Phase fitted and Amplification fitted Hybrid Methods for Solving Second-order Ordinary Differential Equations

F. Samat, F. Ismail and M. Suleiman

Abstract—Two fifth-order explicit hybrid methods are developed. Based on these methods, phase fitted and amplification fitted methods are constructed by vanishing both the phase-lag and the dissipation error. For the phase fitted and amplification fitted methods, computation of the output stage is dependent on the frequency of the problem being solved, thus the methods can only be applied when the frequency is known in advance. Numerical comparisons that have been carried out show the advantage of the new methods for solving several second-order ordinary differential equations with oscillating solutions.

Index Terms—hybrid methods, second-order ordinary differential equations, zero dissipation error, zero phase-lag

I. INTRODUCTION

In this paper, we are interested in the research on numerical methods for solving second order ordinary differential equations of the form

$$y''(x) = f(x, y(x)), y(x_0) = y_0, y'(x_0) = y'_0$$

where the first derivative does not appear explicitly. These problems often arise in engineering and applied sciences such as celestial mechanics, quantum mechanics, elastodynamics, theoretical physics, chemistry and electronics and can be solved by using Runge Kutta Nystrom methods (see for example Senu [1]) and multistep methods. Several authors such as Fatunla, et. al. [2], Chawla [3], Tsitouras [4] and Simos [5] proposed hybrid methods which are obtained from the idea underlying both the Runge Kutta and linear multistep methods.

In the developments of hybrid methods, it is important to increase the order of the methods to achieve higher accuracy. In addition, if the second order ordinary differential equations have oscillating solutions, then it is also essential to consider the phase-lag and the dissipation error that result from comparing the numerical solution with the analytical solution. These are actually two types of truncation errors. The first is the angle between the analytical solution and the numerical solution while the second is the distance from a standard cyclic solution. The study of phase-lag has been initiated by Brusa and Nigro [6]. The research of hybrid methods has been carried out by many authors paying attention to obtain methods with minimal phase-lag or with zero dissipation error (see [7] to [11]).

Consider the class of hybrid methods:

$$y_{n+1} = 2y_{n} - y_{n-1} + h^2 \sum_{j=1}^{p} b_j f(x_n + c_j h, g_j)$$

with $g_i = (1 + c_i) y_{n} - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_j f(x_n + c_j h, g_j)$

This class of methods has been discussed in many papers (for example see [4,12,13]). By assuming $c_1 = 1$ and $c_2 = 0$, Tsitouras [4] derived an eight-order implicit hybrid method. Meanwhile, Franco [13] proposed a class of explicit hybrid methods by assuming $c_1 = -1$ and $c_2 = 0$. In [14], Fang et al. derived one- frequency and two-frequency explicit hybrid methods based on the fifth-order hybrid method in [13]. The coefficients of the new methods in [14] are obtained by vanishing both the phase-lag and the dissipation error.

Here, inspired by Runge Kutta methods, we choose $c_1 = 0$ and $c_2 = 1$. The class of explicit hybrid methods with $c_1 = 0$ and $c_2 = 1$ can be represented by the Butcher tableau:

$$
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & \cdots & 0 \\
1 & a_{11} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a_{11} & a_{12} & \cdots & a_{s,1} & 0 \\
& b_1 & b_2 & \cdots & b_{s-1} & b_s \\
\end{array}
$$

The leading term associated with the local truncation error of a $p$-th order hybrid method is given as

$$e_{p,1}(t_i) = \frac{\alpha(t_i)}{(p+2)!} \left[ 1 + (-1)^p \psi^p(t_i) \right] t_i \in T_2,$$

where $T_2, \alpha(t_i)$ and $\psi^p(t_i)$ are as defined in [12]. The quantity

$$E = \sum_{i=1}^{n_{p+2}} e^2_{p,1}(t_i)$$

where $n_{p+2}$ is the number of trees of order $p+2$, is called the error constant for the $p$-th order method. Based on this class of methods, we derive fifth order explicit hybrid methods.

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with four stages \((s = 4)\). Then, based on these methods, we derive phase fitted and amplification fitted explicit hybrid methods. The phase fitted and amplification fitted methods are obtained by vanishing the phase-lag and the dissipation error. The implementation of the methods is investigated by comparing the accuracy of the methods with that of the base and other existing methods.

II. PHASE-LAG ANALYSIS

Let \( H = \lambda h \) and \( e = \{1 \ 1 \ \ldots \ 1\}^T \). Applying the hybrid methods defined in (1) to equation \( y'' = -\lambda^2 y, \ \lambda > 0 \) yields the recursion

\[
y_{s+1} - S(H^2)y_u + P(H^2)y_{s+1} = 0
\]  

(2)

where

\[
S(H^2) = 2 - H^2b^T \left(1 + H^2A\right)^{-1}(e + e)
\]

and

\[
P(H^2) = 1 - H^2b^T \left(1 + H^2A\right)^{-1} e.
\]

The characteristic equation associated with (2) is

\[
\xi^2 - S(H^2)\xi + P(H^2) = 0
\]  

(3)

According to Houwen and Sommeijer [15], phase-lag is defined as the difference

\[
t = H - \theta(H)
\]

where \( H \) is the phase (or argument) of the exact solution of \( y'' = -\lambda^2 y \) and \( \theta(H) \) is the phase of the principal root of (3). In case for explicit methods, the matrix \( A \) is nilpotent of degree \( s \) (that is \( \lambda^s = 0 \)). Therefore,

\[
\left(1 + H^2A\right)^{-1} = I - H^2A + H^4A^2 - H^6A^3 + \ldots - H^{2s-2} A^{s-1}.
\]

For the hybrid methods corresponding to the characteristic equation (3), the quantity

\[
\phi(H) = H - \arccos \left(\frac{S(H^2)}{2\sqrt{P(H^2)}}\right)
\]

is called phase-lag (or dispersion error) while the quantity

\[
d(H) = 1 - \sqrt{P(H^2)}
\]

is called dissipation (or amplification error). A hybrid method corresponding to (3) is said to have the phase-lag of order \( n \) if \( \phi(H) = O(H^{n+1}) \). If \( \phi(H) = 0 \) then the method is said to be phase fitted or zero dispersive. If \( P(H^2) = 1 \) then \( d(H) = O(H^{n+1}) \) and the method with this property is said to be amplification fitted or zero dissipative. If \( P(H^2) \neq 1 \) and \( d(H) = O(H^{n+1}) \) the method with this property is said to be dissipative of order \( m \).

The interval \((0, H_p)\) is called the interval of periodicity of the method if

\[
P(H^2) = 1 \quad \text{and} \quad |S(H^2)| < 2 \quad \text{for all} \quad H \in (0, H_p)
\]

whereas the method is called \( P \)-stable if

\[
P(H^2) = 1 \quad \text{and} \quad |S(H^2)| < 2 \quad \text{for all} \quad H \in (0, \infty).
\]

The interval \((0, H_a)\) is called the interval of absolute stability if

\[
|P(H^2)| < 1 \quad \text{and} \quad |S(H^2)| < 1 + P(H^2) \quad \text{for all} \quad H \in (0, H_a).
\]

III. CONSTRUCTION OF HYBRID METHODS

A. Construction of Fifth-order Methods

In this section, fifth-order explicit hybrid methods are constructed. The following are order conditions that have to be satisfied (see [12]).

\[
\sum_{i=1}^s b_i = 1
\]

\[
\sum_{i=1}^s b_i c_i = 0
\]

\[
\sum_{i=1}^s b_i c_i^2 = \frac{1}{6}
\]

\[
\sum_{i=1}^s b_i a_i = \frac{1}{12}
\]

\[
\sum_{i=1}^s b_i a_i c_i = 0
\]

\[
\sum_{i=1}^s b_i c_i a_i = \frac{1}{15}
\]

\[
\sum_{i=1}^s b_i c_i^2 a_i = \frac{1}{30}
\]

\[
\sum_{i=1}^s b_i c_i a_i c_j = -\frac{1}{60}
\]

\[
\sum_{i=1}^s \sum_{j=1}^s b_i a_i a_j = \frac{7}{120}
\]

\[
\sum_{i=1}^s \sum_{j=1}^s b_i a_i c_j = \frac{1}{180}
\]

\[
\sum_{i=1}^s \sum_{j=1}^s b_i a_j a_k = \frac{1}{360}
\]

Substituting \( s = 4, c_1 = 0, c_2 = 1, a_{ij} = 0 \ (j \geq i) \) into the above order conditions and solving the resulting equations using Maple software, we obtain

\[
b_1 = \frac{25c_1^2 + 7c_1 - 3}{6c_3(5c_3 + 2)},
\]

\[
c_4 = \frac{5c_3^2 - 2}{5(c_3 + 1)},
\]

\[
b_2 = \frac{5c_3^2 - 2}{6(7 + 10c_3)(-1 + c_1)},
\]

\[
b_3 = \frac{1}{2c_3(-1 + c_1)(10c_3 + 2 + 5c_3^2)},
\]

\[
b_4 = \frac{125c_3^4 + 1}{67 + 10c_3(5c_3 + 3)(10c_3 + 2 + 5c_3^2)} + a_{21},
\]

\[
a_{21} = \frac{2}{3}c_3 - \frac{1}{6}c_3^2 + \frac{1}{2}c_3, \quad a_{32} = \frac{1}{6}c_3(-1 + c_1)(c_3 + 1),
\]

\[
a_{41} = \frac{3750c_3(c_3 + 1)^4}{3750c_3(c_3 + 1)^4},
\]

\[
a_{42} = \frac{(5c_3 + 2)(325c_3 + 570c_3^2 + 240c_3 - 14)}{(5c_3 + 2)(7 + 10c_3)(-1 + 10c_3^2 - 10c_3)}.
\]

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For the first method, the free parameter $c_3$ is chosen so that the error constant $E$, is as small as possible giving us
$$c_3 = -\frac{69}{100}, \quad E = 1.85 \times 10^{-2}$$

The new method will be denoted by EHMS1. Coefficients of EHMS1 method are displayed in Table I.

| TABLE I
| COEFFICIENTS OF EHMS1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 69 | 1 | 0 | 0 | 0 | 0 | 0 |
| 100 | 5 | 1 | 0 | 0 | 0 | 0 |
| 29 | 2000000 | 2000000 | 0 | 0 | 0 | 0 |
| 31 | 191168847 | -468225147 | 32307535143 | 0 | 0 | 0 |

The phase-lag order and the dissipation order for this method are six and five respectively with the following quantities
$$\phi(H) = \frac{71}{1512000}H^7 + O(H^9)$$
$$d(H) = \frac{31}{216000}H^6 + O(H^{12})$$

The interval of absolute stability is $(0, 3.36)$.

For the second method, the free parameter $c_3$ is chosen so that the phase-lag order is eight. This gives us the values
$$c_3 = \frac{25}{28}, \quad E = 7.09 \cdot 10^{-2}$$

The new method is denoted by EHMSII. Coefficients of EHMSII method are shown in Table II.

| TABLE II
| COEFFICIENTS OF EHMSII |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 25 | 0 | 1 | 0 | 0 | 0 | 0 |
| 28 | 43904 | 43904 | 0 | 0 | 0 | 0 |
| 23 | 454986 | 16744 | -13866608 | 0 | 0 | 0 |
| 5 | 15625 | 33125 | -828125 | 0 | 0 | 0 |

This method has phase-lag of order eight and is dissipative of order five with the following quantities
$$\phi(H) = \frac{17}{7257600}H^7 + O(H^9)$$
$$d(H) = \frac{1}{20160}H^6 + O(H^{12})$$

The interval of absolute stability is $(0, 3.94)$.

B. Construction of Phase Fitted and Amplification Fitted Methods

Here, phase fitted and amplification fitted hybrid methods denoted by EHM5IPA and EHM5IIIPA will be derived. The derivations of EHM5IPA and EHM5IIIPA are being based on EHMSI and EHMSII methods respectively. Table III shows coefficients of EHM5IPA method.

| TABLE III
| COEFFICIENTS OF EHM5IPA |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 69 | 334397 | 120497 | 0 | 0 | 0 | 0 |
| 100 | 2000000 | 2000000 | 0 | 0 | 0 | 0 |
| 29 | 173865730 | 6193240 | 292262000 | 0 | 0 | 0 |
| 31 | 191168847 | 468225147 | 32307535143 | 0 | 0 | 0 |

It is noted that some of the values are taken from Table I. The coefficients $b_1$ and $b_2$ are obtained by vanishing the phase-lag and the dissipation error. The quantity $S(H^2)$ has to be equal to $2\cos(H)\sqrt{P(H^2)}$ in order to vanish the phase-lag. Solving the resulting equation, we get
$$b_1 = \frac{1}{1217408400H^2} \left[ -13051620000H^2 + 5582790000H^4 \\ +1217408400H^4 b_2(H) + 3494413H^8 \\ -37311313H^8 + 24348168000 \\ -2438816800H^2 b_2(H) -1040520 \cos(H)547560000 \\ -547560000H^2 b_2(H) -157170H^8 - 205470000H^2 \right]^{1/2}$$

For the dissipation error to vanish, we set $P(H^2)=1$ and then solve the resulting equation giving
$$b_2 = -\frac{31}{108000}H^4 - \frac{761}{2028}.$$
Fig. 1. Behaviour of the coefficients $b_1$ and $b_2$ of the new proposed method; EHM5IPA for several values of $H = \lambda h$.

Let us consider coefficients of EHM5IPA given by Table IV.

Some of the values in Table IV are taken from Table II. Using the similar procedure, by vanishing the phase-lag, we obtain

$$ b_1 = \frac{1}{61437600H^5} \left[ -594272H^2 - 450800H^4 ight. $$

$$ + 61437600H^4 b_1 + 6095H^8 - 176755H^6 + 122875200 $$

$$ - 122875200H^2 b_2 - 2760 \cos(H) \left( 1982030400 ight. $$

$$ - 1982030400H^2 b_2 - 196630H^6 + 179712400H^2 \right]^{1/2} $$

whereas by vanishing the dissipation error,

$$ b_2 = \frac{10080}{1908} $$

The Taylor series expansion for $b_1$ is given by

$$ b_1 = \frac{2791}{3450} + \frac{1}{10080} H^4 - \frac{1}{20160} H^6 + \frac{1}{1814400} H^8 $$

$$ - \frac{1}{239500800} H^{10} + \frac{1}{43589145600} H^{12} - \ldots $$

The behaviour of the coefficients is given in Fig. 2.

**IV. NUMERICAL RESULTS**

All new codes have been applied to some second-order problems to provide numerical comparisons with other competitive codes in the scientific literature. Codes that have been used for numerical comparisons are denoted by:

EHM5I : The first fifth-order explicit hybrid method with four stages derived in this paper. This method has phase-lag of order six and is dissipative of order five. The interval of absolute stability is $(0, 3.36)$.

EHM5II : The second fifth-order explicit hybrid method with four stages derived in this paper. This method has phase-lag of order eight and is dissipative of order five. The interval of absolute stability is $(0, 3.94)$.

EHM5IPA : The phase fitted and amplification fitted explicit hybrid method which is derived based on EHM5I in this paper.

EHM5IIPA : The phase fitted and amplification fitted explicit hybrid method which is derived based on EHM5II in this paper.

FETSH : The fifth-order explicit hybrid method with three stages derived by Franco [13]. This method has phase-lag of order eight and is dissipative of order five. The interval of absolute stability is $(0, 2.84)$ whereas the formula for this method is given by

$$ Y_1 = y_{n+1}, Y_2 = y_n $$

$$ Y_3 = (1 + c_1)Y_n - c_2Y_{n+1} + h^2 (a_{11}f_{n+1} + a_{12}f_n) $$

$$ Y_4 = (1 + c_2)Y_n - c_4Y_{n+1} + h^2 (a_{21}f_{n+1} + a_{22}f_n) $$

$$ + a_{23}f(x_n + c_2h, Y_n) $$

$$ Y_{n+1} = 2Y_n - y_{n+1} + h^2 [b_1f_{n+1} + b_2f_n $$

$$ + b_3f(x_n + c_1h, Y_1) + b_4f(x_n + c_2h, Y_2)] $$

The coefficients of the method can be found in [13].

TSI7: The seventh-order explicit hybrid method with four stages derived in [16]. This method has the form

$$ f_n = f(x_n, y_n) $$

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The coefficients can be found in [16]. According to Tsitouras [16], the coefficients of this method have been selected so that the local truncation error is minimized.

### Problem 1 (non-homogeneous problem)

\[ y^{(n)} + 100y + 99\sin(x) = 1, \quad y(0) = 1, \quad y^{(0)}(0) = 11, \quad 0 \leq x \leq 100 \]

**Solution:**

\[ y(x) = \cos(10x) + \sin(10x) + \sin(x) \]

We choose \( H = 10h \) for EHM5IPA and EHM5IIPA codes.

### Problem 2 (homogeneous problem)

\[ y^{(n)} - 2500y = 0, \quad y(0) = 0, \quad y^{(0)}(0) = 0 \]

**Solution:**

\[ y(x) = \frac{1}{50}\sin(50x) \]

We choose \( H = 50h \) for EHM5IPA and EHM5IIPA codes.

### Problem 3 (almost periodic orbit problem)

\[ \varepsilon(x) + z(x) = \frac{1}{1000}e^{i\alpha}, \quad z(0) = 1, \quad z^{(0)}(0) = 0.9995i, \quad \varepsilon \in \mathbb{C}, \quad 0 \leq x \leq 100 \]

The theoretical solution is \( z(x) = (1 - 0.0005ix)e^{i\alpha} \). If \( z(x) = y_1(x) + iy_2(x), y_1, y_2 \in \mathbb{R} \), then the problem can be transformed into the equivalent form

\[ y_1^{(n)} = -y_1 + \frac{1}{1000}\cos(x), \quad y_1(0) = 1, \quad y_1^{(0)}(0) = 0 \]
\[ y_2^{(n)} = -y_2 + \frac{1}{1000}\sin(x), \quad y_2(0) = 0, \quad y_2^{(0)}(0) = 0.9995 \]

with the theoretical solution:

\[ y_1(x) = \cos(x) + 0.0005x \sin(x) \]
\[ y_2(x) = \sin(x) - 0.0005x \cos(x) \]

We choose \( H = h \) for EHM5IPA and EHM5IIPA codes.

### Problem 4 (perturbed system)

\[ y_1^{(n)} + 100y_1 + \frac{2y_1y_2}{y_1^2 + y_2^2} = f_1(x), \quad y_1(0) = 1, \quad y_1^{(0)}(0) = \varepsilon \]
\[ y_2^{(n)} + 25y_2 + \frac{y_1^2 - y_2^2}{y_1^2 + y_2^2} = f_2(x), \quad y_2(0) = -\varepsilon, \quad y_2^{(0)}(0) = 5 \]

with \( \varepsilon = 10^{-3} \) and

\[ f_1(x) = \left[ (2\cos(10x)\sin(5x) + 2\varepsilon(\sin(5x))\sin(x) - \cos(10x)\cos(x) - \varepsilon^2\sin(10x))\cos^2(5x) + \sin^2(5x) + 2\varepsilon(\cos(x)\cos(10x) - \cos(x)\sin(5x)) + \varepsilon^2 \right] + 99\varepsilon \sin(x) \]
\[ f_2(x) = \left[ (\cos^2(10x) - \sin^2(5x) + 2\varepsilon(\cos(x))\cos(10x) + \cos(\cos(x))\sin(2x) - \varepsilon^2\cos(10x) + \sin^2(5x) + 2\varepsilon(\cos(x))\cos(10x) - \cos(x)\sin(5x)) + \varepsilon^2 \right] - 24\varepsilon \cos(x) \]
Solution:
\[ y_1(x) = \cos(10x) + \varepsilon \sin(x), \quad y_2(x) = \sin(5x) - \varepsilon \cos(x), \]
\[ 0 \leq x \leq 10. \]
For EHM5IPA and EHM5IIPA codes, we choose \( H = 10h \) for the first component and \( H = 5h \) for the second component.

From the numerical results, it is observed that EHM5I is almost as accurate as FETSH method. In addition, the maximum global error for EHM5II is of the same order as that for FETSH method with advantage for EHM5II as EHM5II is more stable for solving Problem 2. For Problem 3, EHM5IPA is the most accurate followed by TSI7. TSI7 is unstable when it was used to solve Problem 2 and Problem 4 for big step-size. Of all methods, EHM5IPA and EHM5IIPA are the most accurate for solving most problems considered. This is because EHM5IPA and EHM5IIPA are being phase fitted and amplification fitted compared to other methods.
In this paper, we investigate the implementation of phase fitted and amplification fitted explicit hybrid methods for solving second-order ordinary differential equations. From numerical observations, we conclude that the phase fitted and amplification fitted explicit hybrid methods are very accurate for solving second-order ordinary differential equations having oscillating solutions. The results also indicate that phase fitting and amplification fitting gives us methods with better accuracy compared to the base methods. Moreover, all of the new methods are capable to solve any physical problems whose solutions are in the oscillatory form. All codes are designed using Microsoft Visual C++ version 6.0 software in HP computer with specification Intel(R)Core(TM)2DuoCPU P8600@2.40GHz.

REFERENCES


