Optimal Investment under Inflation Protection and Optimal Portfolios with Stochastic Cash Flows Strategy

Charles I. Nkeki* and Chukwuma R. Nwozo

Abstract—**In this paper, we considers the optimal portfolio strategies and expected wealth with stochastic cash flows under inflation protection for an investment company (IC). The IC trade on a complete diffusion model, receives a stochastic cash inflows and pays a stochastic outflows to its holder. The cash inflows are invested into a market that is characterized by a cash account, an inflation-linked bond and a stock. The inflation risks associated with the investment could be hedged by investing in inflation-linked bond. The utility function is assumed to be a quasi-concave function of the value of wealth of the IC. It was found that as the market evolve, parts of the inflation-linked bond and stock portfolio values should be transferred to cash account. It was also found that the portfolio processes involved inter-temporal hedging terms that offset any shock to both the stochastic cash inflows and cash outflows.**

Index Terms—**optimal portfolios, stochastic cash flows, cash inflows, cash outflows, inflation protection, quasi-concave.**
AMS Subject Classifications. 91B28, 91B30, 91B70, 93E20.

I. INTRODUCTION

This paper consider optimal portfolios and investment strategies for IC who received continuous-time stochastic cash inflows and pays continuously a stochastic cash outflows to its holder. The cash inflows are invested into a cash account, an inflation-linked bond and a stock. Inflation-linked bonds are bonds with interest rates that varies according to inflation. An inflation-linked bond, for example, may pay a fixed coupon plus an additional coupon with the amount adjusted periodically according to some inflation indicator, such as the Consumer Price Index. If these bonds are held to maturity, then the investor guarantees that the return will exceed the rate of inflation. Inflation-linked bonds exist to provide a low-risk investment vehicle in which the return is guaranteed not to fall below the rate of inflation. They are also called indexed bonds. Inflation-linked bonds are generally less risky than stocks, as they attract interest at a predetermined rate and have guaranteed returns. Inflation-linked bonds can be used to hedge inflation risk.

A non-linear partial differential equation (Hamilton-Jacobi-Equation (HJB)) was derived from the expected utility of wealth and then, the optimal portfolio values for the IC were determined using power utility function. It was assumed that the underlying assets, cash inflows and cash outflows are driven by a standard geometric Brownian motion with constant drifts and volatilities.

Today, inflation risk is of increase. As a result of tremendous increase in inflation risk in nations economy, investment companies have started investing optimally, the inflows paid by the holders into inflation-linked bonds. Although, according to [11], the demand in inflation-linked bonds among private investors is rather low. It is therefore, strongly recommended for firms whose profits are indeed negatively correlated with inflation. Example of such firms are insurance companies, pension funds companies, e.t.c.

In related literature, [4] examined the rationale, nature and financial consequences of two alternative approaches to portfolio regulations for the long-term institutional investor sectors of life insurance and pension funds. [3] considered the deterministic life styling (the gradual switch from equities to bonds according to preset rules) which is a popular asset allocation strategy during the accumulation phase of a defined contribution pension plans which is designed to protect the pension funds from a catastrophic fall in the stock market just prior to retirement. They shown that this strategy, although easy to understand and implement can be highly suboptimal. [1] modeled and analyzed the *ex ante* liquidity premium demanded by the holder of an "illiquid annuity". The annuity is an insurance product that is similar to a pension savings account with both an accumulation and "decumulation" phase. They computed the yield needed to compensate for the utility welfare loss, which is induced by the inability to re-balance and maintain an optimal portfolio when holding an annuity. [5],[6],[7] considered the optimal design of the minimum guarantee in a defined contribution pension fund scheme. They studied the investment in the financial market by assuring that

*Charles I. Nkeki is with the Department of Mathematics, Faculty of Physical Sciences, University of Benin, P. M. B. 1154, Benin City, Edo State, Nigeria. email: nkekicharles2003@yahoo.com.
Chukwuma R. Nwozo is with the Department of Mathematics, Faculty of Science, University of Ibadan, Ibadan, Oyo State, Nigeria. email: crnwozo@yahoo.com.

(Advance online publication: 21 May 2013)

the pension fund optimizes its retribution which is a part of the surplus, that is the difference between the pension fund value and the guarantee. [17] considered the optimal management and inflation protection strategy for defined contribution pension plans using Martingale approach. They derived an analytical expression for the optimal strategy and expresses it in terms of observable market variables. Our aim is to determine the portfolio values and expected terminal wealth for the IC. Also, to determine to which extent cash inflows and outflows should be hedged. [13],[14],[15] studied the variational form of classical portfolio strategy and expected wealth for a pension plan member. They assumed that the growth rate of salary was linear function of time and that the the cash inflow was stochastic.

The remainder of this paper is organized as follows. In section 2, we present the problem formulation and financial market models. Section 3 presents the wealth dynamics of the IC. In section 4, we present the discounted cash inflows and cash outflows processes, the definition of present value of the expected flows of future cash inflows process as well as the expected cash outflows process. Section 5 present the dynamics of the values of the wealth process for the IC. In section 6, we present the optimal portfolio strategies for the IC. In section 7, we present the optimal expected value of wealth at time t and at the terminal period for the IC. Section 8 presents some special cases arising from the problem. Finally, section 9 concludes the paper.

II. PROBLEM FORMULATION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathbb{F}(\mathcal{F}) = \{\mathcal{F}_t : t \in [0,T]\}$, where $\mathcal{F}_t = \sigma(W^I(t), W^S(t) : s \leq t)$, the Brownian motions $W(t) = (W^I(t), W^S(t))'$ is a 2-dimensional process, defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}(\mathcal{F}), \mathbb{P})$, $t \in [0,T]$, where $\mathbb{P}$ is the real world probability measure, t the time period, T the terminal time period. $W^I(t)$ is the Brownian motion with respect to source of uncertainty arising from inflation and $W^S(t)$ is the Brownian motion with respect to source of uncertainty arising from the stock market. $\sigma^S = (\sigma_1^S, \sigma_2^S)$ and $\sigma^I = (\sigma_I, 0)$ are the volatility vector of stock and volatility vector of the inflation-linked bond with respect to changes in $W^S(t)$ and $W^I(t)$. μ is the appreciation rate for stock. Moreover, σ^S and σ^I referred to as the coefficients of the market and are progressively measurable with respect to the filtration $\mathcal{F}$.

We assume that the IC faces a market that is characterized by a risk-free asset (cash account) and two risky assets, all of whom are tradeable. In this paper, we allow the stock price to be correlated to inflation. Also, we correlated the cash inflows and outflows to stock market in other to determine the extent to which cash inflows and outflows should be hedged. The dynamics of the underlying assets are given by (1) to (3)

$$\begin{aligned} &dC(t) = rC(t)dt, \\ &C(0) = 1 \end{aligned} \tag{1}$$

$$\begin{aligned} &dS(t) = \mu S(t)dt + \sigma_1^S S(t)dW^I(t) + \sigma_2^S S(t)dW^S(t), \\ &S(0) = s_0 > 0 \end{aligned} \tag{2}$$

$$\begin{aligned} &dB(t,Q(t)) = (r + \sigma_I\theta^I)B(t,Q(t))dt + \sigma_I B(t,Q(t))dW^I(t), \\ &B(0) = b > 0 \end{aligned} \tag{3}$$

where,
r is the nominal interest rate,
θ^I is the price of inflation risk,
$C(t)$ is the price process of the cash account at time t,
$S(t)$ is stock price process at time t,
$Q(t)$ is the inflation index at time t and has the dynamics:
$dQ(t) = E(q)Q(t)dt + \sigma_I Q(t)dW^I(t)$,
where $E(q)$ is the expected rate of inflation, which is the difference between nominal interest rate, r and real interest rate R (i.e. $E(q) = r - R$).
$B(t,Q(t))$ is the inflation-indexed bond price process at time t.
Then, the volatility matrix

$$\Sigma := \begin{pmatrix} \sigma_I & 0 \\ \sigma_1^S & \sigma_2^S \end{pmatrix} \tag{4}$$

corresponding to the two risky assets and satisfies $det(\Sigma) = \sigma_I\sigma_2^S \neq 0$. Therefore, the market is complete and there exists a unique market price θ satisfying

$$\theta := \begin{pmatrix} \theta^I \\ \theta^S \end{pmatrix} = \begin{pmatrix} \theta^I \\ \dfrac{\mu - r - \theta^I\sigma_1^S}{\sigma_2^S} \end{pmatrix} \tag{5}$$

where θ^S is the market price of stock risks and θ^I is the market price of inflation risks (MPIR). We now define the following exponential process which we assumed to be Martingale in $\mathbb{P}$:

$$Z(t) = \exp(-\theta' W(t) - \frac{1}{2}\|\theta\|^2 t). \tag{6}$$

where, $W(t) = (W^I(t), W^S(t))'$. We assume in this paper that the cash inflows process $\varphi(t)$ at time t and cash outflows process $L(t)$ at time t follow the dynamics, respectively presented in (7) and (8).

$$\begin{aligned} &d\varphi(t) = \varphi(t)(\omega dt + \sigma_1^\varphi dW^I(t) + \sigma_2^\varphi dW^S(t)), \\ &\varphi(0) = \varphi_0 > 0 \end{aligned} \tag{7}$$

$$\begin{aligned} &dL(t) = L(t)(\delta dt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t)), \\ &L(0) = L_0 > 0 \end{aligned} \tag{8}$$

where, $\omega > 0$ is the expected growth rate of the cash inflows and σ_1^φ is the volatility caused by the source of inflation, $W^I(t)$ and σ_2^φ is the volatility caused by the source of uncertainty arises from the stock market, $W^S(t)$, and $\delta > 0$ is the expected growth rate of the cash outflows and σ_1^L is the volatility caused by the source of inflation, $W^I(t)$ and σ_2^L is the volatility caused by the source of uncertainty arises from the stock market, $W^S(t)$.

We now define the cross correlation matrices

$\Sigma^{\varphi,S} := \begin{pmatrix} \sigma_1^\varphi & \sigma_1^S \\ \sigma_2^\varphi & \sigma_2^S \end{pmatrix}, \Sigma^{L,S} := \begin{pmatrix} \sigma_1^L & \sigma_1^S \\ \sigma_2^L & \sigma_2^S \end{pmatrix},$
$\Sigma^{I,\varphi} := \begin{pmatrix} \sigma_I & \sigma_1^\varphi \\ 0 & \sigma_2^\varphi \end{pmatrix}, \Sigma^{I,L} := \begin{pmatrix} \sigma_I & \sigma_1^L \\ 0 & \sigma_2^L \end{pmatrix},$
$\Sigma^{S,I} := \begin{pmatrix} \sigma_1^S & \sigma_I \\ \sigma_2^S & 0 \end{pmatrix}.$

Applying *Itô* Lemma on (7) and (8), we have the following

$$\varphi(t) = \varphi_0 \exp((\omega - \frac{1}{2}\|\sigma^\varphi\|^2)t + \theta' W(t)), \quad (9)$$

where, $\sigma^\varphi = (\sigma_1^\varphi, \sigma_2^\varphi)'$.

$$L(t) = L_0 \exp((\delta - \frac{1}{2}\|\sigma^L\|^2)t + \theta' W(t)), \quad (10)$$

where, $\sigma^L = (\sigma_1^L, \sigma_2^L)'$.

III. THE WEALTH PROCESS

Let $X^{\Delta,\varphi,L}(t)$ be the wealth process at time t, where $\Delta(t) = (\Delta^I(t), \Delta^S(t))$ is the portfolio process at time t and $\Delta^I(t)$ is the proportion of wealth invested in the inflation-linked bond at time t and $\Delta^S(t)$ is the proportion of wealth invested in stock at time t. Then, $\Delta_0(t) = 1 - \Delta^I(t) - \Delta^S(t)$ is the proportion of wealth invested in cash account at time t.

Definition 1. *The portfolio process Δ is said to be self-financing if the corresponding wealth process $X^{\Delta,\varphi,L}(t)$, $t \in [0,T]$, satisfies*

$$\begin{aligned} &dX^{\Delta,\varphi,L}(t) = \Delta^S(t) X^{\Delta,\varphi,L}(t)\frac{dS(t)}{S(t)} \\ &+\Delta^I(t) X^{\Delta,\varphi,L}(t)\frac{dB(t,Q(t))}{B(t,Q(t))} \\ &+(1-\Delta^S(t)-\Delta^I(t)) X^{\Delta,\varphi,L}(t)\frac{dC(t)}{C(t)} \\ &+(\varphi(t)-L(t))dt, \\ &X^{\Delta,\varphi,L}(0) = x. \end{aligned} \quad (11)$$

IV. DISCOUNTED CASH FLOW PROCESSES

Definition 2. *The expected value of discounted future cash inflows process is defined as*

$$\Psi(t) = E[\int_t^T \frac{\Lambda(u)}{\Lambda(t)}\varphi(u)du|\mathcal{F}(t)], \quad (12)$$

where, $\Lambda(t) = \frac{Z(t)}{C(t)} = \exp(-rt)Z(t)$ is the stochastic discount factor which adjusts for nominal interest rate and market price of risks, and $E(.|\mathcal{F}(t))$ is a real world conditional expectation with respect to the Brownian filtration $(\mathcal{F}(t))_{t\geq 0}$. For detail on real world measure $\mathbb{P}$, see [12], [15], [16].

Proposition 1. *Let $\Psi(t)$ be the expected value of the discounted future cash inflows (EVDFCI) process, then*

$$\Psi(t) = \frac{\varphi(t)}{\phi}(\exp[\phi(T-t)] - 1), \quad (13)$$

where $\phi = \omega - r - \theta^\varphi \cdot \theta$.

Proof: By definition,

$$\begin{aligned} \Psi(t) &= E[\int_t^T \frac{\Lambda(u)}{\Lambda(t)}\varphi(u)du|\mathcal{F}(t)] \\ &= \varphi(t)E[\int_t^T \frac{\Lambda(u)\varphi(u)}{\Lambda(t)\varphi(t)}du|\mathcal{F}(t)]. \end{aligned}$$

But, the processes $\Lambda(.)$ and $\varphi(.)$ are geometric Brownian motions and it follows that $\frac{\Lambda(u)\varphi(u)}{\Lambda(t)\varphi(t)}$ is independent of the Brownian filtration $\mathcal{F}(t)$, $u \geq t$. Hence,

$$\begin{aligned} \Psi(t) &= \varphi(t)E[\int_0^{T-t} \Lambda(s)\frac{\varphi(s)}{\varphi(0)}ds] \\ &= \varphi(t)E[\int_0^{T-t} \exp(-rs)Z(s)\exp((\omega \\ &- \frac{1}{2}\|\sigma^\varphi\|^2)s + \sigma^{\varphi\prime}W(s))ds] \\ &= \varphi(t)E[\int_0^{T-t} \exp(-rs - \frac{1}{2}\|\theta\|^2 s - \theta' W(s)) \\ &+ (\omega - \frac{1}{2}\|\sigma^\varphi\|^2)s + \sigma^{\varphi\prime}W(s))ds] \\ &= \varphi(t)E[\int_0^{T-t} \exp((\omega - r)s\exp(-\frac{1}{2}(\|\theta\|^2 \\ &+ \|\sigma^\varphi\|^2)s + (\sigma^\varphi - \theta)' W(s)ds] \\ &= \varphi(t)E[\int_0^{T-t} \exp((\omega - r - \sigma^\varphi\cdot\theta)s\exp(-\frac{1}{2}(\|\sigma^\varphi \\ &- \theta\|^2 s + (\sigma^\varphi - \theta)' W(s)ds] \\ &= \varphi(t)[\int_0^{T-t} \exp(\phi s)ds] \end{aligned}$$

where, $\phi = \omega - r - \sigma^\varphi \cdot \theta$.
Therefore,

$$\Psi(t) = \frac{\varphi(t)}{\phi}[\exp(\phi(T-t) - 1]. \quad (14)$$

Proposition 1 tells us that the value of expected future cash inflows process $\Psi(t)$ is proportional to the instantaneous total cash inflows process $\varphi(t)$. Observe that at time T, the value of the inflow of cash is zero. This is because the value Ψ_0 has been invested while setting up the investment.
Taking the differential of both sides of (14), we obtain

$$\begin{aligned} &d\Psi(t) = \Psi(t)[(r + \sigma_1^\varphi\theta^I + \sigma_2^\varphi\theta^S)dt + \sigma_1^\varphi dW^I(t) \\ &+\sigma_2^\varphi dW^S(t)] - \varphi(t)dt. \end{aligned} \quad (15)$$

Definition 3. *The expected discounted cash outflows process at time t is defined as*

$$\Phi(t) = E\left[\int_t^{T+t} \frac{\Lambda(u)}{\Lambda(t)}L(u)du|\mathcal{F}(t)\right], T \geq t. \quad (16)$$

The contingent claim $L(t)$ that matures at the stopping time $t \in [0,T]$ is an $\mathcal{F}(t)$-measurable non-negative payoff that possesses a finite expectation. As outlined in [14], the value $\Phi(t)$ (the cash outflows process) can be obtained at time t by the real-world pricing formula given in (16).

Proposition 2. *Let $\Phi(t)$ be the expected discounted cash outflows (EDCO) process, then*

$$\Phi(t) = \frac{L(t)}{\beta}\left(1 - \exp[-\beta T]\right), \tag{17}$$

where $\beta = \delta - r - \sigma^L \cdot \theta$.

Proof: By definition,

$$\begin{aligned}\Phi(t) &= E\left[\int_t^{T+t} \frac{\Lambda(u)}{\Lambda(t)} L(u) du | \mathcal{F}(t)\right] \\ &= L(t) E\left[\int_t^{T+t} \frac{\Lambda(u)L(u)}{\Lambda(t)L(t)} du | \mathcal{F}(t)\right].\end{aligned}$$

But, the processes $\Lambda(.)$ and $L(.)$ are geometric Brownian motions and it follows that $\frac{\Lambda(u)L(u)}{\Lambda(t)L(t)}$ is independent of the Brownian filtration $\mathcal{F}(t)$, $u \geq t$. Adopting change of variables, we have

$$\begin{aligned}\Phi(t) &= L(t) E\left[\int_0^T \Lambda(\tau)\frac{L(\tau)}{L(0)} d\tau\right] \\ &= L(t) E\left[\int_0^T \exp(-r\tau) Z(\tau) \frac{L(\tau)}{L(0)} d\tau\right]\end{aligned}$$

Using (10), we have

$$\begin{aligned}&\Phi(t) = L(t)E[\int_0^T \exp(-r\tau)Z(\tau)\exp((\delta \\ &-\frac{1}{2}\|\sigma^L\|^2)\tau + \sigma^{L\prime} W(\tau)) d\tau]\end{aligned} \tag{18}$$

Applying parallelogram law on (18), we have

$$\begin{aligned}&\Phi(t) = L(t)E[\int_0^T \exp(\delta - r)\tau \exp((-\frac{1}{2}\|\sigma^L \\ &-\theta\|^2 - \sigma^L \cdot \theta)\tau + (\sigma^L - \theta)' W(\tau)) d\tau]\end{aligned} \tag{19}$$

Simplifying, (19), we have

$$\begin{aligned}\Phi(t) &= L(t)E[\int_0^T \exp(\delta - r - \sigma^L \cdot \theta)\tau \exp((-\frac{1}{2}\|\sigma^L \\ &- \theta\|^2)\tau + (\sigma^L - \theta)' W(\tau)) d\tau] \\ &= L(t)\int_0^T \exp(\delta - r - \sigma^L \cdot \theta)\tau d\tau \\ &= L(t)\int_0^T \exp(\beta\tau) d\tau\end{aligned}$$

where $\beta = \delta - r - \sigma^L \cdot \theta$.
Therefore,

$$\Phi(t) = \frac{L(t)}{\beta}[1 - \exp(-\beta T)]. \tag{20}$$

Proposition 2 tells us that the expected discounted cash outflows process $\Phi(t)$ is proportional to the instantaneous total cash outflows process $L(t)$.

Lemma 1. *Let $\Phi(t)$ be the expected discounted cash outflows process, then*

$$d\Phi(t) = \Phi(t)[\alpha dt + {\sigma_1}^L dW^I(t) + {\sigma_2}^L dW^S(t)] - L(t)dt,$$

where $\alpha = \dfrac{\delta(1 - \exp(-\beta T)) + \beta}{1 - \exp(-\beta T)}$.

Proof: Taking the differential of both sides of (20), we obtain

$$\begin{aligned}d\Phi(t) &= \left(\frac{1 - \exp(-\beta T)}{\beta}\right) dL(t) \\ &= \left(\frac{1 - \exp(-\beta T)}{\beta}\right) L(t)((\delta + \frac{\beta}{1 - \exp(-\beta T)})dt \\ &+ {\sigma_1}^L dW^I(t) + {\sigma_2}^L dW^S(t)) - L(t)dt \\ &= \left(\frac{1 - \exp(-\beta T)}{\beta}\right) L(t)(\alpha dt \\ &+ {\sigma_1}^L dW^I(t) + {\sigma_2}^L dW^S(t)) - L(t)dt\end{aligned}$$

Therefore,

$$d\Phi(t) = \Phi(t)(\alpha dt + {\sigma_1}^L dW^I(t) + {\sigma_2}^L dW^S(t)) - L(t)dt, \tag{21}$$

where, $\alpha = \dfrac{\delta(1 - \exp(-\beta T)) + \beta}{1 - \exp(-\beta T)}$.
Obviously, the dynamics of the cash outflows and cash inflows processes in this paper have similar features. The difference is that the formal is seen as a form of cash outflow to be received by the holder at the maturity date while the later is seen as a form of cash inflow that is invested optimally by the IC.

V. WEALTH VALUATION OF THE IC

Definition 4. *The value of wealth process of the IC at time t is define as*

$$V(t) = X^{\Delta,\varphi,L}(t) + \Psi(t) - \Phi(t). \tag{22}$$

The value of wealth, $V(t)$ equals the wealth, $X^{\Delta,\varphi,L}(t)$ plus the discounted expected value of future cash inflows, $\Psi(t)$ less the discounted expected value of cash outflows, $\Phi(t)$.

Proposition 3. *The change in wealth of the IC is given by the dynamics*

$$\begin{aligned}&dV(t) = (\Delta^S(t)X^{\Delta,\varphi,L}(t)(\mu - r) \\ &+\Delta^I(t)X^{\Delta,\varphi,L}(t)\theta^I\sigma_I + rX^{\Delta,\varphi,L}(t) \\ &+\Psi(t)(r + \sigma^\varphi \cdot \theta) - \alpha\Phi(t))dt \\ &+(\Delta^S(t)X^{\Delta,\varphi,L}(t)\sigma_1^S + \Delta^I(t)X^{\Delta,\varphi,L}(t)\sigma_I \\ &+\Psi(t)\sigma_1^\varphi - \sigma_1^L\Phi(t))dW^I(t) + (\Delta^S(t)\times \\ &X^{\Delta,\varphi,L}(t)\sigma_2^S + \Psi(t)\sigma_2^\varphi - \sigma_2^L\Phi(t))dW^S(t), \\ &V(0) = v = x + \Psi(0) - \Phi(0).\end{aligned} \tag{23}$$

Proof: Taking the differential of both sides of (22) and substituting in (11) and (15), the result follows. In the next section, we present the optimal portfolio strategies for the IC.

VI. OPTIMAL PORTFOLIO STRATEGIES FOR THE IC

In this section, we consider the optimal portfolio process for the IC. We define the general value function

$$J(t,V_\Delta,\Delta)=E[u(V_\Delta(T))|X^{\Delta,\varphi,L}(t)=X,\\ \Psi(t)=\Psi,\Phi(t)=\Phi]$$

where $V_\Delta(t)$ is the path of $V(t)$ given the portfolio strategy $\Delta(t)=(\Delta^S(t),\Delta^I(t))$. Define $A(V)$ to be the set of all admissible portfolio strategy that are $\mathcal{F}_V$-progressively measurable, and let $U(V(t))$ be a quasi-concave function in $V(t)$ such that

$$U(V(t))=\sup_{\Delta\in A(V)}E[U(V(T))|X^{\Delta,\varphi,L}(t)=X,\\ \Psi(t)=\Psi,\Phi(t)=\Phi]=\sup_{\Delta\in A(V)}J(t,V_\Delta,\Delta) \tag{24}$$

Then $U(t,V)$ satisfies the HJB equation

$$U_t+\sup_{\Delta\in A(V)}\mathcal{H}(t,V,\Delta)=0,\\ \text{subject to:}U(T,v)=\frac{1}{\gamma}v^\gamma. \tag{25}$$

where,

$$\begin{aligned}&\mathcal{H}(t,V,\Delta)=(\Delta^S(t)X(\mu-r)+\Delta^I(t)X\sigma_I\theta^I)U_X\\&+\frac{1}{2}((\sigma_1^S)^2X^2\Delta^S(t)^2+{\sigma_I}^2X^2\Delta^I(t)^2\\&+(\sigma_2^S)^2X^2\Delta^S(t)^2)U_{XX}+\Delta^S(t)\Delta^I(t)X^2\sigma_1^S\sigma_IU_{XX}\\&+\Delta^S(t)\Psi X\sigma_1^S\sigma_1^\varphi U_{X\Psi}-\Delta^S(t)\Phi X\sigma_1^S\sigma_1^LU_{X\Phi}\\&+\Delta^I(t)\Psi X\sigma_I\sigma_1^\varphi U_{X\Psi}-\Delta^I(t)\Phi X\sigma_I\sigma_1^LU_{X\Phi}\\&+\Delta^S(t)\Psi X\sigma_2^S\sigma_2^\varphi U_{X\Psi}-\Delta^S(t)\Phi X\sigma_2^S\sigma_2^LU_{X\Phi}+rXU_X\\&+\Psi(r+\sigma^\varphi\cdot\theta)U_\Psi-\Phi\alpha U_\Phi+\frac{1}{2}(\sigma_1^\varphi)^2\Psi^2U_{\Psi\Psi}\\&-\frac{1}{2}(\sigma_1^L)^2\Phi^2U_{\Phi\Phi}-\Psi\Phi\sigma_1^\varphi\sigma_1^LU_{\Psi\Phi}-\Psi\Phi\sigma_2^\varphi\sigma_2^LU_{\Psi\Phi}\\&+\frac{1}{2}\Psi^2(\sigma_2^\varphi)^2U_{\Psi\Psi}-\frac{1}{2}\Phi^2(\sigma_2^L)^2U_{\Phi\Phi}.\end{aligned} \tag{26}$$

For the explicit solution of (25), see Appendix.
Since U is a quasi-concave function in $V(t)$ and $U(t,V)\in C^{1,2}(R\times[0,T])$, then (25) is well defined. The HJB equation (25) was solve explicitly (See the Appendix). Finding the partial derivative of $\mathcal{H}(t,V,\Delta)$ with respect to $\Delta^S(t)$ and $\Delta^I(t)$ and set to zero, we obtain the following

$$(\Delta^S(t))^*=\frac{-(\mu-r)U_X-\Delta^I(t)^*X\sigma_1^S\sigma_IU_{XX}}{((\sigma_1^S)^2+(\sigma_2^S)^2)XU_{XX}}\\-\frac{\Psi(\sigma_1^S\sigma_1^\varphi+\sigma_2^S\sigma_2^\varphi)U_{X\Psi}+\Phi(\sigma_1^S\sigma_1^L+\sigma_2^S\sigma_2^L)U_{X\Phi}}{((\sigma_1^S)^2+(\sigma_2^S)^2)XU_{XX}} \tag{27}$$

$$(\Delta^I(t))^*=\\ \frac{-\theta^IU_X-\Delta^S(t)^*X\sigma_1^SU_{XX}-\Psi\sigma_1^\varphi U_{X\Psi}+\Phi\sigma_1^LU_{X\Phi}}{X\sigma_IU_{XX}}. \tag{28}$$

Substituting (28) into (27), we obtain the following

$$(\Delta^S(t))^*=\\ \frac{\left(\sigma_1^S\theta^I-(\mu-r)\right)U_X-\Psi\sigma_2^S\sigma_2^\varphi U_{X\Psi}+\Phi\sigma_2^S\sigma_2^LU_{X\Phi}}{(\sigma_2^S)^2XU_{XX}}. \tag{29}$$

Substituting (29) into (28), we obtain the following

$$(\Delta^I)(t)^*=\frac{\left((\mu-r)\sigma_1^S-\theta^I\left((\sigma_2^S)^2+(\sigma_1^S)^2\right)\right)U_X}{X\sigma_I(\sigma_2^S)^2U_{XX}}\\+\frac{\Psi\sigma_2^S|\Sigma^{\varphi,S}|U_{X\Psi}+\Phi\sigma_2^S|\Sigma^{L,S}|U_{X\Phi}}{X\sigma_I(\sigma_2^S)^2U_{XX}}. \tag{30}$$

Proposition 4. *Suppose that $U(V)=\frac{V^\gamma}{\gamma}$, $\gamma<1$,$\gamma\neq0$, then the optimal portfolio values of the IC in stock market and inflation-linked bond are respectively given as*

$$(\Delta^S(t))^*=H_1(t)\left(\frac{\mu-r-\sigma_1^S\theta^I}{(1-\gamma)(\sigma_2^S)^2}\right)\\-H_2(t)\frac{\sigma_2^\varphi}{\sigma_2^S}-H_3(t)\frac{\sigma_2^L}{\sigma_2^S}. \tag{31}$$

$$(\Delta^I(t))^*=H_1(t)\left(\frac{\theta^I\left((\sigma_2^S)^2+(\sigma_1^S)^2\right)-(\mu-r)\sigma_1^S}{(1-\gamma)\sigma_I(\sigma_2^S)^2}\right)\\+H_2(t)\frac{|\Sigma^{\varphi,S}|}{\sigma_2^S\sigma_I}-H_3(t)\frac{|\Sigma^{L,S}|}{\sigma_2^S\sigma_I}. \tag{32}$$

$$(\Delta_0(t))^*=1+H_2(t)\left(\frac{\sigma_2^\varphi}{\sigma_2^S}-\frac{|\Sigma^{\varphi,S}|}{\sigma_2^S\sigma_I}\right)\\+H_3(t)\left(\frac{|\Sigma^{L,S}|}{\sigma_2^S\sigma_I}+\frac{\sigma_2^L}{\sigma_2^S}\right)-\\H_1(t)\left(\frac{\theta^I\left((\sigma_2^S)^2+(\sigma_1^S)^2\right)-(\mu-r)\sigma_1^S}{(1-\gamma)\sigma_I(\sigma_2^S)^2}+\frac{\mu-r-\sigma_1^S\theta^I}{(1-\gamma)(\sigma_2^S)^2}\right). \tag{33}$$

where, $H_1(t)=\frac{X(t)+\Psi(t)-\Phi(t)}{X(t)}$, $H_2(t)=\frac{\Psi(t)}{X(t)}$, $H_3(t)=\frac{\Phi(t)}{X(t)}$.

Proof: Given $U(X,\Psi,\Phi)=\frac{(X+\Psi-\Phi)^\gamma}{\gamma}$. Then, finding the following partial derivatives: U_X, U_Ψ, U_Φ, U_{XX}, $U_{X\Psi}$, $U_{X\Phi}$ and substitute into (29) and (30), the result follows.

Numerical Example 1. *The figures below, represent the portfolio values of the investment in stock, inflation-linked bond and cash, respectively. They are obtained by taking $L_0=120$ Naira, $\delta=0.09$, $\sigma_1^L=0.31$, $\sigma_2^L=0.41$, $\mu=0.1$, $r=0.04$, $\sigma_1^S=0.35$, $\sigma_2^S=0.45$, $\theta^I=0.13$, $\gamma=0.5$, $\varphi_0=100$ Naira, $\omega=0.08$, $\sigma_1^\varphi=0.32$, $\sigma_2^\varphi=0.4$, $\sigma_I=0.418$ and $T=10$ and Mathematica 6.0 was used for the simulations.*

We observed that: If we divide the right hand side of (34) by $H_1(t)$ and then add $\frac{H_2(t)}{H_1(t)}\frac{\sigma_2^\varphi}{\sigma_2^S}$ and $\frac{H_3(t)}{H_1(t)}\frac{\sigma_2^L}{\sigma_2^S}$ to the result, we obtain the classical portfolio strategy, $\frac{\mu-r-\sigma_1^S\theta^I}{(1-\gamma)(\sigma_2^S)^2}$ in the stock market. In the same vain, if we divide the right hand side of (35) by

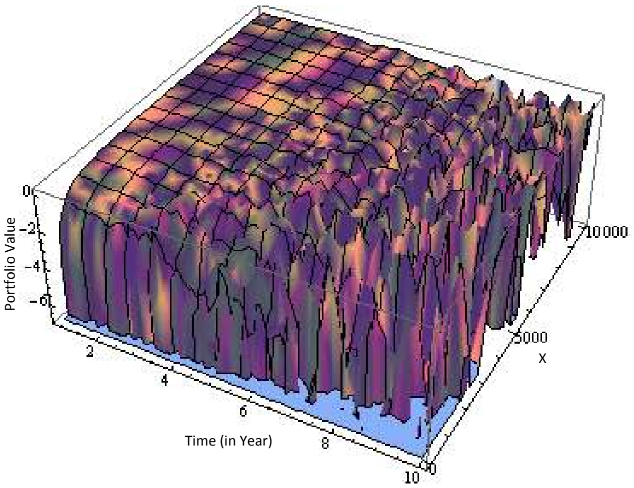

Figure 1: Portfolio value in stock. This figure is obtained by setting $L_0 = 120$ Naira, $\delta = 0.09$, $\sigma_1^L = 0.31$, $\sigma_2^L = 0.41$, $\mu = 0.1$, $r = 0.04$, $\sigma_1^S = 0.35$, $\sigma_2^S = 0.45$, $\theta^I = 0.13$, $\gamma = 0.5$, $\varphi_0 = 100$ Naira, $\omega = 0.08$, $\sigma_1^\varphi = 0.32$, $\sigma_2^\varphi = 0.4$, $\sigma_I = 0.418$ and $T = 10$

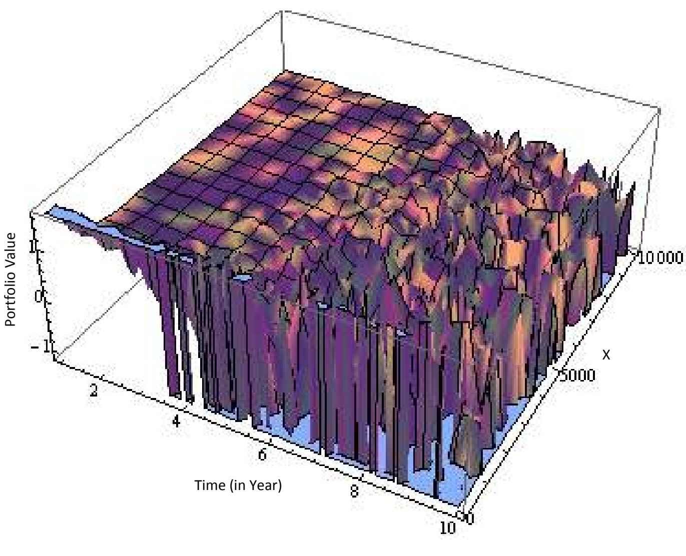

Figure 2: Portfolio value in inflation-linked bond. This figure is obtained by setting $L_0 = 120$ Naira, $\delta = 0.09$, $\sigma_1^L = 0.31$, $\sigma_2^L = 0.41$, $\mu = 0.1$, $r = 0.04$, $\sigma_1^S = 0.35$, $\sigma_2^S = 0.45$, $\theta^I = 0.13$, $\gamma = 0.5$, $\varphi_0 = 100$ Naira, $\omega = 0.08$, $\sigma_1^\varphi = 0.32$, $\sigma_2^\varphi = 0.4$, $\sigma_I = 0.418$ and $T = 10$

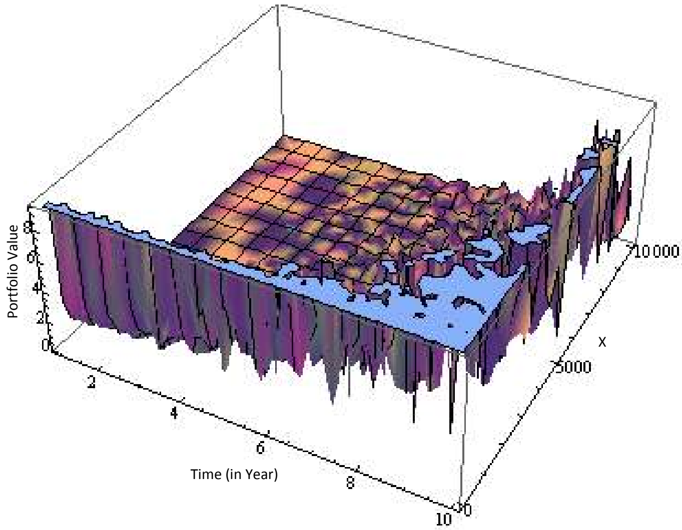

Figure 3: Portfolio value in cash account. This figure is obtained by setting $L_0 = 120$ Naira, $\delta = 0.09$, $\sigma_1^L = 0.31$, $\sigma_2^L = 0.41$, $\mu = 0.1$, $r = 0.04$, $\sigma_1^S = 0.35$, $\sigma_2^S = 0.45$, $\theta^I = 0.13$, $\gamma = 0.5$, $\varphi_0 = 100$ Naira, $\omega = 0.08$, $\sigma_1^\varphi = 0.32$, $\sigma_2^\varphi = 0.4$, $\sigma_I = 0.418$ and $T = 10$

$H_1(t)$ and then add $-\dfrac{H_2(t)}{H_1(t)}\dfrac{|\Sigma^{\varphi,S}|}{\sigma_2^S\sigma_I}$ and $\dfrac{H_3(t)}{H_1(t)}\dfrac{|\Sigma^{L,S}|}{\sigma_2^S\sigma_I}$ to the result, we obtain the classical portfolio strategy, $\dfrac{\theta^I((\sigma_2^S)^2 + (\sigma_1^S)^2) - (\mu - r)\sigma_1^S}{(1-\gamma)\sigma_I(\sigma_2^S)^2}$ in inflation-linked bond. Therefore, the first part of the portfolios in inflation-linked bond and stock market are the variational form of the classical optimal portfolio rule. The second parts of the portfolios containing the function $H_2(t)\dfrac{\sigma_2^\varphi}{\sigma_2^S}$ and $H_2(t)\dfrac{|\Sigma^{\varphi,S}|}{\sigma_2^S\sigma_I}$ hedges the shock associated with the cash inflows whose present value Ψ_0 has been invested while at the beginning of the planning horizon. The third parts with the function $H_3(t)\dfrac{|\Sigma^{L,S}|}{\sigma_2^S\sigma_I}$ and $H_3(t)\dfrac{\sigma_2^L}{\sigma_2^S}$ hedges the shock associated with the cash outflows. Therefore, the second and the third terms in the portfolio values are the inter-temporal hedging terms that offset any shock to both stochastic inflows and outflows, respectively.

The above formulas depend on the optimal wealth value, which consists of the optimal wealth level $X^*(t)$, the expected future cash inflows, $\Psi(t)$ and the expected cash outflows, $\Phi(t)$. The first ones are observable, the second parts reflect the expectation of the IC on the future cash inflows, the third parts reflect the expectation of the IC on the cash outflows, the fourth and fifth terms reflect the inter-temporal hedging terms that simply offset any shock to both the stochastic cash inflows and cash outflows. These inter-temporal hedging terms will in a way protect the IC from a catastrophic fall in the risky assets. Hence, protect the IC from the risk of not meeting

its claims obligations to the holder.
Now, at $t = 0$, we have

$$(\Delta^S(0))^* = H_1(0)[\frac{\mu - r - \sigma_1^S\theta^I}{(1-\gamma)(\sigma_2^S)^2}] - H_2(0)\frac{\sigma_2^\varphi}{\sigma_2^S} - H_3(0)\frac{\sigma_2^L}{\sigma_2^S}. \quad (34)$$

$$(\Delta^I(0))^* = H_1(0)(\frac{\theta^I((\sigma_2^S)^2 + (\sigma_1^S)^2) - (\mu - r)\sigma_1^S}{(1-\gamma)\sigma_I(\sigma_2^S)^2}) + H_2(0)\frac{|\Sigma^{\varphi,S}|}{\sigma_2^S\sigma_I} - H_3(0)\frac{|\Sigma^{L,S}|}{\sigma_2^S\sigma_I}. \quad (35)$$

where, $H_1(0) = \dfrac{x_0 + \Psi(0)}{x_0}$, $H_2(0) = \dfrac{\Psi(0)}{x_0}$, $H_3(0) = \dfrac{\Phi(0)}{x_0}$.

VII. EXPECTED VALUE OF WEALTH (EVW) FOR THE IC

In this section, we consider the optimal value of wealth for the investment company at time t and at the terminal period. From (23), we have that

$$\begin{aligned} &dE[V(t)] = E[\Delta^S(t)^* X^{\Delta,\varphi,L}(t)(\mu - r) \\ &+\Delta^I(t)^* X^{\Delta,\varphi,L}(t)\theta^I\sigma_I + rX^{\Delta,\varphi,L}(t) \\ &+\Psi(t)(r + \sigma_1^\varphi\theta^I + \sigma_2^\varphi\theta^S) - \alpha\Phi(t)]dt, \\ &E[V(0)] = v_0, \end{aligned} \quad (36)$$

where, $v_0 = x_0 + \Psi_0 - \Phi_0$, $\Psi_0 = \dfrac{\varphi_0(\exp(\phi T) - 1)}{\phi}$, $\Phi_0 = \dfrac{L_0(1 - \exp(-\beta T))}{\beta}$.
Substituting (14),(20),(29),(30) into (36) and for simplicity, we set $r = \delta$, to obtain

$$\begin{aligned} &dE[V(t)] = [y_1E(V(t)) + y_2\exp(\omega t)(\exp(\phi(T-t)) - 1) \\ &+y_3\exp(rt)]dt \end{aligned} \quad (37)$$

$E[V(0)] = v_0$,
where,

$$y_1 = \frac{\theta^S(\mu - r)}{1-\gamma} + \frac{\theta^I(\theta^I + \sigma_1^S\theta^S)}{1-\gamma} + r,$$

$$y_2 = \frac{\varphi_0}{\phi}\left(\frac{\theta^I|\Sigma^{S,\varphi}|}{\sigma_2^S} - \frac{\sigma_2^\varphi(\mu - r)}{\sigma_2^S} + \sigma^\varphi\cdot\theta\right),$$

$$y_3 = \frac{L_0(\exp(-\beta T) - 1)}{\beta}\times \left(\frac{\sigma_2^L(\mu - r)}{\sigma_2^S} + \frac{\theta^I|\Sigma^{L,S}|}{\sigma_2^S} - \frac{\sigma^L\cdot\theta}{1 - \exp(-\beta T)}\right),$$

By solving the ODE (37), we find that the expected value of the wealth for the IC under optimal control at time t is

$$\begin{aligned} &E(V(t)) = \\ &\left(v_0 + \frac{y_2}{\omega - y_1} + \frac{y_3}{y_1 - r} + \frac{y_3y_1^2}{(\omega - y_1)(\omega - \phi - y_1)(y_1 - r)}\right) \\ &\times\exp(y_1t) + \frac{y_2\exp(\phi T)(\exp(\omega t) - \exp((\phi + y_1)t))}{\omega - \phi - y_1} \\ &-\frac{y_2\exp((\omega + \phi)t)}{\omega - y_1} - \frac{y_3\exp(rt)}{y_1 - r}. \end{aligned} \quad (38)$$

At terminal time T, we have:

$$\begin{aligned} &E(V(T)) = \\ &\left(v_0 + \frac{y_2}{\omega - y_1} + \frac{y_3}{y_1 - r} + \frac{y_3y_1^2}{(\omega - y_1)(\omega - \phi - y_1)(y_1 - r)}\right) \\ &\times\exp(y_1T) + \frac{y_2\exp(\phi T)(\exp(\omega T) - \exp((\phi + y_1)T))}{\omega - \phi - y_1} \\ &-\frac{y_2\exp((\omega + \phi)T)}{\omega - y_1} - \frac{y_3\exp(rT)}{y_1 - r}. \end{aligned} \quad (39)$$

Therefore, the expected optimal value of final wealth for the IC is given in (39).
Therefore, the optimal value of final wealth for the IC is the sum of the fund, $(v_0 + \dfrac{y_2}{\omega - y_1} + \dfrac{y_3}{y_1 - r} + \dfrac{y_3y_1^2}{(\omega - y_1)(\omega - \phi - y_1)(y_1 - r)})\exp(y_1T)$ that the IC would get investing the whole portfolio always in the risk-less asset and the risky assets, plus the term, $\dfrac{y_2\exp(\phi T)(\exp(\omega T) - \exp((\phi + y_1)T))}{\omega - \phi - y_1} - \dfrac{y_2\exp((\omega + \phi)T)}{\omega - y_1}$ that depend both on the goodness of the risky assets with respect to the risk-less asset and the stochastic cash inflows, plus the term, $\dfrac{y_3\exp(rT)}{y_1 - r}$ that depend on the risk-less asset, the risky assets and the cash outflows. Thus, we observe by the definition of θ and y_1, that the higher the *Sharpe ratio* of the risky asset, θ the higher the expected optimal value of final wealth, everything else being equal. We take $r = 0.04$, $\mu = 0.1$, $\gamma = 0.9$, $\omega = 0.08$, $\sigma_1^S = 0.35$, $\sigma_2^S = 0.45$, $\sigma_I = 0.418$, $\varphi_0 = 10,000$ Naira, $L_0 = 12,000$Naira, and $x_0 = 1$, $\sigma_1^\varphi = 0.32$, $\sigma_2^\varphi = 0.40$, $\sigma_1^L = 0.31$, $\sigma_2^L = 0.41$, T=20 to obtain figure 4. Figure 4 shows the expected value of wealth for the IC at different value of MPIR (θ^I) at time t. At the terminal time, observe that when θ^I equals 0.10, the EVW is obtained to be 34,799,000 Naira; when θ^I is 0.13, the EVW is 43,562,000 Naira and when θ^I equals 0.15, we have that the EVW is 55,107,000 Naira. We conclude therefore that for all other parameters remain fixed, the higher the market price of inflation risks, the higher the expected value of returns for the investment and vice versa. This is only true for $t > 15$. At $t < 15$, observe the the reverse was the case.

VIII. THE SPECIAL CASE $\varphi_0 = 0$, $L_0 = 0$, $\omega = 0$ and $\delta = 0$

By set $\varphi_0 = 0$, $L_0 = 0$, $\omega = 0$ and $\delta = 0$, we obtained the usual portfolio selection problem. It is obvious from

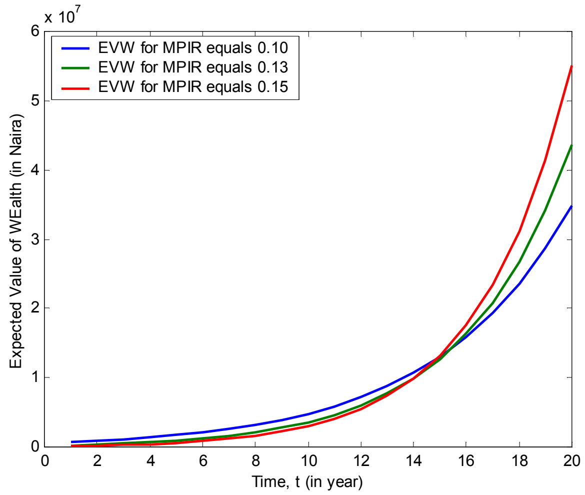

Figure 4: Expected value of wealth for the investment.

the previous analysis, that equalities still hold for (11), if $\varphi_0 = 0$, $L_0 = 0$, $\omega = 0$ and $\delta = 0$ provided that the initial wealth is greater than zero. We now summarize the expected terminal wealth of the investor, taking $\varphi_0 = 0$, $L_0 = 0$, $\omega = 0$ and $\delta = 0$ in (9) and (10).

Corollary 1. *Assume that an investor wants to invest a wealth of $x_0 > 0$ for the time horizon $T > 0$ in a financial market as in section (2.1) and wealth equation (6). Assume that the investor maximizes the expected utility of final wealth at time T. Then,*
(i)$E(V(T)) = v_0 \exp(y_1 T)$;
(ii) $E(X(T)) = x_0 \exp(y_1 T)$;
(iii) $E(V(T)) = E(X(T))$.

Observe that if $y_2 = 0$ and $y_3 = 0$, the expected value of final wealth becomes $x_0 \exp(y_1 T)$. We now have the following corollary.

Corollary 2. *Suppose that $y_2 = 0$, $y_3 = 0$ and $x_0 > 0$ in (39), then*

$$E(V(T)) = x_0 \exp(y_1 T).$$

Corollary 3. *Suppose that $y_2 = 0$, $y_3 = 0$, $\theta^I = 0$ and $x_0 > 0$ in (39), then*

$$E(V(T)) = x_0 \exp(\bar{y}_1 T),$$

where, $\bar{y}_1 = \dfrac{\theta^S(\mu - r)}{1-\gamma} + r.$

Corollary 4. *Suppose that $y_2 = 0$, $y_3 = 0$, $\theta^I = 0$, $\mu = r$ and $x_0 > 0$ in (39), then*

$$E(V(T)) = x_0 \exp(rT).$$

Corollary 5. *Suppose that $y_2 = 0$, $y_3 = 0$, $\theta^S = 0$ and $x_0 > 0$ in (39), then*

$$E(V(T)) = x_0 \exp(\bar{z} T),$$

where, $\bar{z} = \dfrac{(\theta^I)^2}{1-\gamma} + r.$

Corollary 3 presents the expected wealth from the usual portfolio selection problem that involve investment into a riskless asset (cash account) and a stock. Corollary 4 presents the expected wealth from the usual portfolio selection problem that involves investment into a riskless asset alone. Corollary 5 presents the expected wealth from the usual portfolio selection problem that involve investment into a riskless asset and an inflation-linked bond.
Consider another special case were $y_1 = 0$, $y_2 \neq 0$ and $y_3 \neq 0$. In that case, the expected value of wealth for the IC becomes

$$E(V(T)) = v_0 - \frac{y_3}{\delta}(1 - \exp(\delta T)) + y_2 \times \left(\frac{\exp(\phi T)(\exp(\omega T) - \exp(\phi T))}{\omega - \phi} - \frac{\exp((\omega - \phi)T)}{\omega} + \frac{1}{\omega} \right).$$

It means that the cash inflows were either not invested into any of the underlying assets, or there is no contribution to the portfolio of the IC for investing into the underlying assets. In that case, the expected value of wealth for the IC can only be affected by the face value of the cash inflows and cash outflows.

IX. CONCLUSION

The paper examined the optimal portfolios with stochastic cash inflows and outflows and expected terminal wealth for IC. The portfolio values and expected terminal wealth that accrued to the IC were obtained. It was found that the optimal share of portfolios in stock and inflation-linked bond ultimately depend on cash inflows, cash outflows and the optimal wealth level of the investment at time t. It was also found that as the markets evolve, parts of the portfolio values in stock and inflation-linked bond should be transferred to the cash account. The portfolio processes were found to involved inter-temporal hedging terms that offset any shock to the cash inflows and cash outflows.

REFERENCES

[1] S. Brawne, A. M. Milevsky and T. S. Salisbury, Asset allocation and the liquidity premium for illiquid annuities. Manuscript, 2003.

[2] A. J. G. Cairns, D. Blake, and K. Dowd, Optimal dynamic asset allocation for defined-contribution pension plans, *In proceedings of the 10th AFIR International Colloquium*, 2000.

[3] A. J. G. Cairns, D. Blake, and K. Dowd, Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans, *CRIS Discussion paper series, The University of Nottingham*, 2003.

[4] P. E. Davis, Portfolio regulation of life insurance companies and pension funds, Oxford University Press, 2000.

[5] G. Deelstra, M. Grasselli and P. Koehl, Optimal design of the guarantee for defined contribution funds, Manuscript, 2002.

[6] G. Deelstra, M. Grasselli and P. Koehl, Optimal investment strategies in the presence of a minimum guarantee, *Insurance: Mathematics and Economics*, 33(2003), pp. 189-207.

[7] G. Deelstra, M. Grasselli and P. Koehl, Optimal design of the guarantee for defined contribution funds, *Journal of Economics Dynamics and Control*, 28(2004), pp. 2239-2260.

[8] D. Dentcheva and A. Ruszczynski, Portfolio optimization with stochastic dominance constraints, *SIAM Journal on Optimization*, 14, 2003, pp. 548-566.

[9] I. Fisher, The theory of interest, MacMillan press Ltd, London, 1930.

[10] B. Hojgaard and E. Vigna, Mean-variance portfolio selection and efficient frontier for defined contribution pension schemes, Manuscript, 2007.

[11] R. Korn and S. Kruse, Einfache verfahren zur bewertung von inflationsgekopelten finanzprodukten, *Blitter der DGVFM, XXVI*, 3 2004, pp.351-367.

[12] S. Miller, Analytic pricing of contingent claim under the real world measure, *International Journal of Theoretical and Applied Finance*, 11, 2008, pp.841-867.

[13] C. R. Nwozo and C. I. Nkeki, Optimal portfolio and strategic consumption planning in a life-cycle of a pension plan member in a defined contributory pension scheme, *IAENG International Journal of Applied Mathematics*, 41(4) 2011, pp.299-309.

[14] C. I. Nkeki, On optimal portfolio management of the accumulation phase of a defined contributory pension scheme, Ph.D thesis, 2011, Department of Mathematics, University of Ibadan, Ibadan, Nigeria.

[15] C. I. Nkeki and C. R. Nwozo, Variational form of classical portfolio strategy and expected wealth for a defined contributory pension scheme, *Journal of Mathematical Finance*, 2(1), 2012, pp.132-139.

[16] E. Platen, Real world pricing of long terms contracts, QFRC Report 262, University of Technology, Sydney.

[17] A. Zhang, R. Korn and C. O. Ewald, Optimal management and inflation protection for defined contribution pension plans, Worker paper, University of St. Andrews, 2007.

APPENDIX

Therefore, substituting (29) and (30) into (25), we obtain following the HJB equation

$$\begin{aligned}
&U_t + rXU_X + \Psi(r + \sigma^{\varphi}\cdot\theta)U_{\Psi} - \alpha\Phi U_{\Phi}\\
&+\frac{1}{2}((\sigma_1^{\varphi})^2 + (\sigma_2^{\varphi})^2)\Psi^2 U_{\Psi\Psi}\\
&-\Psi\Phi(\sigma_1^{\varphi}\sigma_1^{L} + \sigma_2^{\varphi}\sigma_2^{L})U_{\Psi\Phi}\\
&-\frac{1}{2}\Phi^2((\sigma_1^{L})^2 + (\sigma_2^{L})^2)U_{\Phi\Phi}\\
&+\frac{(\mu - r)\big(a_1U_X^2 - \Psi a_2 U_X U_{X\Psi} + \Phi b_2 U_X U_{X\Phi}\big)}{\sigma_2^S U_{XX}}\\
&+\frac{\theta^I\big(b_1U_X^2 + \Psi a_3 U_X U_{X\Psi} + \Phi a_4 U_X U_{X\Phi}\big)}{(\sigma_2^S)^4 U_{XX}}\\
&+\frac{\Psi\sigma_1^{\varphi}\big(b_1U_XU_{X\Psi} + \Psi a_3 U_{X\Psi}^2 + \Phi a_4 U_{X\Phi}U_{X\Psi}\big)}{(\sigma_2^S)^2 U_{XX}}\\
&-\frac{\Phi\sigma_1^{L}\big(b_1U_XU_{X\Phi} + \Psi a_3 U_{X\Psi}U_{X\Phi} + \Phi a_4 U_{X\Phi}^2\big)}{(\sigma_2^S)^2 U_{XX}}\\
&+\frac{(\sigma_1^S)^2(a_1^2U_X^2 + \Psi^2a_2^2U_{X\Psi}^2 + \Phi^2b_2^2U_{X\Phi}^2 - 2\Psi a_1a_2U_XU_{X\Psi})}{2(\sigma_2^S)^4U_{XX}}\\
&+\frac{(\sigma_1^S)^2(2\Phi a_1b_2U_XU_{X\Phi} - 2\Psi\Phi a_2b_2U_{X\Phi}U_{X\Psi})}{2(\sigma_2^S)^4U_{XX}}\\
&+\frac{(b_1^2U_X^2 + \Psi^2a_3^2U_{X\Psi}^2 + \Phi^2a_4^2U_{X\Phi}^2 + 2\Psi b_1a_3U_XU_{X\Psi}}{2(\sigma_2^S)^4U_{XX}}\\
&+\frac{2\Phi b_1a_4U_XU_{X\Phi} + 2\Psi\Phi a_3a_4U_{X\Phi}U_{X\Psi})}{2(\sigma_2^S)^4U_{XX}}\\
&+\frac{\sigma_1^S\big(a_1b_1U_X^2 + \Psi^2a_1a_3U_XU_{X\Psi} + \Phi a_1a_4U_XU_{X\Phi}\big)}{(\sigma_2^S)^4U_{XX}}\\
&-\frac{\sigma_1^S\big(2\Psi b_1a_2U_XU_{X\Psi} + \Psi^2a_2a_3U_{X\Psi}^2\big)}{(\sigma_2^S)^4U_{XX}}\\
&+\frac{\Psi^2a_4a_2U_{X\Phi}U_{X\Psi} + b_1b_2\Phi U_XU_{X\Phi}}{(\sigma_2^S)^4U_{XX}}\\
&+\frac{a_3b_2\Phi\Psi U_{X\Phi}U_{X\Psi} + \Phi^2a_4b_2U_{X\Phi}^2)}{(\sigma_2^S)^4U_{XX}}\\
&+\left(\frac{\Psi\sigma_1^S\sigma_1^{\varphi}}{(\sigma_2^S)^2U_{XX}} + \frac{\Psi\sigma_2^{\varphi}}{\sigma_2^SU_{XX}}\right)(a_1U_XU_{X\Psi}\\
&-\Psi a_2U_{X\Psi}^2 + \Phi b_2U_{X\Psi}U_{X\Phi})\\
&-\left(\frac{\Phi\sigma_1^S\sigma_1^{L}}{(\sigma_2^S)^2U_{XX}} + \frac{\Phi\sigma_2^{L}}{\sigma_2^SU_{XX}}\right)(a_1U_XU_{X\Phi}\\
&-\Psi a_2U_{X\Psi}U_{X\Phi} + \Phi b_2U_{X\Phi}^2) = 0.
\end{aligned}\tag{40}$$

where $a_1 = \sigma_1^S\theta^I - (\mu - r)$, $b_1 = -(a_1\sigma_1^S + \theta^I(\sigma_2^S)^2)$, $a_2 = \sigma_2^S\sigma_2^{\varphi}$, $b_2 = \sigma_2^S\sigma_2^L$, $a_3 = \sigma_2^S|\Sigma^{\varphi,S}|$, $a_4 = \sigma_2^S|\Sigma^{L,S}|$. We assume the solution of the form $U(t,v) = \dfrac{(vg(t))^{\gamma}}{\gamma}$ such that $g(T) = 1$. Then,

$$U_t = (vg(t))^{\gamma-1}g'(t) \tag{41}$$

$$U_X = v^{\gamma-1}g(t)^{\gamma} \tag{42}$$

$$U_{XX} = (\gamma-1)v^{\gamma-2}g(t)^{\gamma} \tag{43}$$

$$U_{X\Psi} = (\gamma-1)v^{\gamma-2}g(t)^{\gamma} \tag{44}$$

$$U_{X\Phi} = -(\gamma-1)v^{\gamma-2}g(t)^{\gamma} \tag{45}$$

$$U_{\Psi} = v^{\gamma-1}g(t)^{\gamma} \tag{46}$$

$$U_{\Psi\Psi} = (\gamma-1)v^{\gamma-2}g(t)^{\gamma} \tag{47}$$
$$U_{\Psi\Phi} = -(\gamma-1)v^{\gamma-2}g(t)^{\gamma} \tag{48}$$
$$U_{\Phi} = -v^{\gamma-1}g(t)^{\gamma} \tag{49}$$
$$U_{\Phi\Phi} = (\gamma-1)v^{\gamma-2}g(t)^{\gamma} \tag{50}$$

Substituting the partial derivatives (41)-(50) into (40), we obtain the following HJB equation

$$\begin{aligned}
&v\frac{g'(t)}{g(t)} + rv^2 + (\Psi\sigma^{\varphi}\cdot\theta + \sigma^L\cdot\theta\Phi)v \\
&+\frac{\mu-r}{\sigma_2^S}\left(a_1v - \Psi a_2 - \Phi b_2\right)v \\
&+\frac{1}{2}(((\sigma_1^{\varphi})^2 + (\sigma_2^{\varphi})^2)\Psi^2 + 2\Psi\Phi(\sigma_1^{\varphi}\sigma_1^L \\
&+\sigma_2^{\varphi}\sigma_2^L) - \Phi^2((\sigma_1^L)^2 + (\sigma_1^L)^2))(\gamma-1) \\
&+\frac{\theta^I}{(\sigma_2^S)^4}(b_1v + \Psi a_3 - \Phi a_4)v \\
&+\frac{\Psi\sigma_1^{\varphi}}{(\sigma_2^S)^2}(b_1v + (\Psi a_3 - \Phi a_4)(\gamma-1)) \\
&+\frac{\Phi\sigma_1^L}{(\sigma_2^S)^2}(b_1v + (\Psi a_3 + \Phi a_4)(\gamma-1)) \\
&+\frac{\Phi}{\sigma_2^S}\left(\frac{\sigma_1^S\sigma_1^L}{\sigma_2^S} + \sigma_2^L\right)(a_1v - (\Psi a_2 - \Phi b_2)(\gamma-1))v \\
&+\frac{(\sigma_1^S)^2}{2(\sigma_2^S)^4}(a_1^2v + (\Psi a_2 + \Phi b_2)^2\times \\
&(\gamma-1) - 2\Psi a_1a_2 - 2\Phi a_1b_2) \\
&+\frac{1}{2(\sigma_2^S)^4}(b_2^2v^2 + (\Psi a_3 + \Phi b_4)^2(\gamma-1) + 2\Psi b_1a_3 \\
&-2\Phi b_1a_4v) + \frac{1}{2(\sigma_2^S)^4}(a_1b_1v^2 + (\Psi^2a_1a_3 - \Psi b_1a_2 \\
&-\Phi a_1a_4)v - \Psi^2a_2a_3(\gamma-1)) + \frac{1}{(\sigma_2^S)^4}((\Phi^2a_4b_4 - \Phi\Psi b_2a_3 \\
&-\Psi^2a_4a_2)(\gamma-1) - b_1b_2\Phi v) + \frac{\Psi}{\sigma_2^S}\left(\frac{\sigma_1^S\sigma_1^{\varphi}}{\sigma_2^S} + \sigma_1^{\varphi}\right) \\
&\times(a_1v - (\Psi a_2 + \Phi b_2)(\gamma-1)) = 0.
\end{aligned} \tag{51}$$

We now set $\xi = rv^2 + (\Psi\sigma^{\varphi}\cdot\theta + \sigma^L\cdot\theta\Phi)v$

$$\begin{aligned}
&+\frac{\mu-r}{\sigma_2^S}\left(a_1v - \Psi a_2 - \Phi b_2\right)v \\
&+\frac{1}{2}(((\sigma_1^{\varphi})^2 + (\sigma_2^{\varphi})^2)\Psi^2 + 2\Psi\Phi(\sigma_1^{\varphi}\sigma_1^L \\
&+\sigma_2^{\varphi}\sigma_2^L) - \Phi^2((\sigma_1^L)^2 + (\sigma_1^L)^2))(\gamma-1) \\
&+\frac{\theta^I}{(\sigma_2^S)^4}(b_1v + \Psi a_3 - \Phi a_4)v \\
&+\frac{\Psi\sigma_1^{\varphi}}{(\sigma_2^S)^2}(b_1v + (\Psi a_3 - \Phi a_4)(\gamma-1)) \\
&+\frac{\Phi\sigma_1^L}{(\sigma_2^S)^2}(b_1v + (\Psi a_3 + \Phi a_4)(\gamma-1)) \\
&+\frac{\Phi}{\sigma_2^S}\left(\frac{\sigma_1^S\sigma_1^L}{\sigma_2^S} + \sigma_2^L\right)(a_1v - (\Psi a_2 - \Phi b_2)(\gamma-1))v \\
&+\frac{(\sigma_1^S)^2}{2(\sigma_2^S)^4}(a_1^2v + (\Psi a_2 + \Phi b_2)^2\times \\
&(\gamma-1) - 2\Psi a_1a_2 - 2\Phi a_1b_2) \\
&+\frac{1}{2(\sigma_2^S)^4}(b_2^2v^2 + (\Psi a_3 + \Phi b_4)^2(\gamma-1) + 2\Psi b_1a_3 \\
&-2\Phi b_1a_4v) + \frac{1}{2(\sigma_2^S)^4}(a_1b_1v^2 + (\Psi^2a_1a_3 - \Psi b_1a_2 \\
&-\Phi a_1a_4)v - \Psi^2a_2a_3(\gamma-1)) + \frac{1}{(\sigma_2^S)^4}((\Phi^2a_4b_4 - \Phi\Psi b_2a_3 \\
&-\Psi^2a_4a_2)(\gamma-1) - b_1b_2\Phi v) + \frac{\Psi}{\sigma_2^S}\left(\frac{\sigma_1^S\sigma_1^{\varphi}}{\sigma_2^S} + \sigma_1^{\varphi}\right) \\
&\times(a_1v - (\Psi a_2 + \Phi b_2)(\gamma-1)),
\end{aligned}$$

so that (51) becomes

$$\frac{g'(t)}{g(t)} + \frac{\xi}{v} = 0. \tag{52}$$

Solving (52), we have

$$g(t) = g(0)e^{-\frac{\xi}{v}t}. \tag{53}$$

Therefore,

$$U(t,v) = \frac{(vg(0))^{\gamma}}{\gamma}e^{-\frac{\xi\gamma}{v}t}.$$

But, $U(0,v) = \frac{(vg(0))^{\gamma}}{\gamma}$. This is the present value of expected future utility of wealth for the IC.
Hence,

$$U(t,v) = U(0,v)e^{-\frac{\xi\gamma}{v}t}.$$

This is the expected utility of the value of wealth accrued to the IC at time t.