Analytical Method of Average Run Length for Trend Exponential AR(1) Processes in EWMA Procedure

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Abstract— **The Exponentially Weighted Moving Average (EWMA) procedure are used for monitoring and detecting small shifts in the process mean which performs quicker than the Shewhart control chart. Usually, the common assumption of the Statistical Process Control (SPC) is the observations are independent and identically distributed (IID). In practice, however, the observed data are from industry and finance is serially correlated with trend. In this paper, we extend to use CUSUM procedure to compare with EWMA procedure. The performance of latter is superior to the former when the magnitudes of shift are small to moderate. It is shown that EWMA procedure performs better than the CUSUM procedure for the case of trend exponential AR(1) processes.**

Index Terms— **Trend AR(1), Exponentially Weighted Moving Average, Average Run Length, Exponential White Noise**

I. Introduction

The observations are usually independent and identically distributed (IID), but in reality they might be serially correlations with trend. Some researchers have considered the problem of data correlation as it is related to SPC (see [1]). The Exponentially Weighted Moving Average (EWMA) procedures are used to monitor and detect small shifts in the process mean which is quicker than the Shewhart control chart. The control limits and performance measures for EWMA control chart of correlated processes is based on variables or attributes (see [2] and [3]). Recently, several researchers have shown an increasing interest in the formulation and analytical of non-Gaussian models for serially correlated data, e.g., [4] and [5]. Exponential white noise has been studied in the connection with pollution problem (see [6]), and some paper has studied with exponential white noise by [6], [7], [8], [9] and [10]. In our study, an explicit formula for the EWMA chart for trend stationary exponential AR(1) processes is presented. An overview of the EWMA procedure for serially dependent data is given in Section 2. Later, Section 3 reviews the performance method for serially dependent data in EWMA procedure. Next, Section 4 discusses briefly the explicit formula for the average run length (ARL) for trend exponential AR(1) processes in EWMA procedure. In Section 5, a comparison of the performance of the EWMA procedure and CUSUM procedure is made. Finally, Section 6 concludes the discussion in the paper.

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II. A Review of the Trend Exponential AR(1) Processes in EWMA Procedure

In [11], [12], [13], [14] and [15] give detailed explanations EWMA procedure for serially dependent data. In the monitoring of the trend exponential AR(1) process in EWMA procedure, assume that we have the observations $Y_t, \{t = 0,1,2,...\}$ taken over time. The EWMA statistic Z_t is given by:

$$Z_t = (1-\lambda) Z_{t-1} + Y_t \qquad (1)$$

where λ is a smoothing constant $(0 < \lambda < 1)$, the sequence $\{Y_t, t = 1,2,3,...\}$ consists of the trend AR(1) processes and the initial value of Z_0 is usually selected to be the process target of Y_t or the average of random data. The trend AR(1) processes is assumed to be as follow

$$Y_t = \gamma + \beta t + \rho Y_{t-1} + X_t, \quad t \geq 1 \qquad (2)$$

where γ is a constant, β is the trend slope in term of t, and ρ is the autoregressive coefficient $(0 < \rho < 1)$. Let X_t is the independent random error term at time t following $Exp(\alpha)$. The variance of Z_t for the large t will be

$$\sigma_{Z_t}^2 = \left(\frac{\lambda}{2-\lambda}\right) \frac{1+\rho(1-\lambda)}{(1-\rho^2)(1-\rho(1-\lambda))} \sigma_X^2. \qquad (3)$$

Therefore the upper control limit (*UCL*) and lower control limit (*LCL*) for monitoring the process when plotting Z_t versus the time t are

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$$UCL = \mu + L\sigma_X \sqrt{\left(\frac{\lambda}{2-\lambda}\right)\frac{1+\rho(1-\lambda)}{(1-\rho^2)(1-\rho(1-\lambda))}} = B$$
$$LCL = \mu - L\sigma_X \sqrt{\left(\frac{\lambda}{2-\lambda}\right)\frac{1+\rho(1-\lambda)}{(1-\rho^2)(1-\rho(1-\lambda))}} = 0 \quad (4)$$

where L is a constant to be chosen, and σ_X is the standard deviation of a known underlying probability distribution. The process will be declared to be in an out-of-control state when $Z_t > B$. The alarm time for the EWMA in then given by

$$\tau = \inf\{t > 0 : Z_t > B\}. \quad (5)$$

Assume $E_\theta(.)$ denote the expectation at time $\theta,$ where $\theta \le \infty$. The ARLs of the EWMA control chart for the given process are that:

$$ARL_0 = E_\infty(\tau) = T, \quad (6)$$

where T is given (usually large) and

$$ARL_1 = E_1(\tau). \quad (7)$$

III. A REVIEWS OF THE PERFORMANCE METHOD FOR SERIALLY DEPENDENT DATA IN EWMA PROCEDURE

Usually, the performance of the control chart is measured by the average run length (ARL). The ARL_0 is defined as the expectation of false alarm time (τ) before an in-control process is taken to signal to be out of control. For practical purposes, a sufficient large in-control ARL_0 is desired. When the process is out-of-control, the performance of a control chart is usually used as ARL_1. The ARL_1 is defined as the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control. Ideally, ARL_1 should be small.

A control chart based on the Exponentially Weighted Moving Average (EWMA) model was first proposed by [16]. The methods to evaluate the performance of EWMA control charts for serially correlated have been studied by [17]. They used simulation method based on the presence of autocorrelation for EWMA control chart. The ARL, and steady state ARL of EWMA were estimated numerically by [18] using an integral equation approach and a Markov chain approach to investigate EWMA and CUSUM procedures for the process mean when data was described by an AR(1) process with additional random error. The EWMA control charts based on the observations which follow an AR(1) process, plus a random error, and to detect changes in the process mean, or in the process variance, the authors using a simulation approach is discussed by [14]. In [19] presented the ARL of the EWMA control chart for monitoring the mean of an AR(1) process, plus a random error by using an integral equation method. In [20] compared the ARL for the EWMAST chart, the CUSUM residual chart, the EWMA residual chart, the X residual chart, and X chart using simulation. In [21] calculated the ARL of $\overline{X}$ and EWMA charts using analytical and simulation techniques. In [22] studied the EWMA chart with residual-based approaches for detecting process shifts by using simulation. In [23] studied EWMA chart for an AR model and calculated ARL by Markov chain approach. In [24] evaluated the ARL of EWMA charts with heavy tailed distribution for monitoring the mean of the stationary processes by simulation method. In [25] computed exactly *ARL* with the Markov chain approach for a Poisson INAR(1) model of EWMA chart. In [26] designed the ARL performance of autocorrelated process control chart using a Monte Carla simulation. In [27] used finite Markov Chain imbedding technique to investigate the run length properties for control charts when the process observations were autocorrelated. Recently, [28] have derived explicit formula of performance for EWMA control charts for AR(1) process observations with exponential white noise, [29] have derived explicit formula of *ARL* for EWMA control chart for trend stationary exponential AR(1) processes.

IV. EXPLICIT FORMULA FOR TREND EXPONENTIAL AR(1) PROCESSES IN EWMA PROCEDURE

The performance of a control chart is measured by the average run length (ARL). The ARL_0 is defined as the expected of false alarm time (τ) before an in-control process is taken to signal to be out of control. A sufficient large in-control ARL_0 is desired. When the process is out-of-control, the performance of a control chart is usually used as ARL_1. It is the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control. Ideally, ARL_1 should be small. The values of ARL_0 and ARL_1 for an EWMA control chart with exponential white noise observations are derived by [28]. The authors used an integral equation approach and derived a Fredholm integral equation of second type for the *ARL*. The explicit formulas obtained by solving the integral equations are:

$$ARL = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha}}\left(e^{-\frac{B}{\lambda\alpha}} - 1\right)}{\lambda e^{-\frac{a+b+\rho v}{\alpha}} + e^{-\frac{B}{\alpha}} - 1} \quad (8)$$

where γ is a constant, β is the trend slope in term of t, ρ is the autoregressive coefficient $(0<\rho<1)$, α is a parameter of the exponential distribution, λ is a smoothing parameter, u, v are initial values, and B is boundary value.

V. NUMERICAL COMPARISONS OF PERFORMANCE

We present an explicit formula for trend exponential AR(1) processes in EWMA procedure. The numerical results for ARL_0 when $\alpha = \alpha_0$ and ARL_1 when $\alpha = \alpha_1$

are given for the trend exponential AR(1) processes in EWMA procedure was calculated from Eq. (8). To evaluate the performance of a control chart for monitoring trend AR(1) processes in EWMA procedure, we designed the trend exponential AR(1) processes with numerical parameters $0.3 \le \rho \le 0.9, \gamma = 0, \beta = 0.2$ and $Z_0 = Y_0 = 0.1$ with weighting constant $\lambda = 0.3$ is given for an in-control process. The CUSUM procedure was constructed with constants $a = 2,3$ and control limit $h = 3,4$ as suggested by [30]. The characteristics of the control charts measured in terms of ARL are examined for different values of shifts in the mean $\alpha = \alpha_1 = 1.01, 1.03, 1.05, 1.07, 1.09, 1.1, 1.2$.

TABLE 1 THE NUMERICAL RESULTS FOR ARL_0 OBTAINED FROM FORMULA (8) AND NUMERICAL INTEGRAL EQUATION FOR THE TREND EXPONENTIAL AR(1) PROCESSES IN EWMA PROCEDURE WHEN $\alpha_0 = 1$, THE ENTRIES INSIDE THE PARENTHESES ARE THE CPU TIMES IN SECONDS

ρ	B	*ARLs*	
		Explicit formula	Integral equation
0.30	0.2693	99.6997	99.6996 (44.1457)
0.35	0.2678	101.2144	101.2144 (45.4545)
0.40	0.2663	102.4312	102.4312 (44.3564)
0.45	0.2647	99.8073	99.8073 (50.9680)
0.50	0.2632	100.3019	100.3019 (45.7691)
0.55	0.2617	100.4625	100.4625(48.5439)
0.60	0.2602	100.2834	100.2834 (44.6353)
0.65	0.2587	99.7657	99.7656 (45.5038)
0.70	0.2572	98.9174	98.9173 (44.6866)
0.75	0.2558	101.2327	101.2327 (43.4398)
0.80	0.2543	99.6852	99.6852 (43.9520)
0.85	0.2529	101.3722	101.3722 (43.8690)
0.90	0.2514	99.1437	99.1437 (43.5503)

TABLE 2 THE NUMERICAL RESULTS FOR ARL_0 OBTAINED FROM FORMULA (8) AND NUMERICAL INTEGRAL EQUATION FOR THE TREND EXPONENTIAL AR(1) PROCESSES IN EWMA PROCEDURE WHEN $\alpha_0 = 5$, THE ENTRIES INSIDE THE PARENTHESES ARE THE CPU TIMES IN SECONDS

ρ	B	*ARLs*	
		Explicit formula	Integral equation
0.30	1.6830	300.8045	300.8043 (44.2737)
0.35	1.6810	300.8805	300.8802 (43.8778)
0.40	1.6790	300.7862	300.7859 (44.2318)
0.45	1.6770	300.5217	300.5214 (45.1578)
0.50	1.6750	300.0877	300.0875 (43.9042)
0.55	1.6730	299.4855	299.4853 (45.9985)
0.60	1.6710	298.7169	298.7166 (48.5696)
0.65	1.6690	297.7841	297.7839 (45.1732)
0.70	1.6671	302.7630	302.7628 (43.9773)
0.75	1.6651	301.4660	301.4657 (43.7833)
0.80	1.6631	300.0088	300.0085 (44.0086)
0.85	1.6611	298.3959	298.3957 (44.2005)
0.90	1.6592	302.7290	302.7287 (43.4986)

In Table 1, the ARL_0's for EWMA procedure is reported. Consider, the ARL_0's for the chart on EWMA procedure for with parameters $\gamma = 0, \beta = 0.2, \lambda = 0.3$ and the trend AR(1) processes with parameter $0.3 \le \rho \le 0.9$. We compare the numerical results obtained by explicit formulas with the numerical results via integral equations method. Both methods gives ARL_0 for EWMA control chart for trend exponential AR(1) processes when $\alpha_0 = 1$. The explicit formulas give results which are very closed to the numerical integral equations. Notice that, calculations with explicit formula equation (8) are simple and considerable much faster from the point of view of computation times. For example, if we set $\rho = 0.3$, calculations time based on our technique takes less than 1 sec., while the CPU time required to obtain numerical solutions of integral equation for the EWMA run, show inside the brackets is 50-60 times larger. The in-control *ARL*'s for the EWMA procedure used explicit formula and numerical integral equation are reported in Table 2 when $\alpha_0 = 5$.

TABLE 3 COMPARISON OF ARLS VALUES WITH CUSUM FOR TREND EXPONENTIAL AR(1) PROCESSES

ρ	α	EWMA $\lambda = 0.3$	CUSUM $a = 2, h = 3$
0.40	1.00	38.4541	38.5615
	1.01	26.8849	36.8709
	1.03	17.0025	33.8117
	1.05	12.5827	31.1263
	1.07	10.0771	28.7588
	1.09	8.4634	26.6627
	1.10	7.8544	25.7041
	1.20	4.7772	18.5466
0.75	1.00	17.4376	17.4509
	1.01	14.6171	14.0336
	1.03	11.1412	13.1689
	1.05	9.0786	12.4020
	1.07	7.7131	11.7187
	1.09	6.7423	11.1068
	1.10	6.3551	10.8245
	1.20	4.1985	8.6495

TABLE 4 COMPARISON OF ARLS VALUES WITH CUSUM FOR TREND EXPONENTIAL AR(1) PROCESSES

ρ	α	EWMA $\lambda = 0.3$	CUSUM $a = 3, h = 4$
0.40	1.00	437.0680	435.3320
	1.01	72.4236	406.8440
	1.03	27.7236	356.7320
	1.05	17.4499	356.7320
	1.07	12.8821	278.3280
	1.09	10.3013	247.5360
	1.10	9.3906	233.8180
	1.20	5.2439	139.4190
0.75	1.00	256.7971	257.6000
	1.01	63.7623	241.4070
	1.03	26.0144	212.8330
	1.05	16.6214	188.5710
	1.07	12.3577	167.8600
	1.09	9.9230	150.0900
	1.10	9.0594	142.1500
	1.20	5.0976	87.0196

In Tables 3-4, we compare the numerical results obtained by explicit formulas were both gives ARL_0 and ARL_1 for the trend exponential AR(1) processes in EWMA and CUSUM procedures. The trend exponential AR(1) processes with parameters $\rho = 0.4, 0.75$,

$\alpha_1 = 1.01, 1.03, 1.05, 1.07, 1.09, 1.1, 1.2$. The EWMA procedure with parameters $\lambda = 0.3$ and $\gamma = 0, \beta = 0.2$. The CUSUM procedure with parameters constants $a = 2,3$ and control limits $h = 3,4$, respectively. In Table 3-4, it is clear that when a process is correlated, the EWMA procedure performs better than CUSUM procedure when a process is positively autocorrelated.

VI. CONCLUSION

Several control charts or procedures have been proposed for autocorrelated data. In this article, we extend to use CUSUM procedure to compare with EWMA. The performance of latter is superior to the former when the magnitudes of shift are small to moderate. The performance of the analytical results for EWMA compared with the analytical results for CUSUM. The performance of the EWMA procedure proposed by [29] is better than the CUSUM procedure proposed by [30] based on ARL for trend exponential AR(1) processes.

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