Solving Second Order Linear Dirichlet and Neumann Boundary Value Problems by Block Method

Zanariah Abdul Majid, Mohd Mughti Hasni and Norazak Senu

Abstract—In this paper, the direct three-point block one-step methods are considered for solving linear boundary value problems (BVPs) with two different types of boundary conditions which is the Dirichlet and Neumann boundary conditions. This method will solve the second order linear BVPs directly without reducing it to the system of first order equations. The direct solution of these two types of BVPs will be calculated at three points simultaneously using constant step size. This method will be used together with the linear shooting technique to construct the numerical solution. The implementation is based on the predictor and corrector formulas in the $PE(CE)^r$ mode. Numerical results are given to show the performance of this method compared to the existing methods.

Index Terms—dirichlet boundary value problems, neumann boundary value problems, block method

I. INTRODUCTION

MANY problems in science and technology are formulated in boundary value problems as in diffusion, heat transfer, deflection in cables and the modeling of chemical reaction. There are several types of boundary value problems (BVPs) and some of them depend on the boundary condition itself. In this paper, we consider the second order linear two-point BVPs which as follows:

$$y'' = p(x)y' + q(x)y + r(x), \ q(x) > 0, \ [a,b] \tag{1}$$

with the Dirichlet boundary conditions:

$$y(a) = \alpha \ \text{and} \ y(b) = \beta \tag{2}$$

and with the Neumann boundary conditions:

$$y'(a) = \delta \ \text{and} \ y'(b) = \mu. \tag{3}$$

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Theorem:
Suppose the function f in the BVPs in (1) is continuous on the set

$$D = \left\{ \left(x, y, y'\right) \middle| a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty \right\},$$

and the partial derivatives f_y and $f_{y'}$ are also continuous on D. If

(i) $f_y\left(x, y, y'\right) > 0$ for all $\left(x, y, y'\right) \in D$, and

(ii) A constant M exists, with

$$\left| f_{y'}\left(x, y, y'\right) \right| \le M, \text{ for all } \left(x, y, y'\right) \in D$$

Then, the BVPs has a unique solution.

Corollary:
In linear BVPs in (1) satisfies

(i) $p(x), q(x)$, and $r(x)$ are continuous on $[a,b]$,

(ii) $q(x) > 0$ on $[a,b]$,

then, the problem has unique solution.

In literature, there are several researcher that has been conducted the research for solving the BVPs such as Wang *et al.* [14], Emad *et al.* [9] and Bongsoo [13].

Adomian Decomposition Method (ADM) has been widely used by many researchers for solving differential and integral problems. However, this method has dealt with some difficulties for solving the problem involving with the boundary conditions. Thus, Bongsoo [13] in 2008 have proposed a method called the Extended Adomian Decomposition Method (EADM) for solving the two-point linear and nonlinear second order boundary value problems. Bongsoo [13] managed to overcome the problems inhibit in the ADM by creating a new canonical form containing all boundary conditions make it suitable for solving the BVPs.

Finite Difference (FDM), Finite Element (FEM) and Finite Volume (FVM) methods have been proposed by Fang *et al.* [2] in 2002 to solve the two-point BVPs. The comparison in terms of the accuracy has been made between these three methods and the results shown that they are comparable to each other with no remarkable differences. Later in 2006, Caglar *et al.* [1] have proposed a new method called the B-spline Interpolation (CBIM) to compare the accuracy of the results obtained with the methods proposed by Fang *et al.* [2] in 2002. Clearly, the CBIM managed to give better results. A few years later, Hamid *et al.* [3] have proposed a new method called the Extended Cubic B-Spline method (ECBIM) for solving the linear two-point BVPs by applying the same procedure in CBIM in [1] but using the extended version cubic

B-spline. Hamid *et al.* [3] have compared the results obtained by ECBIM with the CBIM in [1], FDM in [2], FEM in [2] and FVM in [2]. The results have shown that this method which is the ECBIM was better in terms of their accuracy.

Many works has been done to solve the BVPs with the Dirichlet boundary condition but there is not much attention pay to the BVPs with the Neumann boundary condition. However, there are several researcher that have shown their interest for solving the BVPs with the Neumann boundary conditions such as Li-Bin *et al.* [11], Ramadan *et al.* [10], Emad *et al.* [9] and Siraj-ul-Islam *et al.* [12].

Ramadan *et al.* [10] in 2007, have proposed the polynomial and nonpolynomial spline approaches to the numerical solution of second order BVPs subjected to Neumann boundary conditions. The numerical results have shown that the accuracy of the nonpolynomial spline method was better than the two polynomial spline methods which are the quadratic and cubic splines. Recently in 2011, Li-Bin *et al.* [11] managed to come out with a new numerical based on the interpolation by quartic spline functions to solve the second order BVPs with Neumann conditions. Li-Bin *et al.* [11] have compared their numerical result with Ramadan *et al.* [10]. The results have shown that the method proposed by Li-Bin *et al.* [11] was better.

Several researchers such as Fatunla [4], Majid *et al.* [5], Rosser [7] and Shampine *et al.* [8] have proposed a block method which computes simultaneously the solution values at different points along the interval. One-step block method being referred as one previous point to obtain the solution.

Recently in 2011, Mukhtar *et al.* [15] have derived the three-point block one-step method in their paper for solving general second order ordinary differential equations (ODEs) directly without reducing it into the system of first order equations. The results showed that it is better in terms of the accuracy and the computed times. Thus, we are motivated to implement this method for solving the two types of linear BVPs subjected to the Dirichlet and Neumann boundary conditions respectively.

In this research, we will extend the idea in [15] for solving (1) with both condition (2) and (3) respectively without reducing it into the system of first-order ODEs by using the direct three-point block one-step methods with the linear shooting techniques for solving second order linear Dirichlet and Neumann BVPs.

II. FORMULATION

The formulation of the direct three-point block one-step method for solving second order linear BVPs will be based in Mukhtar *et al.* [15]. This method will compute three approximation values which is y_{n+1} at x_{n+1} , y_{n+2} at x_{n+2} and y_{n+3} at x_{n+3} simultaneously.

In Fig. 1, the interval of $[a,b]$ need to be divided into a series of block with each block containing three points with the step size $3h$. Then, the solution obtained in the last point within the k th block will be restored as the initial value for the next block. The same procedure will be repeated to compute the solutions until the end of the interval. This method possesses the desirable feature of one-step method in which

being referred to only one previous point to obtain the solutions.
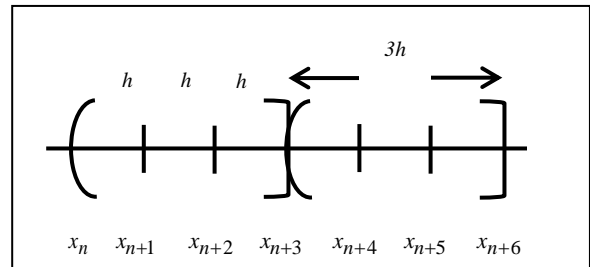

Fig. 1. Three-Point Block One-Step Method

By letting the second order differential equation as follows:

$$y'' = f(x, y, y').$$
(4)

The approximating values for y_{n+1}, y_{n+2} and y_{n+3} was obtained by integrating once and twice over (4) with respect to x. The evaluation of the first point will be approximated by integrating (4) once and twice at both sides over the interval $[x_n, x_{n+1}]$ as follows:

$$\int_{x_n}^{x_{n+1}} y''(x)dx = \int_{x_n}^{x_{n+1}} f(x, y, y')dx,$$
(5)

$$\int_{x_n}^{x_{n+1}}\int_{x_n}^{x} y''(x)dxdx = \int_{x_n}^{x_{n+1}}\int_{x_n}^{x} f(x, y, y')dxdx.$$
(6)

Therefore,

$$y'(x_{n+1}) - y'(x_n) = \int_{x_n}^{x_{n+1}} f(x, y, y')dx,$$
(7)

$$y(x_{n+1}) - y(x_n) - hy'(x_n) = \int_{x_n}^{x_{n+1}} (x_{n+1} - x)f(x, y, y')dx$$
(8)

Thus, Lagrange interpolating polynomial will replace the $f(x, y, y')$ in (7) and (8). The interpolation points involved are (x_n, f_n), (x_{n+1}, f_{n+1}), (x_{n+2}, f_{n+2}) and (x_{n+3}, f_{n+3}) within the block. By doing that, we will obtain the Lagrange interpolating polynomial which as follows:

$$P = \frac{(x - x_{n+1})(x - x_{n+2})(x - x_{n+3})}{(x_n - x_{n+1})(x_n - x_{n+2})(x_n - x_{n+3})} f_n$$
$$+ \frac{(x - x_n)(x - x_{n+2})(x - x_{n+3})}{(x_{n+1} - x_n)(x_{n+1} - x_{n+2})(x_{n+1} - x_{n+3})} f_{n+1}$$
$$+ \frac{(x - x_n)(x - x_{n+1})(x - x_{n+3})}{(x_{n+2} - x_n)(x_{n+2} - x_{n+1})(x_{n+2} - x_{n+3})} f_{n+2}$$
$$+ \frac{(x - x_n)(x - x_{n+1})(x - x_{n+2})}{(x_{n+3} - x_n)(x_{n+3} - x_{n+1})(x_{n+3} - x_{n+2})} f_{n+3}.$$
(9)

Now, by taking $x = x_{n+3} + sh$, $dx = hds$ and replace it into (7) and (8). Then, taking -3 to -2 as the limit of integration in (7) and (8) and the corrector formulae for the first point will be obtained as follows:

$$y'_{n+1} = y'_n + \frac{h}{24}\left(9f_n + 19f_{n+1} - 5f_{n+2} + f_{n+3}\right)$$

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{360}\begin{pmatrix} 97f_n + 114f_{n+1} \\ -39f_{n+2} + 8f_{n+3} \end{pmatrix} \tag{10}$$

Next, the same process will be repeated similar as the corrector formulae for the first point but the integration point is within the interval $[x_{n+1}, x_{n+2}]$ and the limit of integration is from -2 to -1 in order for us to obtain the corrector formula for the second point. Thus, the corrector formulae which is the approximate value of y'_{n+2} and y_{n+2} obtained would be as follows:

$$y'_{n+2} = y'_{n+1} + \frac{h}{24}\left(-f_n + 13f_{n+1} + 13f_{n+2} - f_{n+3}\right)$$

$$y_{n+2} = y_{n+1} + hy'_{n+1} + \frac{h^2}{360}\begin{pmatrix} -8f_n + 129f_{n+1} \\ +66f_{n+2} - 7f_{n+3} \end{pmatrix} \tag{11}$$

The corrector formulae for the third point would be obtained by using the same process to obtain the corrector formulae for the first and second point except their integration points. The integration points would be within the interval $[x_{n+2}, x_{n+3}]$. By taking the limit of integration from -1 to 0, the corrector formulae obtained would be as follows:

$$y'_{n+3} = y'_{n+2} + \frac{h}{24}\left(f_n - 5f_{n+1} + 19f_{n+2} + 9f_{n+3}\right)$$

$$y_{n+3} = y_{n+2} + hy'_{n+2} + \frac{h^2}{360}\begin{pmatrix} 7f_n - 36f_{n+1} + \\ 171f_{n+2} + 38f_{n+3} \end{pmatrix} \tag{12}$$

III. IMPLEMENTATION

The evaluation of the approximation points y_{n+1}, y_{n+2} and y_{n+3} will be based on the $PE(CE)^r$ mode where P, E and C stands for predictor, evaluation and corrector respectively. For each step, r functions evaluation will be used until the convergence is satisfied. The Modified Euler method will be used as the predictor in this algorithm. This method will act as the initial starting point before the corrector formulae take place to compute the approximation values for y_{n+1}, y_{n+2} and y_{n+3}. The same process will be used for each block along the interval until the end of it.

There are two types of BVPs that we need to solve in this research which is the BVPs with the Dirichlet boundary conditions and BVPs with the Neumann boundary conditions. The implementation for solving these two types of BVPs was basically the same but there are slightly differences which can be seen as shown below:

To solve the BVPs, the linear shooting technique will be used together with the direct three-point block one-step method. The BVPs (1) with the Dirichlet boundary conditions will be replaced into two initial value problems (IVP) which as follows:

$$y''_1 = p(x)y'_1 + q(x)y_1 + r(x),\ y_1(a) = \alpha,\ y'_1(a) = 0,$$
$$y''_2 = p(x)y'_2 + q(x)y_2\ \ y_2(a) = 0,\ y'_2(a) = 1. \tag{13}$$

Then, by solving the two IVP which is the nonhomogeneous and homogeneous equation in (13), the linear shooting methodwas obtained which as follows:

$$y(x) = y_1(x) + wy_2(x),\ \text{where}\ w = \frac{\beta - y_1(b)}{y_2(b)}. \tag{14}$$

The same procedure will be used for solving the BVPs with the Neumann boundary conditions with a slightly modifications in terms of the initial conditions and the linear shooting method. First, the BVPs (1) with the Neumann boundary conditions will be replaced into two IVP with their initial conditions which as follows:

$$y''_1 = p(x)y'_1 + q(x)y_1 + r(x),\ y_1(a) = 0,\ y'_1(a) = \alpha,$$
$$y''_2 = p(x)y'_2 + q(x)y_2\ \ y_2(a) = 0,\ y'_2(a) = 1. \tag{15}$$

Then, by performing the linear combination between this two IVP in (15), the linear shooting method will be obtained which as follows:

$$y(x) = y_1(x) + wy_2(x),\ \text{where}\ w = \frac{\beta - y'_1(b)}{y'_2(b)}. \tag{16}$$

This method will be implemented with the constant step size h. The convergence test will be used during the calculation of the approximated solution in the corrector formulae to obtain better accuracy.

The convergence test:

$$|y_{3,r+1} - y_{3,r}| < 0.1 \times TOL \tag{17}$$

where r is the number of iterations and TOL is the tolerance. All problems were tested using the absolute error test. The iterations in the corrector formulae will be repeated until the convergence test was satisfied.

IV. NUMERICAL RESULTS

In this section, six numerical examples are presented. From the six numerical examples that has been tested, there are three

of them come from the BVPs with the Dirichlet boundary condition and another three from the BVPs with the Neumann boundary conditions. The problems will be tested using direct three-point block one-step method (3BVP).

A. Dirichlet Boundary Conditions

Problem 1

$$y''(x) = y(x) + \cos(x), \ [0,1]$$
$$y(0) = 0, \ y(1) = 1,$$

Exact Solution

$$y(x) = \frac{-3\cosh(1) + 3\sinh(1) + \cos(1) + 2}{4\sinh(1)} e^x$$
$$+ \frac{3\cosh(1) + 3\sinh(1) - \cos(1) - 2}{4\sinh(1)} e^{-x} - \frac{\cos(x)}{2}.$$

Source: Bongsoo [13]

Problem 2

$$y''(x) = y'(x) - \exp(x-1) - 1, \ [0,1]$$
$$y(0) = 0, \ y(1) = 1,$$

Exact Solution

$$y(x) = x(1 - \exp(x-1)).$$

Source: Hamid et al. [3]

NOTATIONS

MAXE : Maximum Error of the Computed Solution.

ECBIM(N) : Extended Cubic B-Spline Method Minimizing Using Newton's Method in Hamid et al. [3].

ECBIM(B) : Extended Cubic B-Spline Method Minimizing Using Built-In Function in Hamid et al. [3].

EAD : Extended Adomian Decomposition Method in Bongsoo [13].

3BVP : Implementation of the Direct Three-Point Block One-Step Method For Solving The Linear Dirichlet and Neumann BVPs.

* : Maximum Error of the Computed Solution.

COLHW : Collocation method with the Haar Wavelets in Siraj et al. [12].

SPLINE : Polynomial spline method in Li-Bin et al. [11].

h : Step size.

TS : Total steps.

TABLE I
NUMERICAL RESULTS FOR SOLVING PROBLEM 1 WHEN $h = 0.125$

x	MAXE EAD [13]	TS EAD [13]	MAXE 3BVP	TS 3BVP
$\frac{1}{8}$	4.37×10^{-7}		1.14×10^{-7}	
$\frac{2}{8}$	8.07×10^{-7}		2.22×10^{-7}	
$\frac{3}{8}$	1.05×10^{-6}	8	$^*3.20 \times 10^{-7}$	3
$\frac{4}{8}$	$^*1.14 \times 10^{-6}$		3.12×10^{-7}	
$\frac{5}{8}$	1.05×10^{-6}		2.90×10^{-7}	
$\frac{6}{8}$	8.07×10^{-7}		2.56×10^{-7}	
$\frac{7}{8}$	4.37×10^{-7}		1.30×10^{-7}	

TABLE II
NUMERICAL RESULTS FOR SOLVING PROBLEM 2 WHEN $h = 0.1$

Existing Method $h = 0.1$	MAXE	TS	x	3BVP $h = 0.1$	TS 3BVP
ECBIM(N) [3]	7.91×10^{-6}		0.1	1.13×10^{-7}	
			0.2	2.19×10^{-7}	
			0.3	3.29×10^{-7}	
			0.4	3.74×10^{-7}	
		10	0.5	4.17×10^{-7}	4
			0.6	$^*4.68 \times 10^{-7}$	
			0.7	4.28×10^{-7}	
ECBIM(B) [3]	5.73×10^{-6}		0.8	3.62×10^{-7}	
			0.9	2.62×10^{-7}	

TABLE III
NUMERICAL RESULTS FOR SOLVING PROBLEM 2 WHEN $h = 0.01$

Existing Method $h = 0.01$	MAXE	TS EAD [13]	x	3BVP $h = 0.01$	TS 3BVP
EAD [13]	1.05×10^{-10}	100	0.1	1.18×10^{-11}	34
			0.2	2.15×10^{-11}	
			0.3	2.93×10^{-11}	
			0.4	3.49×10^{-11}	
			0.5	3.84×10^{-11}	
			0.6	$^*3.96 \times 10^{-11}$	
			0.7	3.76×10^{-11}	
			0.8	3.16×10^{-11}	
			0.9	2.00×10^{-11}	

In Table I, the comparison has been made between two types of methods which is EAD and 3BVP. The numerical results have been listed and the maximum error for both

methods has been noted by the asterisk sign. It can be seen that the maximum error for 3BVP was 3.20×10^{-7} and it was better than the EAD which is only 1.14×10^{-6}. Moreover, the total steps for 3BVP was lesser compared to the EAD.

Table II and Table III show the numerical results for problem 2 based on the step size $h = 0.1$ and $h = 0.01$ respectively. In table II, the comparison has been made between the ECBIM(N) and ECBIM(B) and the results obtained show that the maximum error for 3BVP was better with lesser total steps. The same thing goes in table III when the step size was 0.01. In this table, 3BVP gave better results compared to the EAD with a lesser total steps.

B. Neumann Boundary Conditions

Problem 3

$$-y''(x) = (2 - 4x^2)y, \ [0,1]$$
$$y'(0) = 0, \ y'(1) = -\frac{2}{e},$$

Exact Solution
$$y(x) = e^{-x^2}.$$

Source: Siraj et al. [12]

Problem 4

$$y''(x) + xy = (3 - x - x^2 + x^3)\sin(x) + 4x\cos(x), \ [0,1]$$
$$y'(0) = -1, \ y'(1) = 2\sin(1),$$

Exact Solution
$$y(x) = (x^2 - 1)\sin(x).$$

Source: Liu et al. [11]

TABLE IV
NUMERICAL RESULTS FOR PROBLEM 3

j	h	MAXE COLHW	TS COLHW	MAXE 3BVP	TS 3BVP
3	$\frac{1}{16}$	2.90×10^{-4}	16	6.52×10^{-7}	6
4	$\frac{1}{32}$	7.48×10^{-5}	32	7.12×10^{-9}	11
5	$\frac{1}{64}$	1.89×10^{-5}	64	1.69×10^{-9}	11
6	$\frac{1}{128}$	4.76×10^{-6}	128	5.56×10^{-11}	43
7	$\frac{1}{256}$	1.19×10^{-6}	256	5.44×10^{-12}	86
8	$\frac{1}{512}$	2.99×10^{-7}	512	2.88×10^{-13}	171

From Table IV, the local error was computed at their j^{th} collocation point x_j where $j = 1, 2, ..., N$ where N is the last number on the interval. The comparison has been made between the two methods which is COLHW and 3BVP.

It's clear to see that for each different step size, the 3BVP could provide a better result in terms of accuracy at their j^{th} point respectively. For the step size $h = \frac{1}{16}$, the local error was taken at the point number 3 thus giving us the local error for both methods which is the COLHW and 3BVP was 2.90×10^{-4} and 6.52×10^{-7} respectively. Furthermore, the total steps taken by the 3BVP also were nearly half compared to the COLHW.

From this result, it has been shown that the 3BVP could provide a better result for each different step size with lesser total steps.

TABLE V
NUMERICAL RESULTS FOR PROBLEM 4

h	MAXE SPLINE	TS SPLINE	MAXE 3BVP	TS 3BVP
$\frac{1}{8}$	2.22×10^{-4}	8	3.59×10^{-5}	3
$\frac{1}{16}$	5.04×10^{-6}	16	3.49×10^{-6}	6
$\frac{1}{32}$	1.62×10^{-7}	32	1.87×10^{-7}	11
$\frac{1}{64}$	5.57×10^{-9}	64	1.28×10^{-8}	22
$\frac{1}{128}$	1.89×10^{-10}	128	7.76×10^{-10}	43

In Table V, the comparison has been made for each different step size tested at problem 5. The maximum error for both methods which is SPLINE and 3BVP were comparable to each other. In addition to that, the total steps for the 3BVP were lesser as compared to the total steps taken for the SPLINE.

V. CONCLUSIONS

In this research, we have implemented the direct three-point block one-step method together with the linear shooting technique with constant step size which is efficient and suitable for solving the linear Dirichlet and Neumann BVPs directly.

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