AR(1) processes in EWMA procedure. The numerical results for $ARL_{\alpha}$ when $\alpha = \alpha_0$ and $ARL_{1}$ when $\alpha = \alpha_1$.
are given for the trend exponential AR(1) processes in EWMA procedure was calculated from Eq. (8). To evaluate the performance of a control chart for monitoring trend AR(1) processes in EWMA procedure, we designed the trend exponential AR(1) processes with numerical parameters $0.3 \leq \rho \leq 0.9$, $\gamma = 0$, $\beta = 0.2$ and $Z_0 = \gamma_0 = 0.1$ with weighting constant $\lambda = 0.3$ is given for an in-control process. The CUSUM procedure was constructed with constants $\alpha = 2.3$ and control limit $h = 3.4$ as suggested by [30]. The characteristics of the control charts measured in terms of ARL are examined for different values of shifts in the mean $\alpha = \gamma_0 = 1.01, 1.03, 1.05, 1.07, 1.09, 1.1, 1.2.$

**TABLE 1** The numerical results for $ARL_0$ obtained from formula (8) and numerical integral equation for the trend exponential AR(1) processes in EWMA procedure when $\alpha_0 = 1$, the entries inside the parentheses are the CPU times in seconds

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$B$</th>
<th>$ARL_0$</th>
<th>Explicit formula</th>
<th>Integral equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.30</td>
<td>0.2693</td>
<td>99.6997</td>
<td>99.6996 (44.1457)</td>
<td></td>
</tr>
<tr>
<td>0.35</td>
<td>0.2678</td>
<td>101.2144</td>
<td>101.2144 (45.4545)</td>
<td></td>
</tr>
<tr>
<td>0.40</td>
<td>0.2663</td>
<td>102.4312</td>
<td>102.4312 (44.3564)</td>
<td></td>
</tr>
<tr>
<td>0.45</td>
<td>0.2647</td>
<td>99.8073</td>
<td>99.8073 (50.9680)</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>0.2632</td>
<td>100.3019</td>
<td>100.3019 (45.7691)</td>
<td></td>
</tr>
<tr>
<td>0.55</td>
<td>0.2617</td>
<td>100.4625</td>
<td>100.4625 (48.5439)</td>
<td></td>
</tr>
<tr>
<td>0.60</td>
<td>0.2602</td>
<td>100.2834</td>
<td>100.2834 (46.6535)</td>
<td></td>
</tr>
<tr>
<td>0.65</td>
<td>0.2587</td>
<td>99.7657</td>
<td>99.7656 (45.5038)</td>
<td></td>
</tr>
<tr>
<td>0.70</td>
<td>0.2572</td>
<td>98.9174</td>
<td>98.9173 (44.6866)</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>0.2558</td>
<td>101.2327</td>
<td>101.2327 (43.4398)</td>
<td></td>
</tr>
<tr>
<td>0.80</td>
<td>0.2543</td>
<td>99.6852</td>
<td>99.6852 (43.9520)</td>
<td></td>
</tr>
<tr>
<td>0.85</td>
<td>0.2529</td>
<td>101.3722</td>
<td>101.3722 (43.8690)</td>
<td></td>
</tr>
<tr>
<td>0.90</td>
<td>0.2514</td>
<td>99.1437</td>
<td>99.1437 (43.5503)</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 2** The numerical results for $ARL_0$ obtained from formula (8) and numerical integral equation for the trend exponential AR(1) processes in EWMA procedure when $\alpha_0 = 5$, the entries inside the parentheses are the CPU times in seconds

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$B$</th>
<th>$ARL_0$</th>
<th>Explicit formula</th>
<th>Integral equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.30</td>
<td>1.6830</td>
<td>300.8045</td>
<td>300.8043 (44.2737)</td>
<td></td>
</tr>
<tr>
<td>0.35</td>
<td>1.6810</td>
<td>300.8805</td>
<td>300.8802 (44.3778)</td>
<td></td>
</tr>
<tr>
<td>0.40</td>
<td>1.6790</td>
<td>300.7862</td>
<td>300.7859 (44.2318)</td>
<td></td>
</tr>
<tr>
<td>0.45</td>
<td>1.6770</td>
<td>300.5217</td>
<td>300.5214 (45.1578)</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>1.6750</td>
<td>300.0877</td>
<td>300.0873 (45.9042)</td>
<td></td>
</tr>
<tr>
<td>0.55</td>
<td>1.6730</td>
<td>299.4855</td>
<td>299.4853 (45.9985)</td>
<td></td>
</tr>
<tr>
<td>0.60</td>
<td>1.6710</td>
<td>298.7169</td>
<td>298.7164 (48.5696)</td>
<td></td>
</tr>
<tr>
<td>0.65</td>
<td>1.6690</td>
<td>297.7841</td>
<td>297.7839 (45.1732)</td>
<td></td>
</tr>
<tr>
<td>0.70</td>
<td>1.6671</td>
<td>302.7630</td>
<td>302.7628 (43.9773)</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>1.6651</td>
<td>301.4660</td>
<td>301.4657 (43.7833)</td>
<td></td>
</tr>
<tr>
<td>0.80</td>
<td>1.6631</td>
<td>300.0088</td>
<td>300.0085 (44.0086)</td>
<td></td>
</tr>
<tr>
<td>0.85</td>
<td>1.6611</td>
<td>298.3959</td>
<td>298.3957 (44.2005)</td>
<td></td>
</tr>
<tr>
<td>0.90</td>
<td>1.6592</td>
<td>302.7290</td>
<td>302.7287 (43.4986)</td>
<td></td>
</tr>
</tbody>
</table>

In Table 1, the $ARL_0$’s for EWMA procedure is reported. Consider, the $ARL_0$’s for the chart on EWMA procedure for with parameters $\gamma = 0$, $\beta = 0.2$, $\lambda = 0.3$ and the trend AR(1) processes with parameter $0.3 \leq \rho \leq 0.9$. We compare the numerical results obtained by explicit formulas with the numerical results via integral equations.

In Table 2, the $ARL_0$’s for EWMA procedure is reported. Consider, the $ARL_0$’s for the chart on EWMA procedure for with parameters $\gamma = 0$, $\beta = 0.2$, $\lambda = 0.3$ and the trend AR(1) processes with parameter $0.3 \leq \rho \leq 0.9$. We compare the numerical results obtained by explicit formulas with the numerical results via integral equations.

In Tables 3-4, we compare the numerical results obtained by explicit formulas were both gives $ARL_0$, and $ARL_0$ for the trend exponential AR(1) processes in EWMA and CUSUM procedures. The trend exponential AR(1) processes with parameters $\rho = 0.4, 0.75,$
\( \alpha_i = 1.01, 1.03, 1.05, 1.07, 1.09, 1.1, 1.2 \). The EWMA procedure with parameters \( \lambda = 0.3 \) and \( \gamma = 0, \beta = 0.2 \). The CUSUM procedure with parameters constants \( a = 2.3 \) and control limits \( h = 3.4 \), respectively. In Table 3-4, it is clear that when a process is correlated, the EWMA procedure performs better than CUSUM procedure when a process is positively autocorrelated. 

VI. Conclusion

Several control charts or procedures have been proposed for autocorrelated data. In this article, we extend to use CUSUM procedure to compare with EWMA. The performance of latter is superior to the former when the magnitudes of shift are small to moderate. The performance of the analytical results for EWMA compared with the analytical results for CUSUM. The performance of the EWMA procedure proposed by [29] is better than the CUSUM procedure proposed by [30] based on ARL for trend exponential AR(1) processes.

REFERENCES