Interval Oscillation Criteria for a Class of Nonlinear Fractional Differential Equations with Nonlinear Damping Term

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Abstract—**In this paper, based on certain variable transformation, we apply the known (G'/G) method to seek exact solutions for three fractional partial differential equations: the space fractional (2+1)-dimensional breaking soliton equations, the space-time fractional Fokas equation, and the space-time fractional Kaup-Kupershmidt equation. The fractional derivative is defined in the sense of modified Riemann-liouville derivative. With the aid of mathematical software Maple, a number of exact solutions including hyperbolic function solutions, trigonometric function solutions, and rational function solutions for them are obtained.**

Index Terms—**(G'/G) method, fractional partial differential equation, exact solution, variable transformation.**

I. Introduction

Fractional differential equations are generalizations of classical differential equations of integer order, and can find their applications in many fields of science and engineering. In the literature, research on the theory of differential equations, integral equations and matrix equations include various aspects, such as the existence and uniqueness of solutions [1,2], seeking for exact solutions [3,4], numerical method [5-7]. Among these investigations, research on the theory and applications of fractional differential and integral equations has been the focus of many studies due to their frequent appearance in various applications in physics, biology, engineering, signal processing, systems identification, control theory, finance and fractional dynamics, and has attracted much attention of more and more scholars. For example, Bouhassoun [8] extended the telescoping decomposition method to derive approximate analytical solutions of fractional differential equations. Bijura [9] investigated the solution of a singularly perturbed nonlinear system fractional integral equations. Blackledge [10] investigated the application of a certain fractional Diffusion equation, and applied it for predicting market behavior. In [11-18], the existence, uniqueness, stability of solutions, and numerical methods of fractional differential equations were investigated.

In these investigations, we notice that very little attention is paid to oscillation of fractional differential equations. Recent results in this direction include Chen's work [19,20] and Zheng's work [21]. In [19], Chen researched oscillation of the following fractional differential equation:

$$[r(t)(D_-^\alpha y(t))^\eta]' - q(t)f(\int_t^\infty (v-t)^{-\alpha}y(v)dv) = 0,\ t>0,$$

where $r,\ q$ are positive-valued functions, η is the quotient of two odd positive numbers, $\alpha \in (0,1)$, $D_-^\alpha y(t)$ denotes the Liouville right-sided fractional derivative of order α of x, and $D_-^\alpha y(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^\infty (\xi - t)^{-\alpha}y(\xi)d\xi$. Then in [20], under similar conditions to [19], some new oscillation criteria are established for the following fractional differential equation with damping term:

$$D_-^{1+\alpha}y(t) - p(t)D_-^\alpha y(t)$$
$$+q(t)f(\int_t^\infty (v-t)^{-\alpha}y(v)dv) = 0,\ t>0,$$

At the same time, in [21], Zheng researched oscillation of the following nonlinear fractional differential equation with damping term and more general form than the equations mentioned above:

$$[a(t)(D_-^\alpha x(t))^\gamma]' + p(t)(D_-^\alpha x(t))^\gamma$$
$$-q(t)f(\int_t^\infty (\xi-t)^{-\alpha}x(\xi)d\xi) = 0,\ t\in[t_0,\infty).$$

In this paper, we are concerned with oscillation for a class of nonlinear fractional differential equations with nonlinear damping term as follows:

$$D_t^\alpha(r(t)k_1(x(t), D_t^\alpha x(t))) + p(t)k_2(x(t), D_t^\alpha x(t))D_t^\alpha x(t)$$
$$+q(t)f(x(t)) = 0,\ t\ge t_0\ge 0,\ 0<\alpha<1, \tag{1}$$

where $D_t^\alpha(.)$ denotes the modified Riemann-liouville derivative [22] with respect to the variable t, the functions $r,\ q \in C^\alpha([t_0,\infty), R_+),\ p\in C^\alpha([t_0,\infty),[0,\infty))$, and C^α denotes continuous derivative of order α, the function f is continuous satisfying $f(x)/x \ge K$ for some positive constant K and $\forall x\ne 0$, k_1 is continuously differentiable satisfying $k_1^2(u,v)\le Avk_1(u,v)$ for some positive constant A, $\forall v\in R\backslash\{0\}$ and $\forall u\in R$.

The definition and some important properties for the Jumarie's modified Riemann-Liouville derivative of order α are listed as follows (see also in [23-25]):

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, \\ \qquad 0<\alpha<1, \\ (f^{(n)}(t))^{(\alpha-n)},\ n\le\alpha<n+1,\ n\ge 1. \end{cases}$$

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}, \tag{2}$$

$$D_t^\alpha(f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \tag{3}$$

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$$D_t^\alpha f[g(t)] = f'_g[g(t)]D_t^\alpha g(t) = D_g^\alpha f[g(t)](g'(t))^\alpha. \quad (4)$$

As usual, a nonconstant continuable solution $x(t)$ of Eq. (1) is called proper if $\sup_{t\geq t_0} \mid x(t) \mid > 0$. A proper solution $x(t)$ of Eq. (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Eq. (1) is called oscillatory if all its solutions are oscillatory.

We organize the next of this paper as follows. In Section 2, we establish some new interval oscillation criteria for Eq. (1) under the condition for $f(x)$ without monotonicity, while oscillation criteria are established for Eq. (1) under the condition for $f(x)$ with monotonicity in Section 3. We present some examples for the results established in Section 4. Some conclusions are presented at the end of this paper.

For the sake of convenience, in the next of this paper, we denote $\xi_0 = \frac{t_0^\alpha}{\Gamma(1+\alpha)},\ \xi = \frac{t^\alpha}{\Gamma(1+\alpha)},\ R_+ = (0,\infty)$, and let $h_1,\ h_2,\ H \in C([\xi_0,\infty), R)$ satisfying

$$H(\xi,\xi) = 0, H(\xi,s) > 0, \quad \xi > s \geq \xi_0.$$

H has continuous partial derivatives $\frac{\partial H(\xi,s)}{\partial \xi}$ and $\frac{\partial H(\xi,s)}{\partial s}$ on $[\xi_0,\infty)$ such that

$$\frac{\partial H(\xi,s)}{\partial \xi} = -h_1(\xi,s)\sqrt{H(\xi,s)},$$

$$\frac{\partial H(\xi,s)}{\partial s} = -h_2(\xi,s)\sqrt{H(\xi,s)}, \quad \xi > s \geq \xi_0.$$

II. Interval oscillation criteria with $f(x)$ not being monotone

Lemma 2.1. Suppose that $k_2(u,v) : R^2 \to R^2$ is continuous and has the sign of v for all $v \in R\backslash\{0\}$ and all $u \in R$. If $x(t)$ is a non-oscillatory solution of Eq. (1), then $x(t)D_t^\alpha x(t) < 0$ for $t \geq t_*$, where $t_* \geq t_0$ is sufficiently large.

Proof. Suppose $x(t)$ be a proper non-oscillatory solution of Eq. (1). Let $r(t) = \widetilde{r}(\xi),\ x(t) = \widetilde{x}(\xi),\ p(t) = \widetilde{p}(\xi),\ q(t) = \widetilde{q}(\xi)$, where $\xi = \frac{t^\alpha}{\Gamma(1+\alpha)}$. Then by use of Eq. (2) we obtain $D_t^\alpha \xi(t) = 1$, and furthermore by use of the first equality in Eq. (4), we have

$$D_t^\alpha r(t) = D_t^\alpha \widetilde{r}(\xi) = \widetilde{r}'(\xi)D_t^\alpha \xi(t) = \widetilde{r}'(\xi).$$

Similarly we have $D_t^\alpha x(t) = \widetilde{x}'(\xi)$. So Eq. (1) can be transformed into the following form:

$$(\widetilde{r}(\xi)k_1(\widetilde{x}(\xi),\widetilde{x}'(\xi)))' + \widetilde{p}(\xi)k_2(\widetilde{x}(\xi),\widetilde{x}'(\xi))\widetilde{x}'(\xi)$$
$$+\widetilde{q}(\xi)f(\widetilde{x}(\xi)) = 0,\ \xi \geq \xi_0 \geq 0. \quad (5)$$

Then $\widetilde{x}(\xi)$ be a proper non-oscillatory solution of Eq. (5). By [26, Lemma 1], we have $\widetilde{x}(\xi)\widetilde{x}'(\xi) < 0,\ \xi \geq \xi_*$, where $\xi_* \geq \xi_0$ is sufficiently large. Let $\xi_* = \frac{t_*^\alpha}{\Gamma(1+\alpha)}$. Then $x(t)D_t^\alpha x(t) < 0,\ t \geq t_*$, and the proof is complete.

Theorem 2.1. Under the conditions of Lemma 2.1, furthermore, assume that there exists $\rho \in C^\alpha([t_0,\infty), R_+)$ such that for any sufficiently large $T \geq \xi_0$, there exist $a,\ b,\ c$ with $T \leq a < c < b$ satisfying

$$\frac{1}{H(c,a)}\int_a^c H(s,a)K\widetilde{\rho}(s)\widetilde{q}(s)ds$$
$$+\frac{1}{H(b,c)}\int_c^b H(b,s)K\widetilde{\rho}(s)\widetilde{q}(s)ds$$
$$> \frac{A}{4H(c,a)}\int_a^c \widetilde{r}(s)\widetilde{\rho}(s)Q_1^2(s,a)ds$$
$$+\frac{A}{4H(b,c)}\int_c^b \widetilde{r}(s)\widetilde{\rho}(s)Q_2^2(b,s)ds, \quad (6)$$

where $\widetilde{\rho}(\xi) = \rho(t),\ \widetilde{q}(\xi) = q(t),\ \widetilde{r}(\xi) = r(t)$, $Q_1(s,\xi) = h_1(s,\xi) - \frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\sqrt{H(s,\xi)},\ Q_2(\xi,s) = h_2(\xi,s) - \frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\sqrt{H(\xi,s)}$, then Eq. (1) is oscillatory.

Proof: Suppose to the contrary that $x(t)$ be a non-oscillatory solution of Eq. (1). Without loss of generality, we may assume that $x(t) > 0$ on $[T_0,\infty)$, where $T_0 \geq t_0$ is sufficiently large. Furthermore, by Lemma 2.1, there exists $t_* \geq T_0$ such that $D_t^\alpha x(t) < 0,\ t \geq t_*$. So by the assumption on Eq. (1) we have $k_1(x(t), D_t^\alpha x(t)) \geq AD_t^\alpha x(t),\ t \geq t_*$.

Define

$$w(t) = \rho(t)\frac{r(t)k_1(x(t),D_t^\alpha x(t))}{x(t)}. \quad (7)$$

Then for $t \geq t_*$, we have

$$D_t^\alpha w(t) = \frac{D_t^\alpha \rho(t)}{\rho(t)}w(t) - \rho(t)p(t)\frac{k_2(x(t),D_t^\alpha x(t))D_t^\alpha x(t)}{x(t)}$$
$$-\rho(t)q(t)\frac{f(x(t))}{x(t)} - \rho(t)\frac{r(t)k_1(x(t),D_t^\alpha x(t))D_t^\alpha x(t)}{x^2(t)}$$
$$= \frac{D_t^\alpha \rho(t)}{\rho(t)}w(t) - \rho(t)p(t)\frac{k_2(x(t),D_t^\alpha x(t))D_t^\alpha x(t)}{x(t)}$$
$$-\rho(t)q(t)\frac{f(x(t))}{x(t)} - \frac{w^2(t)D_t^\alpha x(t)}{\rho(t)r(t)k_1(x(t),D_t^\alpha x(t))}$$
$$\leq \frac{D_t^\alpha \rho(t)}{\rho(t)}w(t) - \rho(t)p(t)\frac{k_2(x(t),D_t^\alpha x(t))D_t^\alpha x(t)}{x(t)}$$
$$-\rho(t)q(t)\frac{f(x(t))}{x(t)} - \frac{w^2(t)}{A\rho(t)r(t)}. \quad (8)$$

Since $k_2(x(t),D_t^\alpha x(t))D_t^\alpha x(t) \geq 0$, furthermore we obtain

$$D_t^\alpha w(t) \leq -K\rho(t)q(t) + \frac{D_t^\alpha \rho(t)}{\rho(t)}w(t) - \frac{w^2(t)}{Ar(t)\rho(t)},\ t \geq t_*. \quad (9)$$

Let $w(t) = \widetilde{w}(\xi)$. Then $D_t^\alpha w(t) = \widetilde{w}'(\xi),\ D_t^\alpha \rho(t) = \widetilde{\rho}'(\xi)$, and (9) is transformed into the following form

$$\widetilde{w}'(\xi) \leq -K\widetilde{\rho}(\xi)\widetilde{q}(\xi) + \frac{\widetilde{\rho}'(\xi)}{\widetilde{\rho}(\xi)}\widetilde{w}(\xi) - \frac{\widetilde{w}^2(\xi)}{A\widetilde{r}(\xi)\widetilde{\rho}(\xi)},\ \xi \geq \xi_*, \quad (10)$$

where $\xi_* = \frac{t_*^\alpha}{\Gamma(1+\alpha)}$.

Select $a,\ b,\ c$ arbitrarily in $[\xi_*,\infty)$ with $b > c > a$. Substituting ξ with s, multiplying both sides of (10) by

$H(\xi,s)$ and integrating it with respect to s from c to ξ for $\xi\in[c,b)$, we get that

$$\int_c^\xi H(\xi,s)K\widetilde{\rho}(s)\widetilde{q}(s)ds \le -\int_c^\xi H(\xi,s)\widetilde{w}'(s)ds$$
$$+\int_c^\xi H(\xi,s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\widetilde{w}(s)ds-\int_c^\xi H(\xi,s)\frac{\widetilde{w}^2(s)}{A\widetilde{\rho}(s)\widetilde{r}(s)}ds$$
$$=H(\xi,c)\widetilde{w}(c)-\int_c^\xi[(\frac{H(\xi,s)}{A\widetilde{\rho}(s)\widetilde{r}(s)})^{1/2}\widetilde{w}(s)$$
$$+\frac{1}{2}(A\widetilde{\rho}(s)\widetilde{r}(s))^{1/2}Q_2(\xi,s)]^2ds+\int_c^\xi\frac{A\widetilde{\rho}(s)\widetilde{r}(s)}{4}Q_2^2(\xi,s)ds$$
$$\le H(\xi,c)\widetilde{w}(c)+\int_c^\xi\frac{A\widetilde{\rho}(s)\widetilde{r}(s)}{4}Q_2^2(\xi,s)ds. \quad (11)$$

Dividing both sides of the inequality (11) by $H(\xi,c)$ and let $\xi\to b^-$, we obtain

$$\frac{1}{H(b,c)}\int_c^b H(b,s)K\widetilde{\rho}(s)\widetilde{q}(s)ds\le$$
$$\widetilde{w}(c)+\frac{1}{H(b,c)}\int_c^b\frac{A\widetilde{\rho}(s)\widetilde{r}(s)}{4}Q_2^2(b,s)ds. \quad (12)$$

On the other hand, substituting ξ with s, multiplying both sides of (10) by $H(s,\xi)$ and integrating it with respect to s from ξ to c for $\xi\in(a,c]$, we get that

$$\int_\xi^c H(s,\xi)K\widetilde{\rho}(s)\widetilde{q}(s)ds\le-\int_\xi^c H(s,\xi)\widetilde{w}'(s)ds$$
$$+\int_\xi^c H(s,\xi)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\widetilde{w}(s)ds-\int_\xi^c H(s,\xi)\frac{\widetilde{w}^2(s)}{A\widetilde{\rho}(s)\widetilde{r}(s)}ds$$
$$=-H(c,\xi)\widetilde{w}(c)-\int_\xi^c[(\frac{H(s,\xi)}{A\widetilde{\rho}(s)\widetilde{r}(s)})^{1/2}\widetilde{w}(s)$$
$$+\frac{1}{2}(A\widetilde{\rho}(s)\widetilde{r}(s))^{1/2}Q_1(s,\xi)]^2ds+\int_\xi^c\frac{A\widetilde{\rho}(s)\widetilde{r}(s)}{4}Q_1^2(s,\xi)ds$$
$$\le -H(c,\xi)\widetilde{w}(c)+\int_\xi^c\frac{A\widetilde{\rho}(s)\widetilde{r}(s)}{4}Q_1^2(s,\xi)ds. \quad (13)$$

Dividing both sides of the inequality (13) by $H(c,\xi)$ and letting $\xi\to a^+$, we obtain

$$\frac{1}{H(c,a)}\int_a^c H(s,a)K\widetilde{\rho}(s)\widetilde{q}(s)ds\le$$
$$-\widetilde{w}(c)+\frac{1}{H(c,a)}\int_a^c\frac{A\widetilde{\rho}(s)\widetilde{r}(s)}{4}Q_1^2(s,a)ds. \quad (14)$$

A combination of (12) and (14) yields

$$\frac{1}{H(c,a)}\int_a^c H(s,a)K\widetilde{\rho}(s)\widetilde{q}(s)ds$$
$$+\frac{1}{H(b,c)}\int_c^b H(b,s)K\widetilde{\rho}(s)\widetilde{q}(s)ds$$
$$\le\frac{A}{4H(c,a)}\int_a^c\widetilde{r}(s)\widetilde{\rho}(s)Q_1^2(s,a)ds$$
$$+\frac{A}{4H(b,c)}\int_c^b\widetilde{r}(s)\widetilde{\rho}(s)Q_2^2(b,s)ds,$$

which contradicts to (6). So the proof is complete.

Theorem 2.2. Under the conditions of Theorem 2.1, furthermore, suppose (6) does not hold. If for any $l\ge\xi_0$,

$$\lim_{\xi\to\infty}\sup\int_l^\xi\left[H(s,l)K\widetilde{\rho}(s)\widetilde{q}(s)-\frac{A\widetilde{r}(s)\widetilde{\rho}(s)}{4}Q_1^2(s,l)\right]ds>0 \quad (15)$$

and

$$\lim_{\xi\to\infty}\sup\int_l^\xi\left[H(\xi,s)K\widetilde{\rho}(s)\widetilde{q}(s)-\frac{A\widetilde{r}(s)\widetilde{\rho}(s)}{4}Q_2^2(\xi,s)\right]ds>0, \quad (16)$$

then Eq. (1) is oscillatory.

Proof: For any $T\ge\xi_0$, let $a=T$. In (15) we choose $l=a$. Then there exists $c>a$ such that

$$\int_a^c\left[H(s,a)K\widetilde{\rho}(s)\widetilde{q}(s)-\frac{A\widetilde{r}(s)\widetilde{\rho}(s)}{4}Q_1^2(s,a)\right]ds>0. \quad (17)$$

In (16) we choose $l=c$. Then there exists $b>c$ such that

$$\int_c^b\left[H(b,s)K\widetilde{\rho}(s)\widetilde{q}(s)-\frac{A\widetilde{r}(s)\widetilde{\rho}(s)}{4}Q_2^2(b,s)\right]ds>0. \quad (18)$$

Combining (17) and (18) we obtain (6). The conclusion thus comes from Theorem 2.1, and the proof is complete.

In Theorems 2.1-2.2, if we choose $H(\xi,s)=(\xi-s)^\lambda$, $\xi\ge s\ge\xi_0$, where $\lambda>1$ is a constant, then we obtain the following two corollaries.

Corollary 2.1. Under the conditions of Theorem 2.1, if for any sufficiently large $T\ge\xi_0$, there exist $a,\ b,\ c$ with $T\le a<c<b$ satisfying

$$\frac{1}{(c-a)^\lambda}\int_a^c(s-a)^\lambda K\widetilde{\rho}(s)\widetilde{q}(s)ds$$
$$+\frac{1}{(b-c)^\lambda}\int_c^b(b-s)^\lambda K\rho(s)\widetilde{q}(s)ds$$
$$>\frac{A}{4(c-a)^\lambda}\int_a^c\widetilde{r}(s)\widetilde{\rho}(s)(s-a)^{\lambda-2}\left(\lambda+\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}(s-a)\right)^2ds$$
$$+\frac{A}{4(b-c)^\lambda}\int_c^b\widetilde{r}(s)\widetilde{\rho}(s)(b-s)^{\lambda-2}\left(\lambda-\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}(b-s)\right)^2ds, \quad (19)$$

then Eq. (1) is oscillatory.

Corollary 2.2. Under the conditions of Theorem 2.2, if for any $l\ge\xi_0$,

$$\lim_{\xi\to\infty}\sup\int_l^\xi[(s-l)^\lambda K\widetilde{\rho}(s)\widetilde{q}(s)$$
$$-\frac{A\widetilde{r}(s)\widetilde{\rho}(s)}{4}(s-l)^{\lambda-2}(\lambda+\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}(s-l))^2]ds>0 \quad (20)$$

and

$$\lim_{\xi\to\infty}\sup\int_l^\xi[(\xi-s)^\lambda K\widetilde{\rho}(s)\widetilde{q}(s)$$
$$-\frac{A\widetilde{r}(s)\widetilde{\rho}(s)}{4}(\xi-s)^{\lambda-2}(\lambda-\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}(\xi-s))^2]ds>0, \quad (21)$$

then Eq. (1) is oscillatory.

Theorem 2.3. Under the conditions of Theorem 2.1, furthermore, suppose (6) does not hold. If for any $T \geq \xi_0$, there exist $a,\ b$ with $b > a \geq T$ such that for any $u \in C[a,b]$, $u'(t) \in L^2[a,b]$, $u(a) = u(b) = 0$, the following inequality holds:

$$\int_a^b [u^2(s)K\widetilde{q}(s)\widetilde{\rho}(s) - A\widetilde{r}(s)\widetilde{\rho}(s)(u'(s) + \frac{1}{2}u(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)})^2]ds > 0, \tag{22}$$

then Eq. (1) is oscillatory.

Proof: Suppose to the contrary that $x(t)$ be a non-oscillatory solution of Eq. (1). Without loss of generality, we may assume that $x(t) > 0$ on $[T_0, \infty)$, where $T_0 \geq t_0$ is sufficiently large. Similar to the proof of Theorem 2.1, we obtain (10). Select $a,\ b$ arbitrarily in $[\xi_*, \infty)$ with $b > a$. Substituting ξ with s, multiplying both sides of (10) by $u^2(s)$, integrating it with respect to s from a to b and using $u(a) = u(b) = 0$, we get that

$$\begin{aligned}
&\int_a^b u^2(s)K\widetilde{q}(s)\widetilde{\rho}(s)ds \leq -\int_a^b u^2(s)\widetilde{w}'(s)ds \\
&-\int_a^b u^2(s)\frac{\widetilde{w}^2(s)}{A\widetilde{r}(s)\widetilde{\rho}(s)}ds + \int_a^b u^2(s)\widetilde{w}(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}ds \\
&= 2\int_a^b u(s)u'(s)\widetilde{w}(s)ds - \int_a^b u^2(s)\frac{\widetilde{w}^2(s)}{A\widetilde{r}(s)\widetilde{\rho}(s)}ds \\
&+ \int_a^b u^2(s)\widetilde{w}(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}ds \\
&= -\int_a^b \{[\sqrt{\frac{1}{A\widetilde{r}(s)\widetilde{\rho}(s)}}u(s)\widetilde{w}(s) \\
&-\sqrt{A\widetilde{r}(s)\widetilde{\rho}(s)}(u'(s) + \frac{1}{2}u(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)})]^2 \\
&+A\widetilde{r}(s)\widetilde{\rho}(s)(u'(s) + \frac{1}{2}u(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)})^2\}ds.
\end{aligned}$$

Moreover,

$$\int_a^b [u^2(s)K\widetilde{q}(s)\widetilde{\rho}(s) - A\widetilde{r}(s)\widetilde{\rho}(s)(u'(s) + \frac{1}{2}u(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)})^2]ds \leq 0, \tag{23}$$

which contradicts to the assumption (22). So every solution of Eq. (1) is oscillatory, and the proof is complete.

III. Interval oscillation criteria with $f(x)$ being monotone

Theorem 3.1. Under the conditions of Lemma 2.1, furthermore, assume $f \in C^1[R, R]$ satisfying $f'(x) \geq \mu > 0$ for $x \neq 0$, and for any sufficiently large $T \geq \xi_0$, there exist $a,\ b,\ c$ with $T \leq a < c < b$ such that

$$\begin{aligned}
&\frac{1}{H(c,a)}\int_a^c H(s,a)\widetilde{\rho}(s)\widetilde{q}(s)ds + \frac{1}{H(b,c)}\int_c^b H(b,s)\widetilde{\rho}(s)\widetilde{q}(s)ds \\
&> \frac{A}{4\mu H(c,a)}\int_a^c \widetilde{r}(s)\widetilde{\rho}(s)Q_1^2(s,a)ds \\
&+\frac{A}{4\mu H(b,c)}\int_c^b \widetilde{r}(s)\widetilde{\rho}(s)Q_2^2(b,s)ds,
\end{aligned} \tag{24}$$

where $\widetilde{\rho},\ \widetilde{q},\ \widetilde{r},\ Q_1, Q_2$ are defined as Theorem 2.1, then Eq. (1) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ be a non-oscillatory solution of Eq. (1). Without loss of generality, we may assume that $x(t) > 0$ on $[T_0, \infty)$, where $T_0 \geq t_0$ is sufficiently large. Furthermore, by Lemma 2.1, there exists $t_* \geq T_0$ such that $D_t^\alpha x(t) < 0,\ t \geq t_*$. So by the assumption on Eq. (1) we have $k_1(x(t), D_t^\alpha x(t)) \geq AD_t^\alpha x(t),\ t \geq t_*$.

Define

$$w(t) = \rho(t)\frac{r(t)k_1(x(t), D_t^\alpha x(t))}{f(x(t))}. \tag{25}$$

Then for $t \geq t_*$, we have

$$\begin{aligned}
D_t^\alpha w(t) &= \frac{D_t^\alpha \rho(t)}{\rho(t)}w(t) - \rho(t)p(t)\frac{k_2(x(t), D_t^\alpha x(t))D_t^\alpha x(t)}{f(x(t))} \\
&-\rho(t)q(t) - \rho(t)\frac{r(t)k_1(x(t), D_t^\alpha x(t))f'(x(t))D_t^\alpha x(t)}{f^2(x(t))} \\
&= \frac{D_t^\alpha \rho(t)}{\rho(t)}w(t) - \rho(t)p(t)\frac{k_2(x(t), D_t^\alpha x(t))D_t^\alpha x(t)}{f(x(t))} \\
&-\rho(t)q(t) - \frac{w^2(t)f'(x(t))D_t^\alpha x(t)}{\rho(t)r(t)k_1(x(t), D_t^\alpha x(t))} \\
&\leq \frac{D_t^\alpha \rho(t)}{\rho(t)}w(t) - \rho(t)p(t)\frac{k_2(x(t), D_t^\alpha x(t))D_t^\alpha x(t)}{f(x(t))} \\
&-\rho(t)q(t) - \frac{\mu w^2(t)}{A\rho(t)r(t)} \\
&\leq \frac{D_t^\alpha \rho(t)}{\rho(t)}w(t) - \rho(t)q(t) - \frac{\mu w^2(t)}{A\rho(t)r(t)}.
\end{aligned} \tag{26}$$

We notice that (26) is similar to (9). So by similar process from (9) to the end in Theorem 2.1, we can deduce the desired result.

Theorem 3.2. Under the conditions of Theorem 3.1, furthermore, suppose (24) does not hold. If for any $l \geq \xi_0$,

$$\limsup_{\xi\to\infty}\int_l^\xi \left[H(s,l)\widetilde{\rho}(s)\widetilde{q}(s) - \frac{A\widetilde{r}(s)\widetilde{\rho}(s)}{4\mu}Q_1^2(s,l)\right]ds > 0 \tag{27}$$

and

$$\limsup_{\xi\to\infty}\int_l^\xi \left[H(\xi,s)\widetilde{\rho}(s)\widetilde{q}(s) - \frac{A\widetilde{r}(s)\widetilde{\rho}(s)}{4\mu}Q_2^2(\xi,s)\right]ds > 0, \tag{28}$$

then Eq. (1) is oscillatory.

The proof of Theorem 3.2 is similar to Theorem 2.2, and we omit it here.

In Theorems 3.1-3.2, if we choose $H(\xi, s) = (\xi - s)^\lambda,\ \xi \geq s \geq \xi_0$, where $\lambda > 1$ is a constant, then we obtain the following two corollaries, which are similar to Corollaries

2.1-2.2.

Corollary 3.1. Under the conditions of Theorem 3.1, if for any sufficiently large $T \geq \xi_0$, there exist $a,\ b,\ c$ with $T \leq a < c < b$ satisfying

$$\frac{1}{(c-a)^\lambda}\int_a^c (s-a)^\lambda \widetilde{\rho}(s)\widetilde{q}(s)ds + \frac{1}{(b-c)^\lambda}\int_c^b (b-s)^\lambda \rho(s)\widetilde{q}(s)ds$$
$$> \frac{A}{4\mu(c-a)^\lambda}\int_a^c \widetilde{r}(s)\widetilde{\rho}(s)(s-a)^{\lambda-2}\left(\lambda+\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}(s-a)\right)^2 ds$$
$$+\frac{A}{4\mu(b-c)^\lambda}\int_c^b \widetilde{r}(s)\widetilde{\rho}(s)(b-s)^{\lambda-2}\left(\lambda-\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}(b-s)\right)^2 ds, \tag{29}$$

then Eq. (1) is oscillatory.

Corollary 3.2. Under the conditions of Theorem 3.2, if for any $l \geq \xi_0$,

$$\lim_{\xi\to\infty}\sup\int_l^\xi [(s-l)^\lambda K\widetilde{\rho}(s)\widetilde{q}(s) - \frac{A\widetilde{r}(s)\widetilde{\rho}(s)}{4\mu}(s-l)^{\lambda-2}(\lambda+\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}(s-l))^2]ds > 0 \tag{30}$$

and

$$\lim_{\xi\to\infty}\sup\int_l^\xi [(\xi-s)^\lambda K\widetilde{\rho}(s)\widetilde{q}(s) - \frac{A\widetilde{r}(s)\widetilde{\rho}(s)}{4\mu}(\xi-s)^{\lambda-2}(\lambda-\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}(\xi-s))^2]ds > 0, \tag{31}$$

then Eq. (1) is oscillatory.

Theorem 3.3. Under the conditions of Theorem 3.1, furthermore, suppose (24) does not hold. If for any $T \geq \xi_0$, there exist $a,\ b$ with $b > a \geq T$ such that for any $u \in C[a,b],\ u'(t) \in L^2[a,b],\ u(a) = u(b) = 0$, the following inequality holds:

$$\int_a^b [u^2(s)\widetilde{q}(s)\widetilde{\rho}(s) - \frac{A\widetilde{r}(s)\widetilde{\rho}(s)}{\mu}(u'(s)+\frac{1}{2}u(s)\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)})^2]ds > 0, \tag{32}$$

then Eq. (1) is oscillatory.

IV. Applications

Example 1. Consider the nonlinear fractional differential equation with damping term

$$D_t^\alpha(\sin^2(\frac{t^\alpha}{\Gamma(1+\alpha)})e^{-x^2(t)}D_t^\alpha x(t)) + t^2x^4(t)(D_t^\alpha x(t))^2 + x(t)(1+x^2(t)) = 0,\ t\geq 2,\ 0<\alpha<1. \tag{33}$$

In fact, if we set in Eq. (1) $t_0 = 2,\ r(t) = \sin^2(\frac{t^\alpha}{\Gamma(1+\alpha)}),\ p(t) = t^2,\ q(t)\equiv 1,\ k_1(x(t), D_t^\alpha x(t)) = e^{-x^2(t)}D_t^\alpha x(t),\ k_2(x(t), D_t^\alpha x(t)) = x^4(t)D_t^\alpha x(t),\ f(x) = x + x^3$, then we obtain (33).

So $k_1^2(x(t), D_t^\alpha x(t)) = e^{-2x^2(t)}(D_t^\alpha x(t))^2 \leq e^{-x^2(t)}(D_t^\alpha x(t))^2$, which implies $A = 1$. Furthermore, $f'(x) = 1+3x^2 \geq 1,\ t\geq 0$, and then $\mu = 1$. Since $\xi = \frac{t^\alpha}{\Gamma(1+\alpha)}$, then $\widetilde{r}(\xi) = r(t) = \sin^2(\frac{t^\alpha}{\Gamma(1+\alpha)}) = \sin^2\xi$.

In (32), letting $\widetilde{\rho}(s)\equiv 1,\ a = 2k\pi,\ b = 2k\pi+\pi,\ u(s) = \sin s$, then $u(a) = u(b) = 0$. Considering $\widetilde{q}(s)\equiv 1$, we obtain

$$\int_{2k\pi}^{2k\pi+\pi}\left(\sin^2 s - \sin^2 s\cos^2 s\right)ds = \int_{2k\pi}^{2k\pi+\pi}\sin^4 s ds > 0.$$

Therefore, Eq. (33) is oscillatory by Theorem 3.3.

Example 2. Consider the nonlinear fractional differential equation with damping term

$$D_t^\alpha\left((\frac{t^\alpha}{\Gamma(1+\alpha)})^{\frac{2}{3}}\frac{1}{1+x^2(t)}D_t^\alpha x(t)\right) + \ln(5+t^2)\cos^2 t\,(D_t^\alpha x(t))^2 + \frac{x(t)(2+x^2(t))}{1+x^2(t)} = 0,\ t\geq 5,\ 0<\alpha<1. \tag{34}$$

In fact, if we set in Eq. (1) $t_0 = 5,\ r(t) = (\frac{t^\alpha}{\Gamma(1+\alpha)})^{\frac{2}{3}},\ p(t) = \ln(5+t^2),\ q(t)\equiv 1,\ k_1(x(t), D_t^\alpha x(t)) = \frac{1}{1+x^2(t)}D_t^\alpha x(t),\ k_2(x(t), D_t^\alpha x(t)) = \cos^2(t)(D_t^\alpha x(t))^2,\ f(x) = \frac{x(2+x^2)}{1+x^2}$, then we obtain (34).

So $k_1^2(x(t), D_t^\alpha x(t)) = (\frac{1}{1+x^2(t)})^2(D_t^\alpha x(t))^2 \leq \frac{1}{1+x^2(t)}(D_t^\alpha x(t))^2$, which implies $A = 1$. Furthermore, $\widetilde{r}(\xi) = r(t) = (\frac{t^\alpha}{\Gamma(1+\alpha)})^{\frac{2}{3}} = \xi^{\frac{2}{3}}$.

We notice that it is complicated in obtaining the lower-bound of $f'(x)$, while one can easily see $f(x)/x \geq 1$. So $K = 1$, and in (20)-(21), after letting $\widetilde{\rho}(s)\equiv 1,\ \lambda = 2$, considering $\widetilde{q}(s)\equiv 1$, we obtain

$$\lim_{\xi\to\infty}\sup\int_l^\xi [(s-l)^\lambda K\widetilde{\rho}(s)\widetilde{q}(s) - \frac{A\widetilde{r}(s)\widetilde{\rho}(s)}{4}(s-l)^{\lambda-2}(\lambda+\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}(s-l))^2]ds$$
$$= \lim_{\xi\to\infty}\sup\int_l^\xi \left[(s-l)^2 - s^{\frac{2}{3}}\right]ds = \infty$$

and

$$\lim_{\xi\to\infty}\sup\int_l^\xi [(\xi-s)^\lambda K\widetilde{\rho}(s)\widetilde{q}(s) - \frac{A\widetilde{r}(s)\widetilde{\rho}(s)}{4}(\xi-s)^{\lambda-2}(\lambda-\frac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}(\xi-s))^2]ds$$
$$= \lim_{\xi\to\infty}\sup\int_l^\xi \left[(\xi-s)^2 - s^{\frac{2}{3}}\right]ds = \infty.$$

So according to Corollary 2.2 we deduce that Eq. (34) is oscillatory.

Remark. We note that the oscillatory character of the two examples above are not deducible from previously known oscillation criteria in the literature.

V. Conclusions

We have established some new interval oscillation criteria for a class of nonlinear fractional differential equations with nonlinear damping term. As one can see, the variable transformation used in ξ is very important, which ensures that certain fractional differential equations can be turned into another ordinary differential equations of integer order, whose oscillation criteria can be established by generalized Riccati transformation, inequality and integration average technique. Finally, we note that this approach can also be applied to research oscillation for other fractional differential equations involving the modified Riemann-liouville derivative.

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