An Optimal Quantity Discount Policy for Deteriorating Items with a Single Wholesaler and Two Retailers

Hidefumi Kawakatsu, Toshimichi Homma and Kiyoshi Sawada

***Abstract*—In Japanese large-scale super markets, the same items are sold at different departments in the same store. In many cases, managers of these departments independently order the items since they are under competition and are separately evaluated by their superiors. They can possibly reduce their costs if either of them purchases the items for two departments in cooperation with each other. This study considers the quantity discount problem between a single seller (wholesaler) and two buyers (retailers). The seller attempts to increase her/his profit by controlling the buyer's order quantity through a quantity discount strategy. The buyers try to maximize their profits by considering both whether to cooperate with each other and whether to accept the seller's offer. We formulate the above problem as a Stackelberg game between a single seller and two buyers to analyze the existence of the seller's optimal quantity discount pricing policy, which maximizes her total profit per unit of time. The same problem is also formulated as a cooperative game. Numerical examples are presented to illustrate the theoretical underpinnings of the proposed model.**

***Index Terms*—quantity discounts, deteriorating items, total profit, Stackelberg game.**

I. Introduction

In Japanese large-scale super-markets, the same items are sold at different departments in the same store. For instance, assorted sushi boxes are sold both at the ready-made dish department (sozai corner) and the fis department. Managers of these departments independently order the fishe as ingredients for sushi since they are under competition and are separately evaluated by their superiors. They can possibly reduce their costs if either of them purchases the items for two departments in cooperation with each other. Several super-markets have recently applied this strategy.

This study discusses the quantity discount problem[1], [2], [3], [4], [5], [6] between a single seller (wholesaler) and two buyers (retailers). The wholesaler purchases items from upper-leveled supplier and sells them to two retailers. The wholesaler attempts to increase her profi by controlling the retailers' order quantities through a quantity discount strategy. The retailers try to maximize their profit by considering both whether to cooperate with each other and whether to accept the wholesaler's offer. We consider the case where the retailers deal in perishable items such as fresh fruits, fishes sushi boxes and vegetables, and where both the wholesaler's and the retailers' inventory levels are continuously depleted due to the combined effects of its demand and deterioration. Yang [7] and Kawakatsu[8] have developed the model to determine an optimal pricing and a ordering policy for deteriorating items with quantity discounts. However, they focused on the quantity discount problem between a single seller and a single buyer.

Our previous work[9] has formulated the above problem as a Stackelberg game between a single wholesaler and two retailers to analyze the existence of the wholesale's optimal quantity discount pricing policy, which maximizes her/his total profi per unit of time. In this study, we also formulate the same problem as a cooperative game. Numerical examples are presented to illustrate the theoretical underpinnings of the proposed model.

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II. Notation and Assumptions

The wholesaler uses a quantity discount strategy in order to improve her/his profit The wholesaler proposes, for the retailers, an order quantity per lot along with the corresponding discounted wholesale price, which induces the retailers to alter their replenishment policies. We consider the two options throughout the present study as follows:

Option V_1: The retailer i $(i = 1, 2)$ does not adopt the quantity discount proposed by the wholesaler. When the retailer i chooses this option, she/he purchases the products from the wholesaler at an initial price in the absence of the discount, and she/he determines her/himself an optimal order quantity which maximizes her/his own total profi per unit of time.

Option V_2: The retailer i accepts the quantity discount proposed by the wholesaler.

The main notations used in this paper are listed below:

$Q_i^{(j)}$**:** the order quantity per lot for the retailer i under Option $V_j (i = 1, 2,\ j = 1, 2)$.

$T_i^{(j)}$**:** the length of the order cycle for the retailer i under Option V_j.

h_i**:** the inventory holding cost for the retailer i per item and unit of time.

a_i**:** the ordering costs per lot for the retailer i.

θ_i**:** the deterioration rate of the retailer i's inventory.

p_b**:** the retailers' unit selling price, i.e., unit purchasing price for their customers.

μ_i**:** the constant demand rate of the product for the retailer i.

c_s**:** the wholesaler's unit acquisition cost (unit purchasing cost from the upper-leveled manufacturer).

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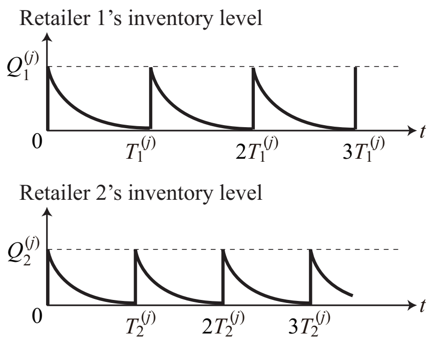

Fig. 1. Transition of Retailers' Inventory Level

p_s: the wholesaler's initial unit selling price, i.e., the retailers' unit acquisition cost in the absence of the discount.
y: the discount rate for the wholesale price proposed by the wholesaler. The wholesaler therefore offers a unit discounted price of $(1-y)p_s$ $(0 \le y < 1)$.
$S^{(j)}$: the wholesaler's order quantity per lot under Option V_j.
h_s: the wholesaler's inventory holding cost per item and unit of time.
a_s: the wholesaler's ordering cost per lot.
θ_s: the deterioration rate of the wholesaler's inventory.

The assumptions in this study are as follows:

1) The retailers' inventory levels are continuously depleted due to the combined effects of its demand and deterioration. In contrast, the wholesaler's inventory is only depleted by deterioration except when the wholesaler ships the products to the retailer. The retailer i's inventory level $I_B(t)$ and the wholesaler's inventory level $I_S(t)$ at time t can respectively be expressed by the following differential equations [9]:

$$dI_B(t)/dt = -\theta_i I_B(t) - \mu_i, \tag{1}$$
$$dI_S(t)/dt = -\theta_s I_S(t), \tag{2}$$

with a boundary conditions $I_B(T_i^{(j)}) = 0$ and $I_S(jT_R^{(j)}) = z_k(T_i^{(j)})$ under Option V_j $(j = 1, 2)$, where $z_k(T_i^{(j)})$ denotes the remaining inventory at the end of the kth shipping cycle, which will be define in Section VI.
2) The rate of replenishment is infinit and the delivery is instantaneous.
3) Backlogging and shortage are not allowed.
4) The quantity of the item can be treated as continuous for simplicity.
5) Both the wholesaler and the retailers are rational and use only pure strategies.

III. RETAILERS' PROFITS

In our previous study[9], we have formulated the retailers profit per unit of time under Option V_1 and V_2 available to the retailers. This section summarizes the results associated with the retailer's profit and their optimal order quantities which maximize their total profit under those options.

A. Under Option V_1

If the retailer i chooses Option V_1, her/his order quantity per lot and her/his unit acquisition cost are respectively given by $Q_i^{(1)} = Q\left(T_i^{(1)}\right)$ and p_s, where p_s is the unit initial price in the absence of the discount. In this case, she/he determines her/himself the optimal order quantity $Q_i^{(1)} = Q_i^*$ which maximizes her/his total profi per unit of time.

Under assumption 1), Kawakatsu et al.[9] have formulated the retailer i's total profi per unit of time, which is given by

$$\begin{aligned} \pi_i^{(1)}\left(T_i^{(1)}\right) &= \frac{p_b \int_0^{T_i^{(1)}} \mu_i dt - p_s Q\left(T_i^{(1)}\right) - h_i \int_0^{T_i^{(1)}} I_B(t)dt - a_i}{T_i^{(j)}} \\ &= \rho_i(p_b\theta_i + h_i) - \frac{\left(p_s + \frac{h_i}{\theta_i}\right) Q\left(T_i^{(1)}\right) + a_i}{T_i^{(1)}}, \end{aligned} \tag{3}$$

where $\rho_i = \mu_i/\theta_i$.

We have also shown that there exists a unique finit $T_i^{(1)} = T_i^{(1)^*}$ (> 0) which maximizes $\pi_i^{(1)}(T_i^{(1)})$ in Eq. (3). The optimal order quantity is therefore given by

$$Q_i^{(1)^*} = Q\left(T_i^{(1)*}\right) = \rho_i \left[e^{\theta_i T_i^{(1)^*}} - 1\right]. \tag{4}$$

The total profi per unit of time becomes

$$\pi_i^{(1)^*} = \rho_i \left[(p_b\theta_i + h_i) - \theta_i \left(p_s + \frac{h_i}{\theta_i}\right) e^{\theta_i T_i^{(1)^*}}\right]. \tag{5}$$

B. Under Option V_2

If the retailer i chooses Option V_2, the order quantity and unit discounted wholesale price are respectively given by $Q_i^{(2)} = Q\left(T_i^{(2)}\right) = \rho_i\left[e^{\theta_i T_i^{(2)}} - 1\right]$ and $(1-y)p_s$. The retailer i's total profi per unit of time can therefore be expressed by

$$\begin{aligned} \pi_i^{(2)}\left(T_i^{(2)}, y\right) &= \rho_i(p_b\theta_i + h_i) \\ &\quad - \frac{\left[(1-y)p_s + \frac{h_i}{\theta_i}\right] Q\left(T_i^{(2)}\right) + a_i}{T_i^{(2)}}. \end{aligned} \tag{6}$$

Let $p^{(1)}$ and $p^{(2)}$ be define by $p^{(1)} = p_s$ and $p^{(2)} = (1-y)p_s$, respectively, then $\pi_i^{(1)}\left(T_i^{(1)}\right)$ in Eq. (3) and $\pi_i^{(2)}\left(T_i^{(2)}, y\right)$ in Eq. (6) can be rewritten as follows:

$$\pi_i^{(j)} = \rho_i(p_b\theta_i + h_i) - \frac{\left[p^{(j)} + \frac{h_i}{\theta_i}\right] Q\left(T_i^{(j)}\right) + a_i}{T_i^{(j)}}. \tag{7}$$

IV. RETAILERS' OPTIMAL POLICY UNDER THE COOPERATIVE GAME

This section discusses a cooperative game between two retailers. In this study, we focus on the situation where there are two departments in the same store, and therefore we assume that the transportation cost of the product from one retailer to the other is zero. This signifie that the retailers can possibly reduce their costs by adopting the strategy that either of the retailers purchases the products from the wholesaler and stocks them, and then she/he distributes the products to the other retailer.

Based on these observations, the joint profi function per unit of time can therefore be expressed by

$$J\left(T_i^{(j)}\right) = \frac{\mu_1+\mu_2}{\theta_i}(p_b\theta_i+h_i) - \frac{\left(p_s+\frac{h_i}{\theta_i}\right)Q\left(T_i^{(j)}\right)+a_i.}{T_i^{(j)}}. \quad (8)$$

A. Under Option V_1

Under Option V_1, we can prove that there exist a unique finit positive $T_i^{(1)}=T_i^{(1)^*}$, which maximizes $J\left(T_i^{(j)}\right)$ in Eq. (8), and the maximum joint profi becomes

$$J^{(1)^*} = \max_{i=1,2}\hat{J}_i^{(1)}, \quad (9)$$

where

$$\hat{J}_i^{(1)} = \frac{\mu_1+\mu_2}{\theta_i} \times\left[(p_b\theta_i+h_i)-(p_s\theta_i+h_i)\,e^{\theta_i T_i^{(1)^*}}\right]. \quad (10)$$

Equation (10) signifie a local maximum value of the joint profi when the retailer i is in charge of ordering and inventory management.

Let R denote the retailer who is in charge of ordering and inventory control and bargains with the wholesaler on behalf of two retailers, and then R is given by

$$R = \begin{cases} 1, & \text{if } \hat{J}_1^{(1)} \geq \hat{J}_2^{(1)}, \\ 2, & \text{if } \hat{J}_1^{(1)} < \hat{J}_2^{(1)}. \end{cases} \quad (11)$$

The analysis with respect to comparing $J_1^{(1)}$ with $J_2^{(1)}$ becomes considerably complicated since Eq. (10) includes the term $T_i^{(1)^*}$ which is determined by a nonlinear equation solution. Neglecting higher order terms of θ_i in the expansion of $e^{\theta_i T_i^{(1)}}$, we have $e^{\theta_i T_i^{(1)}} \approx 1+\theta_i T_i^{(1)}+[\theta_i T_i^{(1)}]^2/2$.

In this case, $J^{(1)^*}$ in Eq. (9) can be expressed as

$$J^{(1)^*} = \begin{cases} \hat{J}_1^{(1)}, & \text{if } a_1(p_s\theta_1+h_1) \leq a_2(p_s\theta_2+h_2), \\ \hat{J}_2^{(1)}, & \text{if } a_1(p_s\theta_1+h_1) > a_2(p_s\theta_2+h_2). \end{cases} \quad (12)$$

It can also be shown in this case that $\hat{J}_R^{(1)} > \sum_{i=1}^{2}\pi_i^{(1)^*}$. We therefore focus on the case where $\hat{J}_R^{(1)} > \sum_{i=1}^{2}\pi_i^{(1)^*}$ in the following sections.

B. Under Option V_2

The wholesaler offers the quantity discount to the retailer R ($R=1,2$) which is define in Eq. (11).

Under Option V_2, the retailer R's joint profi per unit of time can be expressed by

$$J^{(2)}\left(T_R^{(2)},y\right) = \frac{\mu_1+\mu_2}{\theta_R}(p_b\theta_R+h_R) - \frac{\left[(1-y)p_s+\frac{h_R}{\theta_R}\right]Q\left(T_R^{(2)}\right)+a_R}{T_R^{(2)}}. \quad (13)$$

V. RETAILERS' OPTIMAL RESPONSE AND SHAPLEY VALUE IMPUTATION

A. Retailers' optimal response

This subsection discusses the retailer R's optimal response. The retailer R prefers Option V_1 over Option V_2 if $J^{(1)^*} > J^{(2)}$, but when $J^{(1)^*} < J^{(2)}\left(T_R^{(2)},y\right)$, she/he prefers V_2 to V_1. The retailer R is indifferent between the two options if $J^{(1)^*} = J^{(2)}\left(T_R^{(2)},y\right)$, which is equivalent to

$$y = \frac{1}{p_s Q\left(T_R^{(2)}\right)} \times\left\{\left[Q\left(T_R^{(2)}\right)-\rho\theta_R T_R^{(2)}e^{\theta_R T_R^{(1)*}}\right] \times\left(p_s+\frac{h_R}{\theta_R}\right)+a_R\right\}. \quad (14)$$

Let us denote, by $\psi\left(T_R^{(2)}\right)$, the right-hand-side of Eq. (14). It can easily be shown from Eq. (14) that $\psi(T_2)$ is increasing in $T_R^{(2)}$ $(\geq T_R^{(1)*})$.

The maximum value of the joint profi is given by

$$J_\pi^* = \begin{cases} J^{(1)^*}, & \text{if } J^{(1)^*} \geq J^{(2)}\left(T_R^{(2)},y\right), \\ J^{(2)}\left(T_R^{(2)},y\right), & \text{if } J^{(1)^*} < J^{(2)}\left(T_R^{(2)},y\right). \end{cases} \quad (15)$$

B. Shapley value imputation

We focus on the case where two retailers maximize their joint profi and share their cooperative profi according to the Shapley value[10], [11]. In this subsection, we determine the retailers' allocation of profi based on the concept of Shapley value. The Shapley value is one of the commonly used sharing mechanisms in static cooperation games with transferable payoff[10], [11].

Let x_i denote the retailer i's allocation of the cooperative profi ($i=1,2$). In this study, $\boldsymbol{x}=(x_1,x_2)$ can be called an imputation [9], and then x_1 and x_2 are respectively given by

$$x_1 = \frac{\pi_1^{(1)^*}+J_\pi^*-\pi_2^{(1)^*}}{2}, \quad (16)$$

$$x_2 = \frac{J_\pi^*-\pi_1^{(1)^*}+\pi_2^{(1)^*}}{2}. \quad (17)$$

VI. WHOLESALER'S TOTAL PROFIT AND OPTIMAL POLICY

The retailers adopt the cooperative strategy to increase their profit as mentioned in Section IV. The wholesaler can therefore regard the retailers as a single retailer since either of the retailers is in charge of ordering and inventory management. In this case, the wholesaler's total profi per unit of time can be formulated in the same manner as our previous formulation[8]. For this reason, in the following we briefl summarize the results associated with the wholesaler's profit under Option V_1 and V_2 and her/his optimal policy.

The length of the wholesaler's order cycle is given by $N^{(j)}T_R^{(j)}$ under Option V_j ($j=1,2$), where $N^{(j)}$ is a positive integer. This is because the wholesaler can possibly improve her/his total profi by increasing the length of her/his order cycle from $T_R^{(j)}$ to $N^{(j)}T_R^{(j)}$.

The wholesaler's inventory is only depleted by deterioration except when the wholesaler ships the products to the retailer R, as in assumption (1). The wholesaler's inventory level, $I_S(t)$, at time t can therefore be expressed by the following differential equation:

$$dI_S(t)/dt = -\theta_s I_S(t), \tag{18}$$

with a boundary condition $I_S(jT_R^{(j)}) = z_k(T_R^{(j)})$ under Option V_j, where $z_k(T_R^{(j)})$ denotes the remaining inventory at the end of the kth shipping cycle.

In this case, the wholesaler's total profi per unit of time under Option V_1 is given by

$$\begin{aligned} P^{(1)}&\left(N^{(1)}, T_R^{(1)^*}\right) \\ &= \frac{\left(p_s + \frac{h_s}{\theta_s}\right) Q\left(T_R^{(1)^*}\right)}{T_R^{(1)^*}} \\ &\quad - \frac{\left(c_s + \frac{h_s}{\theta_s}\right) S\left(N^{(1)}, T_R^{(1)^*}\right) + a_s}{N^{(1)} T_R^{(1)^*}}. \end{aligned} \tag{19}$$

In contrast, under Option V_2, the wholesaler's total profi per unit of time becomes

$$\begin{aligned} P^{(2)}&\left(N^{(2)}, T_R^{(2)}, y\right) \\ &= \frac{\left[(1-y)p_s + \frac{h_s}{\theta_s}\right] Q\left(T_R^{(2)}\right)}{T_R^{(2)}} \\ &\quad - \frac{\left(c_s + \frac{h_s}{\theta_s}\right) S\left(N^{(2)}, T_R^{(2)}\right) + a_s}{N^{(2)} T_R^{(2)}}, \end{aligned} \tag{20}$$

where

$$Q\left(T_R^{(j)}\right) = \frac{\mu_1 + \mu_2}{\theta_R}\left(e^{\theta_R T_R^{(j)}} - 1\right), \tag{21}$$

$$S\left(N^{(j)}, T_R^{(j)}\right) = Q\left(T_R^{(j)}\right) \frac{e^{N^{(j)}\theta_s T_R^{(j)}} - 1}{e^{\theta_s T_R^{(j)}} - 1}. \tag{22}$$

The wholesaler's optimal values for $T_R^{(2)}$ and y can be obtained by maximizing her/his total profi per unit of time considering the retailer R's optimal response which was discussed in Subsection V-A. Henceforth, let Ω_j $(j = 1, 2)$ be define by

$$\begin{aligned} \Omega_1 &= \left\{\left(T_R^{(2)}, y\right) \mid y \le \psi\left(T_R^{(2)}\right)\right\}, \\ \Omega_2 &= \left\{\left(T_R^{(2)}, y\right) \mid y \ge \psi\left(T_R^{(2)}\right)\right\}. \end{aligned}$$

Figure 2 depicts the region of Ω_j $(j = 1, 2)$ on the $\left(T_R^{(2)}, y\right)$ plane.

A. Optimal Policy under Option V_1

If $\left(T_R^{(2)}, y\right) \in \Omega_1 \setminus \Omega_2$ in Fig. 2, the retailer R will naturally select Option V_1. In this case, the wholesaler can maximize her/his total profi per unit of time independently of T_2 and y on the condition of $\left(T_R^{(2)}, y\right) \in \Omega_1 \setminus \Omega_2$. Hence, the wholesaler's locally maximum total profi per unit of time in $\Omega_1 \setminus \Omega_2$ becomes

$$P^{(1)^*} = \max_{N^{(1)} \in N} P^{(1)}\left(N^{(1)}, T_R^{(1)^*}\right), \tag{23}$$

where N signifie the set of positive integers.

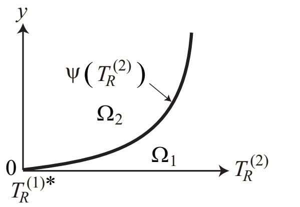

Fig. 2. Characterization of retailer's optimal responses

B. Optimal Policy under Option V_2

On the other hand, if $\left(T_R^{(2)}, y\right) \in \Omega_2 \setminus \Omega_1$, the retailer R's optimal response is to choose Option V_2. Then the wholesaler's locally maximum total profi per unit of time in $\Omega_2 \setminus \Omega_1$ is given by

$$P^{(2)^*} = \max_{N^{(2)} \in N} \hat{P}^{(2)}\left(N^{(2)}\right), \tag{24}$$

where

$$\begin{aligned} \hat{P}^{(2)}&\left(N^{(2)}\right) \\ &= \max_{\left(T_R^{(2)}, y\right) \in \Omega_2 \setminus \Omega_1} P^{(2)}\left(N^{(2)}, T_R^{(2)}, y\right). \end{aligned} \tag{25}$$

More precisely, we should use "sup" instead of "max" in Eq. (25).

For a given $N^{(2)}$, we show below the existence of the wholesaler's optimal quantity discount pricing policy $\left(T_R^{(2)}, y\right) = \left(T_R^{(2)^*}, y^*\right)$ which attains Eq. (25). It can easily be proven that $P^{(2)}\left(N^{(2)}, T_R^{(2)}, y\right)$ in Eq. (20) is strictly decreasing in y, and consequently the wholesaler can attain $\hat{P}^{(2)}\left(N^{(2)}\right)$ in Eq. (25) by letting $y \to \psi\left(T_R^{(2)}\right) + 0$. By letting $y = \psi\left(T_R^{(2)}\right)$ in Eq. (20), the total profi per unit of time on $y = \psi\left(T_R^{(2)}\right)$ becomes

$$\begin{aligned} P^{(2)}&\left(N^{(2)}, T_R^{(2)}\right) \\ &= \frac{\mu_1 + \mu_2}{\theta_R}\left(p_s\theta_R + h_R\right) e^{\theta_R T_R^{(1)^*}} \\ &\quad - \frac{1}{N^{(2)} T_R^{(2)}} \cdot \Big[C \cdot S\left(N^{(2)}, T_R^{(2)}\right) \\ &\qquad - H\left(N^{(2)}\right) Q\left(T_R^{(2)}\right) + a_b N^{(2)} + a_s\Big], \end{aligned} \tag{26}$$

where

$$C = (c_s + h_s/\theta_s), \tag{27}$$

$$H\left(N^{(2)}\right) = (h_s/\theta_s - h_R/\theta_R) N^{(2)}. \tag{28}$$

Let us now defin $L\left(N^{(2)}\right)$ as follows:

$$\begin{aligned} L&\left(N^{(2)}\right) \\ &\equiv \frac{C\theta_s T_R^{(2)} Q\left(T_R^{(2)}\right)}{\left(e^{\theta_s T_2} - 1\right)^2} \\ &\quad \times \Big\{N^{(2)} e^{N^{(2)}\theta_s T_R^{(2)}}\left[e^{\theta_s T_R^{(2)}} - 1\right] \\ &\qquad - e^{\theta_s T_R^{(2)}}\left[e^{N^{(2)}\theta_s T_R^{(2)}} - 1\right]\Big\} \end{aligned}$$

$$+\left[\rho\theta_R e^{\theta_R T_R^{(2)}} T_R^{(2)} - Q\left(T_R^{(2)}\right)\right]$$
$$\times\left[C\frac{e^{N^{(2)}\theta_s T_R^{(2)}}-1}{e^{\theta_s T_R^{(2)}}-1} - H\left(N^{(2)}\right)\right]. \quad (29)$$

We here summarize the results of analysis in relation to the optimal quantity discount policy which attains $\hat{P}^{(2)}\left(N^{(2)}\right)$ in Eq. (25) when $N^{(2)}$ is fi ed to a suitable value.

1) $N^{(2)} = 1$:
In this case, there exists a unique finit $\tilde{T}_R^{(2)}$ $(> T_R^{(1)^*})$ which maximizes $P^{(2)}\left(N^{(2)}, T_R^{(2)}\right)$ in Eq. (26), and therefore $\left(T_R^{(2)^*}, y^*\right)$ is given by

$$\left(T_R^{(2)^*}, y^*\right) \to \left(\tilde{T}_R^{(2)}, \tilde{y}\right), \quad (30)$$

where $\tilde{y} = \psi\left(\tilde{T}_R^{(2)}\right)$.
The wholesaler's total profi then becomes

$$\hat{P}^{(2)}\left(N^{(2)}\right) = \frac{\mu_1+\mu_2}{\theta_R}\left[(p_s\theta_R + h_R)\, e^{\theta_R T_R^{(1)^*}} - (c_s\theta_R + h_R)\, e^{\theta_R T_R^{(2)^*}}\right]. \quad (31)$$

2) $N^{(2)} \geq 2$:
Let us defin $\tilde{T}_R^{(2)}$ $(> T_R^{(1)^*})$ as the unique solution (if it exists) to

$$L(T_R^{(2)}) = a_R N^{(2)} + a_s. \quad (32)$$

In this case, the optimal quantity discount pricing policy is given by Eq. (30).

C. Optimal Policy under Option V_1 and V_2

In the case of $\left(T_R^{(2)}, y\right) \in \Omega_1 \cap \Omega_2$, the retailer is indifferent between Option V_1 and V_2. For this reason, this study confine itself to a situation where the wholesaler does not use a quantity discount policy $\left(T_R^{(2)}, y\right) \in \Omega_1 \cap \Omega_2$.

VII. WHOLESALER'S OPTIMAL POLICY UNDER THE COOPERATIVE GAME

This section discusses a cooperative game between the wholesaler and the retailer R. We focus on the case where the wholesaler and the retailer R attempt to maximize their joint profi per unit of time. We here introduce some additional notations $N^{(3)}$ and $T^{(3)}$, which correspond to $N^{(2)}$ and $T_R^{(2)}$ respectively, under Option V_2 in the previous section. Let $J_P\left(N^{(3)}, T^{(3)}, y\right)$ express the joint profi function per unit of time for the wholesaler and the retailer R, i.e., let $J_P\left(N^{(3)}, T^{(3)}, y\right) = P^{(2)}\left(N^{(3)}, T^{(3)}, y\right) + J^{(2)}\left(T^{(3)}, y\right)$, we have

$$\begin{aligned} &J_P\left(N^{(3)}, T^{(3)}, y\right) \\ &= \frac{\mu_1+\mu_2}{\theta_R}(p_b\theta_R + h_R) \\ &\quad -\frac{1}{N^{(3)}T^{(3)}}\left[C\cdot S\left(N^{(3)}, T^{(3)}\right)\right. \\ &\quad \left. -H\left(N^{(3)}\right)Q\left(T^{(3)}\right) + \left(N^{(3)}a_R + a_s\right)\right]. \quad (33)\end{aligned}$$

It can easily be proven from Eq. (33) that $J_P\left(N^{(3)}, T^{(3)}, y\right)$ is independent of y and we have $J_P\left(N^{(3)}, T^{(3)}, y\right) = P^{(2)}\left(N^{(2)}, T_R^{(2)}\right) + J^{(1)*}$. This signifie that the optimal quantity discount policy $\left(T^{(3)}, y\right) = \left(T^{(3)*}, y^*\right)$ which maximizes $J_P\left(N^{(3)}, T^{(3)}, y\right)$ in Eq. (33) is given by Eq. (30) as shown in Section VI. This is simply because, in this study, the inventory holding cost is assumed to be independent of the value of the item.

TABLE I
SENSITIVITY ANALYSIS

(a) Under Option V_1

a_s	$S^{(1)*}$	$N^{(1)*}$	$P^{(1)*}$	$x_1^{(1)}$	$x_2^{(1)}$
500	73.696	1	2216.637	1621.957	1309.561
1000	152.282	2	2140.522	1621.957	1309.561
2000	152.282	2	2062.686	1621.957	1309.561
3000	152.282	2	1984.851	1621.957	1309.561

(b) Under Option V_2

a_s	$S^{(2)*}$	$N^{(2)*}$	$P^{(2)*}$	$x_1^{(2)}$	$x_2^{(2)}$
500	127.866	1	2261.602	1621.957	1309.561
1000	145.928	1	2218.253	1621.957	1309.561
2000	176.955	1	2143.79	1621.957	1309.561
3000	203.659	1	2079.699	1621.957	1309.561

VIII. NUMERICAL EXAMPLES

Table I reveals the results of sensitively analysis in reference to $x_1^{(j)}$, $x_2^{(j)}$, $S^{(j)^*}(= S(N^{(j)*}T_R^{(j)*}))$ and $P^{(j)^*}(= P(N^{(j)*}T_R^{(j)*}))$ under Option V_j $(j = 1, 2)$ for $(p_b, p_s, c_s, h_s, \theta_s, a_s) = (600, 300, 100, 1, 0.01, 1000)$, $(h_1, \theta_1, a_1, \mu_1) = (1.1, 0.013, 1200, 6)$ and $(h_2, \theta_2, a_2, \mu_2) = (1.5, 0.015, 1300, 5)$ when $a_s = 500$, 1000, 2000 and 3000. In this case, we obtain $\pi_1^{(1)^*} = 1526.521$ and $\pi_2^{(1)^*} = 1214.124$, which are independent of a_s.

In Table I(a) indicates that both $S^{(1)^*}$ and $N^{(1)^*}$ are non-decreasing in a_s. As mentioned in Section II, under Option V_1, the retailer R does not adopt the quantity discount offered by the wholesaler, which signifie that the wholesaler cannot control the retailer R's ordering schedule. In this case, the wholesaler's cost associated with ordering should be reduced by increasing her/his own length of order cycle and lot size by means of increasing $N^{(1)}$. Table I(a) also implies $x_i^{(1)} > \pi_i^{(1)^*}$ $(i = 1, 2)$.

Table I(b) shows that, under Option V_2, $S^{(2)*}$ increases with a_s, in contrast, $N^{(2)*}$ takes a constant value, i.e., $N^{(2)*} = 1$. Under Option V_2, the retailer accepts the quantity discount proposed by the wholesaler. The wholesaler's lot size can therefore be increased by stimulating the retailer to alter her/his order quantity per lot through the quantity discount strategy. If the wholesaler increases $N^{(2)}$ one step, her/his lot size also significantl jumps up since $N^{(2)}$ takes a positive integer. Under this option, the wholesaler should increase her/his lot size using the quantity discount rather than increasing $N^{(2)}$ when a_s takes larger values. Table I reveals that we have $P^{(1)*} < P^{(2)*}$. This indicates that using the quantity discount strategy can increase the wholesaler's total profi per unit of time. We can notice in Table I that $x_i^{(1)} = x_i^{(2)}$ $(i = 1, 2)$ for each value of a_s. This signifie that the retailers' profit do not increase even if they accept the quantity discount proposed by the wholesaler.

IX. Conclusion

In this study, we have discussed a quantity discount problem between a single wholesaler and two retailers under circumstances where both the wholesaler's and the retailers' inventory levels of the product are depleted not only by demand but also by deterioration. In Japanese large-scale super markets, the same items are sold at different departments at the same store. In many cases, managers of these departments independently order the items since they are under competition and are separately evaluated by their superiors. They can possibly reduce their costs if either of them purchases the items for two departments in cooperation with each other.

The wholesaler is interested in increasing her/his profi by controlling the retailers' order quantities through the quantity discount strategy. The retailers attempt to maximize their profit by considering both whether to cooperate with each other and whether to accept the wholesaler's proposal. The analysis with respect to comparing the cooperative solution with non-cooperative one becomes considerably complicated since the local maximum values of the players' total profit per unit of time cannot be expressed as closed form expressions. For this reason, we have shown that the retailers can increase their profit by means of adopting the cooperative strategy in the case where higher order terms of the deterioration rate in the expansion of the exponential can be ignored. Focusing on such a situation, the wholesaler can regard the retailers as a single retailer since either of the retailers is in charge of ordering and inventory management. In this case, we can formulate the above problem as a Stackelberg game between the wholesaler and the retailers in the same manner as our previous formulation[8], [9]. In this study, we have also formulated the same problem as a cooperative game between the wholesaler and the retailer R. The result of our analysis reveals that the wholesaler is indifferent between the cooperative and non-cooperative options. This is because the inventory holding cost is assumed to be independent of the value of the item. The relaxation of such a restriction is an interesting extension.

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