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Exact Solutions for Three Fractional Partial Differential Equations by the (G'/G) Method

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Abstract—**In this paper, based on certain variable transformation, we apply the known (G'/G) method to seek exact solutions for three fractional partial differential equations: the space fractional (2+1)-dimensional breaking soliton equations, the space-time fractional Fokas equation, and the space-time fractional Kaup-Kupershmidt equation. The fractional derivative is defined in the sense of modified Riemann-liouville derivative. With the aid of mathematical software Maple, a number of exact solutions including hyperbolic function solutions, trigonometric function solutions, and rational function solutions for them are obtained.**

Index Terms—**(G'/G) method, fractional partial differential equation, exact solution, variable transformation.**

I. Introduction

In the literature, research on the theory of differential equations, integral equations and matrix equations include various aspects, such as the existence and uniqueness of solutions [1,2], seeking for exact solutions [3,4], numerical method [5-7]. Among these investigations, research on the theory and applications of fractional differential and integral equations has been the focus of many studies due to their frequent appearance in various applications in physics, biology, engineering, signal processing, systems identification, control theory, finance and fractional dynamics, and has attracted much attention of more and more scholars. For example, Bouhassoun [8] extended the telescoping decomposition method to derive approximate analytical solutions of fractional differential equations. Bijura [9] investigated the solution of a singularly perturbed nonlinear system fractional integral equations. Blackledge [10] investigated the application of a certain fractional Diffusion equation, and applied it for predicting market behavior.

Among the investigations for fractional differential equations, research for seeking exact solutions and approximate solutions of fractional differential equations is a hot topic. Many powerful and efficient methods have been proposed so far (for example, see [11-24]) including the fractional variational iteration method, the Adomian's decomposition method, the homotopy perturbation method, the fractional sub-equation method, the finite difference method, the finite element method and so on. Using these methods, solutions with various forms for some given fractional differential equations have been established.

In [25], we extended the known (G'/G) method [26-28] to solve fractional partial differential equations, and obtained some exact solutions for the space-time fractional generalized Hirota-Satsuma coupled KdV equations and the time-fractional fifth-order Sawada-Kotera equation successfully. In this paper, we will furthermore test the validity of the (G'/G) method by applying it to solve other fractional partial differential equations. The fractional derivative is defined in the sense of modified Riemann-liouville derivative. We list the definition and some important properties for the modified Riemann-Liouville derivative of order α as follows [22-24,29]:

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, \\ \qquad 0<\alpha<1, \\ (f^{(n)}(t))^{(\alpha-n)},\ n\le\alpha<n+1,\ n\ge 1. \end{cases}$$

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}, \tag{1}$$

$$D_t^\alpha (f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \tag{2}$$

$$D_t^\alpha f[g(t)] = f'_g[g(t)]D_t^\alpha g(t) = D_g^\alpha f[g(t)](g'(t))^\alpha. \tag{3}$$

The rest of this paper is organized as follows. In Section 2, we give the description of the (G'/G) method for solving fractional partial differential equations. Then in Section 3 we apply this method to seek exact solutions for the space fractional (2+1)-dimensional breaking soliton equations, the space-time fractional Fokas equation, and the space-time fractional Kaup-Kupershmidt equation. Some conclusions are presented at the end of the paper.

II. Description of the (G'/G) method for solving fractional partial differential equations

In this section we give the description of the $(\frac{G'}{G})$ method for solving fractional partial differential equations.

Suppose that a fractional partial differential equation, say in the independent variables $t, x_1, x_2, ..., x_n$, is given by

$$\begin{aligned} &P(u_1, ...u_k,\ D_t^\alpha u_1, ..., D_t^\alpha u_k, D_{x_1}^\beta u_1, ..., \\ &D_{x_1}^\beta u_k, ..., D_{x_n}^\gamma u_1, ..., D_{x_n}^\gamma u_k, ...) = 0, \end{aligned} \tag{4}$$

where $u_i = u_i(t, x_1, x_2, ..., x_n),\ i = 1, ..., k$ are unknown functions, P is a polynomial in u_i and their various partial derivatives including fractional derivatives.

Step 1. Execute certain variable transformation

$$u_i(t, x_1, x_2, ..., x_n) = U_i(\xi), \quad \xi = \xi(t, x_1, x_2, ..., x_n), \tag{5}$$

such that Eq. (4) can be turned into the following ordinary differential equation of integer order with respect to the variable ξ:

$$\widetilde{P}(U_1, ..., U_k,\ U_1', ..., U_k', U_1'', ..., U_k'',\ ...) = 0. \tag{6}$$

Step 2. Suppose that the solution of (6) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$U_j(\xi) = \sum_{i=0}^{m_j} a_{j,i}(\frac{G'}{G})^i,\ j = 1, 2, ..., k, \tag{7}$$

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where $G = G(\xi)$ satisfies the second order ODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{8}$$

and $\lambda,\ \mu,\ a_{j,i},\ i = 0, 1, ..., m_j,\ j = 1, 2, ..., k$ are constants to be determined later, $a_{j,m} \neq 0$. The positive integer m_j can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (6).

By the generalized solutions of Eq. (8) we have

$$(\frac{G'}{G}) = \begin{cases} -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \\ (\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi}), \\ \qquad \lambda^2 - 4\mu > 0, \\ -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \\ (\frac{-C_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2}\xi}), \\ \qquad \lambda^2 - 4\mu < 0, \\ -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}, \qquad \lambda^2 - 4\mu = 0, \end{cases} \tag{9}$$

where $C_1,\ C_2$ are arbitrary constants.

Step 3. Substituting (7) into (6) and using (8), collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of (6) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\lambda,\ \mu,\ a_{j,i},\ i = 0, 1, ..., m_j,\ j = 1, 2, ..., k$.

Step 4. Solving the equations system in Step 3, and using (9), we can construct a variety of exact solutions for Eq. (4).

III. Application of the Proposed Method to Space Fractional (2+1)-Dimensional Breaking Soliton Equations

We consider the space fractional (2+1)-dimensional breaking soliton equations

$$\begin{cases} u_t + a\frac{\partial^{2\alpha+\beta}u}{\partial x^{2\alpha}y^{\beta}} + 4au\frac{\partial^{\alpha}v}{\partial x^{\alpha}} + 4a\frac{\partial^{\alpha}u}{\partial x^{\alpha}}v = 0, \\ \frac{\partial^{\beta}u}{\partial y^{\beta}} = \frac{\partial^{\alpha}v}{\partial x^{\alpha}}, \end{cases}, \tag{10}$$

where $0 < \alpha,\ \beta \leq 1$. Eqs. (10) are variation of the following (2+1)-dimensional breaking soliton equations equations [30-33]

$$\begin{cases} u_t + au_{xxy} + 4auv_x + 4au_xv = 0, \\ u_y = v_x. \end{cases} \tag{11}$$

For Eqs. (11), some periodic wave solutions, non-traveling wave solutions, and Jacobi elliptic function solutions were found in [20-23]. But we notice no research has been paid for Eqs. (10) so far. In the following, we will apply the described method in Section 2 to Eqs. (10).

To begin with, we suppose $u(x, y, t) = U(\xi),\ v(x, y, t) = V(\xi)$, where $\xi = ct + \frac{k_1}{\Gamma(1+\alpha)}x^{\alpha} + \frac{k_2}{\Gamma(1+\beta)}y^{\beta} + \xi_0$, $k_1,\ k_2,\ c,\ \xi_0$ are all constants with $k_1,\ k_2,\ c \neq 0$. Then by use of (1) and the first equality in Eq. (3), we obtain $D_x^{\alpha}u = D_x^{\alpha}U(\xi) = U'(\xi)D_x^{\alpha}\xi = k_1U'(\xi),\ D_y^{\beta}u = D_y^{\beta}U(\xi) = U'(\xi)D_y^{\beta}\xi = k_2U'(\xi),\ u_t = cU'(\xi)$, and then Eqs. (10) can be turned into

$$\begin{cases} cU' + ak_1^2k_2U''' + 4ak_1UV' + 4ak_1VU' = 0, \\ k_2U' = k_1V'. \end{cases} \tag{12}$$

Suppose that the solution of Eqs. (12) can be expressed by

$$\begin{cases} U(\xi) = \sum_{i=0}^{m_1} a_i(\frac{G'}{G})^i, \\ V(\xi) = \sum_{i=0}^{m_2} b_i(\frac{G'}{G})^i. \end{cases} \tag{13}$$

Balancing the order of U''' and UV', U' and V' in (12) we have $m_1 = m_2 = 2$. So

$$\begin{cases} U(\xi) = a_0 + a_1(\frac{G'}{G})^1 + a_2(\frac{G'}{G})^2, \\ V(\xi) = b_0 + b_1(\frac{G'}{G})^1 + b_2(\frac{G'}{G})^2. \end{cases} \tag{14}$$

Substituting (14) into (12), using Eq. (8) and collecting all the terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations yields:

$$a_0 = a_0,\ a_1 = -\frac{3}{2}k_1^2\lambda,\ a_2 = -\frac{3}{2}k_1^2,$$

$$b_0 = -\frac{8ak_1^2k_2\mu + ak_1^2k_2\lambda^2 + c + 4aa_0k_2}{4ak_1},$$

$$b_1 = -\frac{3}{2}k_1k_2\lambda,\ b_2 = -\frac{3}{2}k_1k_2,$$

where $\lambda,\ \mu,\ a_0$ are arbitrary constants.

Substituting the result above into Eqs. (14), and combining with (9) we can obtain the following exact solutions for Eqs. (10).

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solutions:

$$\begin{cases} u_1(x,y,t) = a_0 - \frac{3}{2}k_1^2\lambda[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \\ \quad (\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi})] \\ \quad -\frac{3}{2}k_1^2[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \\ \quad (\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi})]^2, \\ v_1(x,y,t) = -\frac{8ak_1^2k_2\mu + ak_1^2k_2\lambda^2 + c + 4aa_0k_2}{4ak_1} \\ \quad -\frac{3}{2}k_1k_2\lambda[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \\ \quad (\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi})] \\ \quad -\frac{3}{2}k_1k_2[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \\ \quad (\frac{C_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi})]^2, \end{cases} \tag{15}$$

where $\xi = ct + \frac{k_1}{\Gamma(1+\alpha)}x^\alpha + \frac{k_2}{\Gamma(1+\beta)}y^\beta + \xi_0$.

In particular, if we take $C_2 = 0$, then we obtain the following solitary wave solutions, which are shown in Figs. 1-2.

$$\begin{cases} u_2(x,y,t) = a_0 + \frac{3}{8}k_1^2\lambda^2 \\ \quad -\frac{3}{8}k_1^2(\lambda^2-4\mu)[\tanh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi]^2, \\ v_2(x,y,t) = -\frac{8ak_1^2k_2\mu + ak_1^2k_2\lambda^2 + c + 4aa_0k_2}{4ak_1} \\ \quad +\frac{3}{8}k_1k_2\lambda^2 - \frac{3}{8}k_1k_2(\lambda^2-4\mu)[\tanh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi]^2, \end{cases}$$

When $\lambda^2 - 4\mu < 0$, we obtain the periodic function solutions:

$$\begin{cases} u_3(x,y,t) = a_0 - \frac{3}{2}k_1^2\lambda[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \\ \quad (\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi})] \\ \quad -\frac{3}{2}k_1^2[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \\ \quad (\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi})]^2, \\ v_3(x,y,t) = -\frac{8ak_1^2k_2\mu + ak_1^2k_2\lambda^2 + c + 4aa_0k_2}{4ak_1} \\ \quad -\frac{3}{2}k_1k_2\lambda[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \\ \quad (\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi})] \\ \quad -\frac{3}{2}k_1k_2[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \\ \quad (\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi})]^2, \end{cases} \tag{16}$$

where $\xi = ct + \frac{k_1}{\Gamma(1+\alpha)}x^\alpha + \frac{k_2}{\Gamma(1+\beta)}y^\beta + \xi_0$.

When $\lambda^2 - 4\mu = 0$, we obtain the rational function solutions:

$$\begin{cases} u_4(x,y,t) = a_0 - \frac{3}{2}k_1^2\lambda[-\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi}] \\ \quad -\frac{3}{2}k_1^2[-\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi}]^2, \\ v_4(x,y,t) = -\frac{8ak_1^2k_2\mu + ak_1^2k_2\lambda^2 + c + 4aa_0k_2}{4ak_1} \\ \quad -\frac{3}{2}k_1k_2\lambda[-\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi}] \\ \quad -\frac{3}{2}k_1k_2[-\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi}]^2, \end{cases} \tag{17}$$

where $\xi = ct + \frac{k_1}{\Gamma(1+\alpha)}x^\alpha + \frac{k_2}{\Gamma(1+\beta)}y^\beta + \xi_0$.

Remark 1. The established solutions above for the space fractional (2+1)-dimensional breaking soliton equations are new exact solutions so far in the literature.

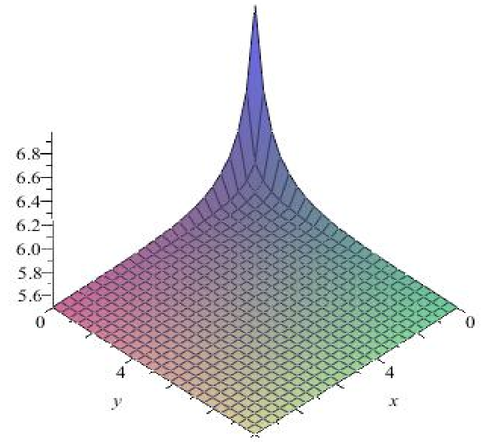

Fig. 1 The solitary wave solution u_2 with $a_0 = 1,\ \lambda = 4,\ \mu = 3,$ $k_1 = k_2 = 1,\ \xi_0 = 0,\ \alpha = \beta = \frac{1}{2},\ c = 1,\ t = 0.1$

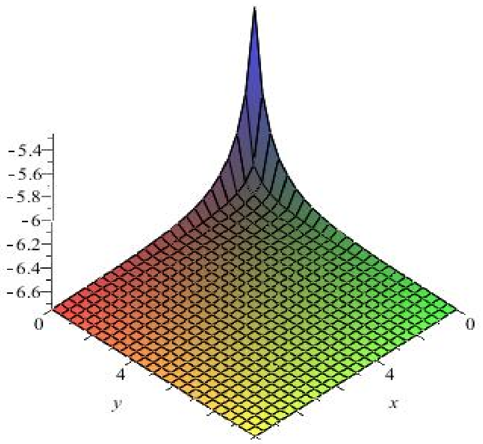

Fig. 2 The solitary wave solution v_2 with $a_0 = 1,\ \lambda = 4,\ \mu = 3,$ $a = k_1 = k_2 = 1,\ \xi_0 = 0,\ \alpha = \beta = \frac{1}{2},\ c = 1,\ t = x_2 = y_2 = 0.1$

IV. Application of the proposed method to space-time fractional Fokas equation

We consider the space-time fractional Fokas equation

$$4\frac{\partial^{2\alpha}q}{\partial t^\alpha \partial x_1^\alpha} - \frac{\partial^{4\alpha}q}{\partial x_1^{3\alpha}\partial x_2^\alpha} + \frac{\partial^{4\alpha}q}{\partial x_2^{3\alpha}\partial x_1^\alpha} + 12\frac{\partial^\alpha q}{\partial x_1^\alpha}\frac{\partial^\alpha q}{\partial x_2^\alpha}$$
$$+12q\frac{\partial^{2\alpha}q}{\partial x_1^\alpha \partial x_2^\alpha} - 6\frac{\partial^{2\alpha}q}{\partial y_1^\alpha \partial y_2^\alpha} = 0,\ 0 < \alpha \le 1. \tag{18}$$

In [24], the authors solved Eq. (18) by a fractional Riccati sub-equation method, and obtained some exact solutions for it. Now we will apply the described method in Section 3 to Eq. (18).

Suppose $q(x,y,t) = U(\xi)$, where $\xi = \frac{c}{\Gamma(1+\alpha)}t^\alpha + \frac{k_1}{\Gamma(1+\alpha)}x_1^\alpha + \frac{k_2}{\Gamma(1+\alpha)}x_2^\alpha + \frac{l_1}{\Gamma(1+\alpha)}y_1^\alpha + \frac{l_2}{\Gamma(1+\alpha)}y_2^\alpha + \xi_0,\ k_1,\ k_2,\ l_1,\ l_2,\ c,\ \xi_0$ are all constants with $k_1,\ k_2,\ l_1,\ l_2,\ c \neq 0$. Then by use of (1) and the first equality in Eq. (3), Eq. (18) can be turned into

$$4ck_1U'' - k_1^3k_2U^{(4)} + k_2^3k_1U^{(4)} + 12k_1k_2(U')^2$$
$$+12k_1k_2UU'' - 6l_1l_2U'' = 0. \tag{19}$$

Suppose that the solution of Eq. (19) can be expressed by

$$U(\xi) = \sum_{i=0}^{m} a_i(\frac{G'}{G})^i, \tag{20}$$

where $G = G(\xi)$ satisfies Eq. (8). By Balancing the order between the highest order derivative term and nonlinear term in Eq. (19) we can obtain $m = 2$. So we have

$$U(\xi) = a_0 + a_1(\frac{G'}{G}) + a_2(\frac{G'}{G})^2. \tag{21}$$

Substituting (21) into (19) and collecting all the terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, yields:

$$a_0 = \frac{k_1^3k_2\lambda^2 - k_1k_2^3\lambda^2 + 8k_1^3k_2\mu - 8k_1k_2^3\mu - 4ck_1 + 6l_1l_2}{12k_1k_2},$$

$$a_1 = \lambda(k_1^2 - k_2^2),\ a_2 = k_1^2 - k_2^2.$$

Substituting the result above into Eq. (21), and combining with (9) we can obtain the following exact solutions to Eq. (18).

When $\lambda^2 - 4\mu > 0$,

$$\begin{aligned} &q_1(t, x_1, x_2, y_1, y_2) = \\ &\frac{k_1^3k_2\lambda^2 - k_1k_2^3\lambda^2 + 8k_1^3k_2\mu - 8k_1k_2^3\mu - 4ck_1 + 6l_1l_2}{12k_1k_2} \\ &+\lambda(k_1^2 - k_2^2)[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \\ &(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi})] \\ &+(k_1^2 - k_2^2)[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \\ &(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi})]^2, \end{aligned} \tag{22}$$

where $\xi = \frac{c}{\Gamma(1+\alpha)}t^\alpha + \frac{k_1}{\Gamma(1+\alpha)}x_1^\alpha + \frac{k_2}{\Gamma(1+\alpha)}x_2^\alpha + \frac{l_1}{\Gamma(1+\alpha)}y_1^\alpha + \frac{l_2}{\Gamma(1+\alpha)}y_2^\alpha + \xi_0$.

When $\lambda^2 - 4\mu < 0$,

$$\begin{aligned} &q_2(t, x_1, x_2, y_1, y_2) = \\ &\frac{k_1^3k_2\lambda^2 - k_1k_2^3\lambda^2 + 8k_1^3k_2\mu - 8k_1k_2^3\mu - 4ck_1 + 6l_1l_2}{12k_1k_2} \\ &+\lambda(k_1^2 - k_2^2)[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \\ &(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi})] \\ &+(k_1^2 - k_2^2)[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \\ &(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi})]^2, \end{aligned} \tag{23}$$

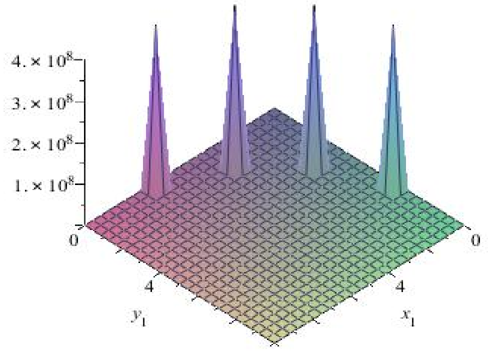

Fig. 3 The periodic solution q_2 in (23) with $\lambda = 2,\ \mu = 2,\ k_1 = 2,\ k_2 = 1,\ l_1 = l_2 = 1,\ C_2 = \xi_0 = 0,\ \alpha = \frac{1}{2},\ c = 1,\ t = x_2 = y_2 = 0.1$

where $\xi = \frac{c}{\Gamma(1+\alpha)}t^\alpha + \frac{k_1}{\Gamma(1+\alpha)}x_1^\alpha + \frac{k_2}{\Gamma(1+\alpha)}x_2^\alpha + \frac{l_1}{\Gamma(1+\alpha)}y_1^\alpha + \frac{l_2}{\Gamma(1+\alpha)}y_2^\alpha + \xi_0$.

In Fig. 3, the periodic function solution (23) is demonstrated with some certain parameters.

When $\lambda^2 - 4\mu = 0$,

$$\begin{aligned} &q_3(t, x_1, x_2, y_1, y_2) = \\ &\frac{k_1^3k_2\lambda^2 - k_1k_2^3\lambda^2 + 8k_1^3k_2\mu - 8k_1k_2^3\mu - 4ck_1 + 6l_1l_2}{12k_1k_2} + \\ &\lambda(k_1^2-k_2^2)[-\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi}] + (k_1^2-k_2^2)[-\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi}]^2, \end{aligned} \tag{24}$$

where $\xi = \frac{c}{\Gamma(1+\alpha)}t^\alpha + \frac{k_1}{\Gamma(1+\alpha)}x_1^\alpha + \frac{k_2}{\Gamma(1+\alpha)}x_2^\alpha + \frac{l_1}{\Gamma(1+\alpha)}y_1^\alpha + \frac{l_2}{\Gamma(1+\alpha)}y_2^\alpha + \xi_0$.

Remark 2. As one can see, the established solutions for the space-time fractional Fokas equation above are different from the results in [24], and are new exact solutions so far to our best knowledge.

V. Application of the Proposed Method to Space-Time Fractional Kaup-Kupershmidt Equation

We consider the following space-time fractional Kaup-Kupershmidt equation

$$\begin{aligned} &D_t^\alpha u + D_x^{5\beta}u + 45u^2D_x^\beta u - \frac{75}{2}D_x^\beta uD_x^{2\beta}u \\ &-15uD_x^{3\beta}u = 0,\ 0 < \alpha,\ \beta \le 1, \end{aligned} \tag{25}$$

which is a variation of the following Kaup-Kupershmidt equation [34-36]:

$$u_t + u_{xxxxx} + 45u_xu^2 - \frac{75}{2}u_xu_{xx} - 15uu_{xxx} = 0, \tag{26}$$

To begin with, we suppose $u(x,t) = U(\xi)$, where $\xi = \frac{c}{\Gamma(1+\alpha)}t^\alpha + \frac{k}{\Gamma(1+\beta)}x^\beta + \xi_0$, $k,\ c,\ \xi_0$ are all constants with $k,\ c \neq 0$. Then by use of (1) and the first equality in Eq. (3), Eq. (25) can be turned into

$$cU' + k^5U^{(5)} + 45kU^2U' - \frac{75}{2}k^3U'U'' - 15k^3UU''' = 0. \tag{27}$$

Suppose that the solution of Eq. (27) can be expressed by

$$U(\xi)=\sum_{i=0}^{m}a_i(\frac{G'}{G})^i. \tag{28}$$

Balancing the order of $U^{(5)}$ and UU''' in Eq. (27) we have $m=2$. So

$$U(\xi)=a_0+a_1(\frac{G'}{G})+a_2(\frac{G'}{G})^2. \tag{29}$$

Substituting (29) into (27), using Eq. (8) and collecting all the terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, yields:

Case 1:

$$a_2=8k^2,\ a_1=8k^2\lambda,\ a_0=\frac{2}{3}k^2(\lambda^2+8\mu),\ k=k,$$

$$c=-11k^5(-8\lambda^2\mu+16\mu^2+\lambda^4),$$

$$g=\frac{1664}{9}k^7\mu^3+\frac{104}{3}k^7\lambda^4\mu-\frac{416}{3}k^7\lambda^2\mu^2. \tag{30}$$

Substituting the result above into Eq. (29) and combining with (9) we can obtain the following exact solutions to Eq. (25).

When $\lambda^2-4\mu>0$,

$$u_1(x,t)=-2k^2\lambda^2+2k^2(\lambda^2-4\mu)$$
$$\left(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right)^2$$
$$+\frac{2}{3}k^2(\lambda^2+8\mu), \tag{31}$$

where $\xi=\frac{-11k^5(-8\lambda^2\mu+16\mu^2+\lambda^4)}{\Gamma(1+\alpha)}t^\alpha+\frac{k}{\Gamma(1+\beta)}x^\beta+\xi_0$.

In particular, when $\lambda>0,\mu=0,C_1\neq0,C_2=0$, we can deduce the soliton solutions of the Kaup-Kupershmidt equation as follows:

$$u_2(x,t)=2k^2\lambda^2sech^2(\frac{\lambda\xi}{2})+\frac{2}{3}k^2\lambda^2. \tag{32}$$

When $\lambda^2-4\mu<0$,

$$u_3(x,t)=-2k^2\lambda^2+2k^2(4\mu-\lambda^2)$$
$$\left(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right)^2$$
$$+\frac{2}{3}k^2(\lambda^2+8\mu), \tag{33}$$

where $\xi=\frac{-11k^5(-8\lambda^2\mu+16\mu^2+\lambda^4)}{\Gamma(1+\alpha)}t^\alpha+\frac{k}{\Gamma(1+\beta)}x^\beta+\xi_0$.

When $\lambda^2-4\mu=0$,

$$u_4(x,t)=-2k^2\lambda^2+\frac{8k^2C_2^2}{(C_1+C_2\xi)^2}+\frac{2}{3}k^2(\lambda^2+8\mu), \tag{34}$$

where $\xi=\frac{-11k^5(-8\lambda^2\mu+16\mu^2+\lambda^4)}{\Gamma(1+\alpha)}t^\alpha+\frac{k}{\Gamma(1+\beta)}x^\beta+\xi_0$.

Case 2:

$$a_2=k^2,\ a_1=k^2\lambda,\ a_0=\frac{1}{12}k^2(\lambda^2+4\mu),\ k=k,$$

$$c=-\frac{1}{16}k^5(-8\lambda^2\mu+16\mu^2+\lambda^4),$$

$$g=-\frac{2}{9}k^7\mu^3-\frac{1}{24}k^7\lambda^4\mu+\frac{1}{6}k^7\lambda^2\mu^2+\frac{1}{288}k^7\lambda^6.$$

Substituting the result above into Eq. (29) and combining with (9) we can obtain the following exact solutions to Eq. (25).

When $\lambda^2-4\mu>0$,

$$u_5(x,t)=-\frac{1}{4}k^2\lambda^2+\frac{1}{4}k^2(\lambda^2-4\mu)$$
$$\left(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right)^2$$
$$+\frac{1}{12}k^2(\lambda^2+4\mu), \tag{35}$$

where $\xi=\frac{-\frac{1}{16}k^5(-8\lambda^2\mu+16\mu^2+\lambda^4)}{\Gamma(1+\alpha)}t^\alpha+\frac{k}{\Gamma(1+\beta)}x^\beta+\xi_0$.

In particular, when $\lambda>0,\mu=0,C_1\neq0,C_2=0$, we can deduce the soliton solutions of the Kaup-Kupershmidt equation as follows:

$$u_6(x,t)=\frac{1}{4}k^2\lambda^2sech^2(\frac{\lambda\xi}{2})+\frac{1}{12}k^2\lambda^2. \tag{36}$$

When $\lambda^2-4\mu<0$,

$$u_7(x,t)=-\frac{1}{4}k^2\lambda^2+\frac{1}{4}k^2(4\mu-\lambda^2)$$
$$\left(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right)^2$$
$$+\frac{1}{12}k^2(\lambda^2+4\mu), \tag{37}$$

where $\xi=\frac{-\frac{1}{16}k^5(-8\lambda^2\mu+16\mu^2+\lambda^4)}{\Gamma(1+\alpha)}t^\alpha+\frac{k}{\Gamma(1+\beta)}x^\beta+\xi_0$.

When $\lambda^2-4\mu=0$,

$$u_8(x,t)=-\frac{1}{4}k^2\lambda^2+\frac{k^2C_2^2}{(C_1+C_2\xi)^2}+\frac{1}{12}k^2(\lambda^2+4\mu), \tag{38}$$

where $\xi = \dfrac{-\frac{1}{16}k^5(-8\lambda^2\mu + 16\mu^2 + \lambda^4)}{\Gamma(1+\alpha)}t^\alpha + \dfrac{k}{\Gamma(1+\beta)}x^\beta + \xi_0$.

Remark 3. The established solutions in Eqs. (31)-(38) are new exact solutions for the space-time fractional Kaup-Kupershmidt equation.

VI. Conclusion

We have applied the known (G'/G) method to solve the space fractional (2+1)-dimensional breaking soliton equations, the space-time fractional Fokas equation, and the space-time fractional Kaup-Kupershmidt equation. Based on certain fractional transformation, such fractional partial differential equations can be turned into ordinary differential equations of integer order, the solutions of which can be expressed by a polynomial in $(\frac{G'}{G})$, where G satisfies the ODE $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$. With the aid of mathematical software, a variety of exact solutions for these fractional partial differential equations are obtained. Being concise and powerful, we note that this approach can also be applied to solve other fractional partial differential equations.

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