Pricing Portfolio Credit Derivatives with Stochastic Recovery and Systematic Factor

Yuko Otani and Junichi Imai

Abstract—**In this paper, a model for pricing portfolio credit derivatives with nested Archimedean copulas, stochastic recovery rates, and an exogenous systematic factor is presented. The model explains the dependence between default probabilities and recovery rates, both of which are affected by risk of changing in portfolio value because of modifications of a systematic factor. We call this risk "economic change risk ". The advantage of the model is that the systematic factor is able to be set optimally. This leads that when we compute prices, we can consider the economic state directly using the proposed model. The paper demonstrates pricing basket credit default swaps (basket CDSs) as an example of portfolio credit derivatives. The effects of basket CDS prices and the dependence structures from the change in economic change risk through numerical experiments are investigated. The results of the experiments show that the model with economic change risk evaluates higher spreads and stronger dependence than those in the existing researches. Moreover, the effects of economic change risk differ according to the choice of copulas and recovery rates.**

Index Terms—**portfolio credit derivatives, copulas, recovery rates, systematic factor.**

I. Introduction

In recent financial crises such as the global credit crunch and the European sovereign crisis, credit risk exposures have seriously affected the global economy. Credit risk is the risk that the value of a portfolio changes because of unexpected changes in the credit quality of issuers or trading partners, and can be traded using credit derivatives. The most popular credit derivatives are CDSs (credit default swaps), for which prices are treated as measures of credit risk. It is known that credit risk has a positive dependence structure. In other words, defaults are more likely to occur after one asset defaults. This becomes more significant when systematic risk becomes higher. Systematic risk is derived from the factors that influence some sector of the whole economy, such as the industrial sector, a nation or the entire world. Macroeconomic factors like stock indexes and GDP capture the state of systematic factors. According to [9], subprime mortgage risk appeared during the credit crunch because of the decrease in housing prices. Moreover, according to [13], principal component analysis shows that sovereign debt credit risk depends mainly on the American stock index. These facts indicate that housing prices and the American stock index describes the state of systematic factors that affect credit risk.

To describe the dependence structure between defaults in reference assets, various models are proposed. One of the most famous models is the copula model developed by [12]. It is well-known that all multivariate distribution functions can be decomposed into their marginal distribution functions and their copulas. In the copula model, a default time for each asset is expressed using copula functions, and loss distributions are calculated directly using numerical experiments. [12] uses a Gaussian copula that has the dependence structure of the multivariate normal distribution function. The Gaussian copula is widely used in practice because it is easy to generate samples from it. However, [16] points out that the Gaussian copula is not able to capture the dependence structure under extreme events, and thus a non-Gaussian copula model is important. [20] introduce a model using Archimedean copulas, some of which are suitable for expressing dependence structures under extreme events. Besides, [18] provides a large homogeneous portfolio approximation in an Archimedean setup. Archimedean copulas offer two advantages when evaluating homogeneous portfolios. First, one can generate samples from Archimedean copulas easily. Second, in Archimedean copulas only one parameter dominates the dependence structures. However, these models cannot evaluate non-homogeneous portfolios. Hence [8] propose a model using nested Archimedean copulas to take different dependence structures according to industrial sectors into account. Nested Archimedean copulas consist of several Archimedean copulas, and so are able to express more complex dependence structures.

The models mentioned above assume that recovery rates are constant. However, according to [1] and [2], recovery rates and default probabilities are negatively correlated and both are affected by systematic factors. To relax the assumption, models with stochastic recovery rates have been introduced. [4] develop a Gaussian copula model where default times and recovery rates are negatively correlated and depend on a systematic factor. Most stochastic recovery rate models are constructed using a Gaussian copula, and thus cannot express the dependence structure under extreme events. To overcome this drawback, [7] introduce a stochastic recovery rate model with nested Archimedean copulas, which expresses not only the dependence within default times and recovery rates but also the correlation between default times and recovery rates. However, this model does not consider systematic factors. Therefore, no model exists that expresses the dependence structure under extreme events and has stochastic recovery rates and systematic factors.

This paper proposes a model for pricing portfolio credit derivatives with nested Archimedean copulas, stochastic recovery rates, and a systematic factor which is generated exogenously. The model can explain the dependence structures between default probabilities and recovery rates, which are affected by the risk that the value of portfolios changes because of modifications of a systematic factor. In this paper, this risk is called "economic change risk". As in [7], nested

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Archimedean copulas describe the relationship between default times and recovery rates. The copula suggested by [5] is used to model economic change risk; when economic change risk is high, default times shorten and recovery rates become lower. The proposed model is a more realistic pricing model because of consideration of an exogenous systematic factor. The advantage of the model is that the systematic factor is able to be set optimally. This leads that when we compute prices, we can consider economic state directly using the proposed model. The proposed model captures the fact that if the economic condition becomes poor, default times and recovery rates of assets will become worse. In this paper, basket CDS pricing is demonstrated as an example of valuation portfolio credit derivatives, noting that this model can also evaluate other portfolio credit derivatives.

The effect of pricing basket CDSs is investigated by the change in economic change risk. The effect of the spread is analyzed numerically, since the price of basket CDSs is no longer available in the explicit form under our assumption. The correlation coefficients among assets and between default times and recovery rates are also computed in order to analyze the effect of the dependence structure. In our numerical experiments, we deal with first-to-default swaps, which are basket CDSs where protection is paid for the first default. The results are analyzed from three aspects: how high economic change risk is, what types of copula should be employed, and whether recovery rates are deterministic or stochastic. Two types of copulas are considered: Gaussian and Gumbel. The Gumbel copula is an Archimedean copula and has a dependence structure that is more likely to increase the values of variables if one variable becomes extremely high. Thanks to this dependence structure, in our model the Gumbel copula is suitable for expressing the fact that default times and recovery rates are likely to decrease simultaneously. The results of the numerical experiments show that the model with economic change risk evaluates higher spreads and stronger dependence than those in the literature. Moreover, the effects of economic change risk differ according to the choice of copulas and recovery rates are shown.

This paper is organized as follows. The valuation model for basket CDSs is discussed in Section II. The concepts of nested Archimedean copulas and copulas with economic change risk are introduced. Next, the definitions of default times, recovery rates, and economic change risk are explained. Furthermore, to calculate spreads of basket CDSs, default legs and premium legs are mentioned. In Section III, numerical experiments are presented along with their results and considerations. First, the settings for the experiments and the parameters are explained. We then compute the spreads of first-to-default swaps and the correlation coefficients between assets and between default times and recovery rates from different models and compare them. Sensitivity analyses are also performed and considerations are discussed. Section IV concludes the paper.

II. Valuation of basket CDSs

A. Copulas

Before explaining the setup for the valuation of basket CDSs, copulas which are used to express dependence structure and economic change risk are introduced. We review

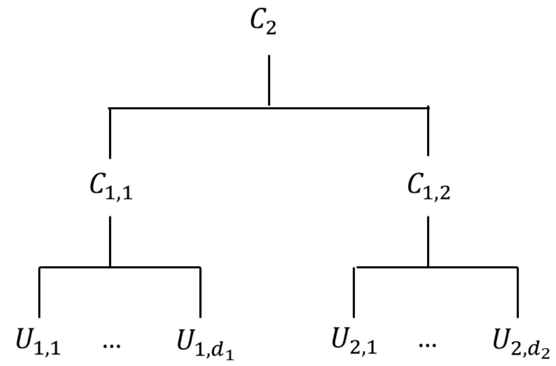

Fig. 1. The structure of two-level nested Archimedean copulas

nested Archimedean copulas, which were first introduced by [10], and have been used in the field of finance by [17]. In studies of pricing credit derivatives, [8] construct a pricing model for collateralized debt obligations (CDOs) using nested Archimedean copulas. The model expresses different dependence structures according to sectors. Furthermore, [7] evaluate prices of CDOs using nested Archimedean copulas and model the relationship between default times and recovery rates.

Fig. 1 shows the structure of d-dimensional two-level nested Archimedean copulas used by [7], where d is some natural number. Assuming that $U_{1,1},\ldots,U_{1,d_1},U_{2,1},\ldots,U_{2,d_2}$ are uniform random variables on $[0,1]$, where $d_1+d_2=d$, the dependence structure of $U_{1,1},\ldots,U_{1,d_1}$ and that of $U_{2,1},\ldots,U_{2,d_2}$ are different, and the variables are correlated to one another. Denoting by $\phi_{1,1},\phi_{1,2},\phi_2$ the generators of the copulas $C_{1,1},C_{1,2},C_2$, the copula of $U_{1,1},\ldots,U_{1,d_1},U_{2,1}\ldots,U_{2,d_2}$, C, are given by

$$\begin{aligned} & C(u_{1,1},\ldots,u_{1,d_1},u_{2,1},\ldots,u_{2,d_2}) \\ &= \phi_2^{-1}\Big(\phi_2\circ\phi_{1,1}^{-1}\big(\phi_{1,1}(u_{1,1})+\cdots+\phi_{1,1}(u_{1,d_1})\big) \\ &\qquad +\phi_2\circ\phi_{1,2}^{-1}\big(\phi_{1,2}(u_{2,1})+\cdots+\phi_{1,2}(u_{2,d_2})\big)\Big), \quad (1) \end{aligned}$$

where $\phi_{1,1}^{-1},\phi_{1,2}^{-1},\phi_2^{-1}$ are the inverse functions of the generators.

The function C of Equation (1) is a copula only if the parameters satisfy some condition. According to [17], if $\phi_{1,1},\phi_{1,2},\phi_2$ are generators for the same kind of copula, this condition is $\theta_{1,s}\geq\theta_2, s\in\{1,2\}$, where $\theta_{1,1},\theta_{1,2},\theta_2$ are the parameters of each generator. One of the methods for sampling from nested Archimedean copulas is suggested by [15]. This algorithm is based on the algorithm of [14], which uses the Laplace-Stielthes transform of distribution functions on $\mathbb{R}^+$. [6] also introduce the approach to find compatible generators and to simulate nested Archimedean copulas.

To describe effects of economic change risk, we adapt new non-exchangeable hierarchical copulas constructed by modifying some factor of the original copula suggested by [5]. $U_1,\ldots,U_d$ denote uniform random variables on $[0,1]$ belonging to some hierarchical copula C_0, and Z denotes

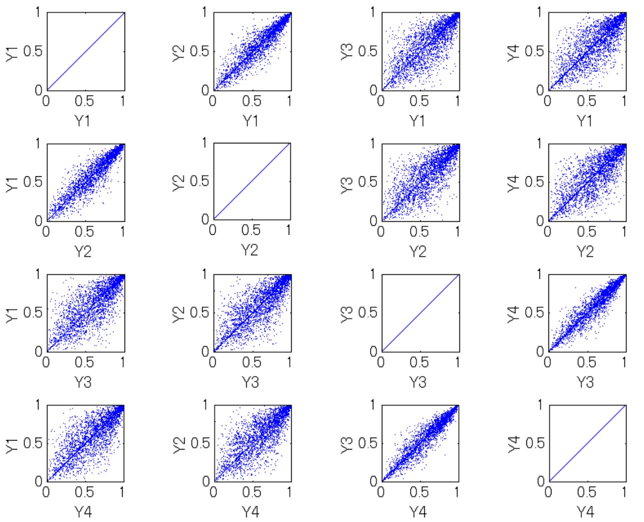

Fig. 2. Simulated points from the four-dimensional nested Gumbel copula with economic change risk

the economic change risk with distribution function $F_z(t)$ that affects the components of the copula C_0. We define $Y_i = \max\{U_i, Z\}, i \in \{1, \ldots, d\}$. Y_i are the components of a new copula C which is the copula C_0 influenced by economic change risk. In other words, $Y_i, i \in \{1, \ldots, d\}$ belongs to the following copula C.

$$C(u_1, \ldots, u_d) = C_0(u_1, \ldots, u_d) F_z(\min_i u_i). \qquad (2)$$

[5] prove that the function C given by Equation (2) is a copula.

Fig. 2 shows 5000 simulated points from the four-dimensional nested Gumbel copula with the parameters $\theta_{1,1} = 2.0$, $\theta_{1,2} = 2.5$, $\theta_2 = 1.8$, and with a uniform random variable Z. It is noticed from Fig. 2 that different components tend to be equal for copulas with economic change risk.

B. Setups for valuation

1) Definitions of default times, recovery rates, and the systematic factor: In the following, the definitions of default times and recovery rates are presented based on the model of [7]. The present model expresses the relationships between default times, recovery rates, and the systematic factor with the copulas of [5].

Assume that C_0 is a copula of $2n$ uniform random variables $U_1^D, \ldots, U_n^D, U_1^L, \ldots, U_n^L$, where n is some natural number. The copula C_0 has the dependence structure represented in Fig. 1, and so $U_1^D, \ldots, U_n^D$ has a different dependence from $U_1^L, \ldots, U_n^L$. We now expose C_0 to the economic change risk $Z \in [0, 1]$, and define $Y_i^D = \max\{U_i^D, Z\}, Y_i^L = \max\{U_i^L, Z\}, i = 1, \ldots, n$. Assuming $F_Z(t)$ is the distribution function of Z, the copula C of $Y_1^D, \ldots, Y_n^D, Y_1^L, \ldots, Y_n^L$ is given by Equation (2).

A probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ is assumed, where $\mathbb{Q}$ denotes some given risk-neutral measure. Moreover, let $(\mathcal{F}_t)_{t\in[0,T]}$ with $\mathcal{F} = \cup_{t\in[0,T]} \mathcal{F}_t \subset \mathcal{G}$, $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$, denote the background filtration representing information about the financial market except for information on occurrence or non-occurrence of default events. A portfolio of n assets with payments depending on the occurrence of defaults is considered. Denote by $\lambda_i(t)$ the $\mathcal{F}_t$-measurable default intensity of asset i at time t. The default times τ_i,$i = 1, \ldots, n$, are given by

$$\tau_i = \inf\Big\{t \in \mathbb{R}^+ |\exp\Big\{-\int_0^t \lambda_i(s)ds\Big\} \le Y_i^D\Big\},$$

where Y_i^D is a uniform random variable on $[0, 1]$ and independent of $\mathcal{F}$.

The recovery rate R_i for a default event are given by $1 - LGD_i$, where the losses given default, LGD_i, are assumed to be identically distributed according to a distribution function F_L with support $[0, 1]$. If no default has happened, R_i is set to equal 1. Therefore, the recovery rate of asset i is given by

$$R_i = \begin{cases} 1 - LGD_i, & \tau_i \le T \\ 1, & \tau_i > T \end{cases}.$$

Assuming that U_i^L is uniformly distributed on $[0, 1]$ and independent of $\mathcal{F}$, we can set $LGD_i = F_L^{\leftarrow}(Y_i^L)$, with $F_L^{\leftarrow}(x) = \inf\{z \in \mathbb{R} | F_L(z) \ge x\}$ denoting the generalized inverse function of F_L. Moreover, we can set $R_i = h(Y_i^L, \tau_i)$ with

$$h(x, \tau) = \begin{cases} 1 - F_L^{\leftarrow}(x), & \tau_i \le T \\ 1, & \tau_i > T \end{cases}.$$

If the economic change risk Z affects U_i^D, U_i^L, then the default times become shorter and the losses given default become higher. Therefore, the larger Z is, the worse the state of the economy is. Z affects not only dependence structures among assets but individual assets because default times and recovery rates are determined by Y_i^D and Y_i^L separately.

2) Distributions of recovery rates and the systematic factor: In this paper, the Kumaraswamy distribution, suggested by [11], is chosen as the loss distribution F_L and the distribution of economic change risk, F_Z. [7] use the Kumaraswamy distribution as the loss distribution F_L. According to [19], the beta distribution is commonly used as the loss distribution. However, the beta distribution is numerically too expensive for Monte Carlo pricing because the distribution function and the inverse distribution function of the beta distribution cannot be expressed explicitly. In contrast, the Kumaraswamy distribution has a closed-form distribution function and inverse distribution function, and thus is more suitable for Monte Carlo pricing.

The density function of the Kumaraswamy distribution f_{Kum} is given by

$$f_{\text{Kum}}(x) = abx^{a-1}(1 - x^a)^{b-1},$$

where $0 \le x \le 1$ and $a, b > 0$. The Kumaraswamy distribution is flexible and supports various shapes, such as skewed and U-shaped distributions, as the parameters a and b change; its distribution function F_{Kum} is given explicitly by

$$F_{\text{Kum}}(x) = 1 - (1 - x^a)^b,$$

with inverse function F_{Kum}^{-1} given by

$$F_{\text{Kum}}^{-1}(x) = \Big(1 - (1 - x)^{1/b}\Big)^{1/a}.$$

The parameters of the loss distribution are set to $a = 2.65$, $b = 2.13$ as in [7]. This is determined from the expectation and the standard deviation. When we estimate

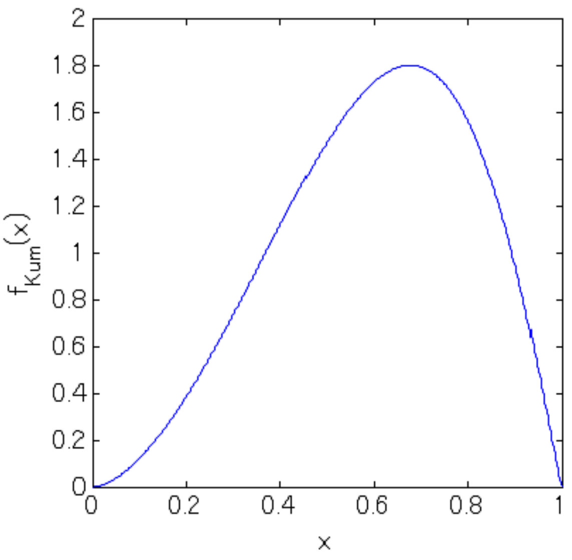

Fig. 3. Density of the Kumaraswamy distribution ($a = 2.65$, $b = 2.13$)

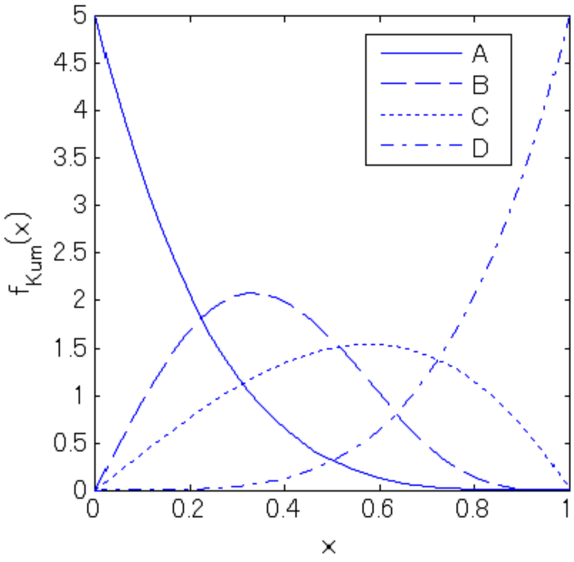

Fig. 4. Densities of the Kumaraswamy distribution (A, B, C, D)

TABLE I
STATES OF THE SYSTEMATIC FACTOR, AND THE PARAMETERS AND EXPECTATIONS OF THE DENSITY FUNCTIONS

State of the systematic factor	Parameter a, b	Expectation
A	$a = 1$, $b = 5$	1/6
B	$a = 2$, $b = 5$	256/693
C	$a = 2$, $b = 2$	8/15
D	$a = 5$, $b = 1$	5/6

default intensities under a risk-neutral measure from the spreads of index CDSs, the most liquid credit derivatives, it is generally assumed that the expectation of the recovery rates conditioned on default is 40 %. Therefore, the expectation of the losses given default conditioned on default is 60 %. Furthermore, [3] show that the standard deviations of the loss rates are 20 %. The parameters are set to the above values to match the two conditions. The shape of the density function with these parameters is shown in Fig. 3.

For the distribution of economic change risk, a high economic change risk corresponds to a poor condition for the systematic factor. In this paper, four sets of parameters for the Kumaraswamy distribution with each set representing a state of the economy are chosen: very good (A), good (B), poor (C), and very poor (D). Each set of parameters and the expectation are given in Table I. The shape of each distribution is displayed in Fig. 4.

C. *Premium legs and default legs*

The model introduced above can be used to valuate various kinds of portfolio credit derivatives. This paper demonstrates pricing of basket CDSs as an example. CDSs are swaps where the buyer receives protection from the seller if defaults occur in a reference asset. The buyer pays regularly premiums for the seller. If protection of CDSs is complete, then the premium payment is also finished. Basket CDSs are CDSs for which the reference asset is a portfolio. In particular, basket CDSs for which protection is paid at the first default are called first-to-default swaps. In the following, we consider the basket CDS with a reference portfolio of n assets providing protection for the m-th default, with $D_{\text{in}} \le m < D_{\text{out}}$, $1 \le D_{\text{in}} < D_{\text{out}} \le n+1$. The spread is computed so that the discounted expectation of the premium leg is equal to that of the default leg.

First the premium leg is shown. t_i, $i = 1, \ldots, I$ denotes the premium payment dates with $t_I = T$ where T is the maturity date of the basket default swap. X represents the daily premium (the spread), $\Delta_{i-1,i}$ is the length of the period $[t_{i-1}, t_i]$, $B(t)$ is the discount factor for maturity time t, and $N(t)$ is the number of defaults by time t. The premium is paid on the unpaid protected nominal. For simplicity, the nominal is assumed to be equal to 1. At some payment date t_i, if $N(t_i) < D_{\text{in}}$, then the premium is due on a full nominal of $D_{\text{out}} - D_{\text{in}}$. If $D_{\text{in}} \le N(t_i) < D_{\text{out}}$, then the premium is paid on the nominal of $D_{\text{out}} - N(t_i)$. If $N(t_i) \ge D_{\text{out}}$, then no premium is paid. The discounted expectation of the premium leg is given by

$$\sum_{i=1}^{I} \Delta_{i-1,i} X B(t_i) \Big((D_{\text{out}} - D_{\text{in}}) \mathbb{I}_{\{N(t_i) < D_{\text{in}}\}} + \sum_{k=D_{\text{in}}}^{D_{\text{out}}} (D_{\text{out}} - k) \mathbb{I}_{\{N(t_i)=k\}} \Big), \tag{3}$$

where $\mathbb{I}$ is an identity function.

Moreover, the accrued premium should be taken into account if default happens between the payment dates. Let us denote by $t_{i(m)}$ the payment date immediately after τ_m, the default time of the asset m, so that $t_{i(m)-1} \le \tau_m < t_{i(m)}$. The asset m is assumed to be the $\Pi(m)$-th default asset in case of a default event. If $D_{\text{in}} \le \Pi(m) < D_{\text{out}}$, then the accrued premium should be paid when the asset m defaults. The discounted expectation of the accrued premium is given by

$$\sum_{i=1}^{I} \sum_{m=1}^{n} X(\tau_m - t_{i(m)-1}) B(\tau_m) \mathbb{I}_{\{t_{i(m)-1} \le t_m < \tau_{i(m)}\}} \times \mathbb{I}_{\{D_{\text{in}} \le \Pi(m) < D_{\text{out}}\}}. \tag{4}$$

The discounted expectation of the premium leg is equal to Equation (3) plus Equation (4).

Next, the default leg is shown. If the asset m defaults before maturity and $D_{\text{in}} \leq \Pi(m) < D_{\text{out}}$, then the default leg for the asset m is given by LGD_m. Therefore, the discounted expectation of the default leg is given by

$$\sum_{m=1}^{n} LGD_m B(\tau_m) \mathbb{I}_{\{\tau_m \leq T\}} \mathbb{I}_{\{D_{\text{in}} \leq \Pi(m) < D_{\text{out}}\}}.$$

The CDS spread X is determined to match the two legs. Denoting the discounted expectation of the premium leg and the default leg by $PL(X)$ and DL respectively, the spread X is given by

$$X = \frac{DL}{PL(1)}.$$

III. Numerical Experiments

In this section we derive spreads of basket CDSs and the correlation coefficients between assets and between default times and recovery rates, and investigate the effects of the systematic factor on them are numerically derived. We begin with an explanation of the setup of the numerical experiments. Next, the results are shown, and the spreads and the correlation coefficients from the models with and without the systematic factor are compared. The sensitivities of the expectation of economic change risk, the copula parameters, and the range of protection are also examined.

A. Setup

The numerical experiments in this paper investigate how changes in the systematic factor affect the spread of basket CDSs and the dependence structure. We calculate the spreads numerically using the Monte Carlo method because the explicit form of the loss distribution of reference assets cannot be driven. The spread of each asset, the correlation coefficients between default times, and those between default times and recovery rates are also computed. The results are analyzed from three perspectives: the size of the economic change risk, the types of copulas, and the types of recovery rates. Both the Gaussian and Gumbel copulas are used. The former type has no tail dependence, whereas the latter type has upper tail dependence and captures correlations which the Gaussian copula cannot express. A copula with upper tail dependence indicates the fact that the default times and the recovery rates are likely to become earlier and smaller simultaneously. Moreover, the systematic factor affects the deterministic recovery rates model (DR) and the stochastic recovery rates model (SR) separately. In the following, we call DR with the systematic factor "DRS" and SR with the systematic factor "SRS". The four shapes for the distribution function of economic change risk discussed in Section II. B. are used, where each shape corresponds to some state of the systematic factor (A, B, C, D).

The parameters are set as follows. The maturity of the basket CDS is $T = 5$, and the premium pays four times a year. The number of reference assets is $n = 20$. For the sake of simplicity, the default intensities are assumed to be constant, $\lambda = 0.0167$, corresponding to the case where the spread of the CDS index is equal to 100 bp under the risk-neutral measure. This assumption may be generalized to allow stochastic default intensities. We deal with the first-to-default swap, that is, $D_{\text{in}} = 1$ and $D_{\text{out}} = 2$. The nominal is equal to 1, and the risk free rate is 1%. In both DR and DRS models, the loss rates are $LGD_i = 0.6, i \in \{1, \dots, n\}$.

TABLE II
Copula parameters of the numerical experiments

Model	Parameter 1	Parameter 2
Ga DR	0.300	-
Ga SR	0.341	0.230
Gu DR	1.253	-
Gu SR	1.143	1.110

The copula parameters are set to have the same spread as the Gaussian DR. The correlation coefficient of the Gaussian DR is $\rho = 0.3$. For simplicity it is assumed that $U_1^D, \dots, U_n^D$ and $U_1^L, \dots, U_n^L$ in SR belong to the same copula. The spread of the Gaussian DR is calculated as 1,109 bp (the range of the confidence interval at the 95% level is 2.6 bp) from one million sets of samples. Table II shows the copula parameters that are adjusted to have the same spread allowing an error of 2.6 bp. Note that in SR, parameter 1 corresponds to the parameters of $C_{1,1}$ and $C_{1,2}$ in Fig. 1, and parameter 2 corresponds to the parameter of C_2.

B. Results

Table III shows the spreads of the first-to-default swap, those of each asset, the correlation coefficient between default times, those between default times and recovery rates calculated from one million sets of samples using the Monte Carlo method. The numbers in parentheses represents the ranges of their confidence intervals at the 95% level. The results are analyzed from the three perspectives mentioned earlier.

First, we remark on the size of the economic change risk. The correlation coefficients of models with economic change risk are larger than those of models without it, especially in cases C and D. The spreads of models with economic change risk are also larger. These imply that the model is capturing the fact that credit risk and its dependence become high and strong when the state of the systematic factor becomes poor.

Second, the effects of copulas are discussed. In DR and SR, the correlation coefficients between default times are different in the choice of copulas because the dependence structures of the two copulas diverge. In DRS and SRS, the difference decreases as economic change risk increases. As concerning the spreads, in DRS, the spread of the first-to-default swap of the Gaussian copula, as well as the Gumbel copula, are almost the same in cases A, B, and C, whereas the spread of the Gaussian copula is higher in case D. In contrast, the spreads of the Gumbel copula in SRS are higher regardless of the state of the systematic factor. The choice of copulas affects the spreads of models with economic change risk even though the correlation coefficients of them are similar.

Focusing now on the recovery rates, the correlation coefficients between default times of models of DRS increase in the same way as those of models of SRS as economic change risk increases. The spreads in DRS are the same as those of DR in cases A and B, and are higher in cases C and D. In SRS, the change in the spreads is greater than in DRS, even

TABLE III
NUMERICAL RESULTS OF SPREADS (BP) AND THE CORRELATION COEFFICIENTS

Model	Spread of the first-to-default swap	Spread of each asset	Correlation coefficient between default times	Correlation coefficient between default times and recovery rates
Ga DR	1109 (2.9)	20.5 (0.04)	0.26 (0.001)	-
Gu DR	1106 (2.0)	20.7 (0.07)	0.22 (0.001)	-
Ga SR	1109 (2.5)	23.5 (0.08)	0.30 (0.001)	0.21 (0.000)
Gu SR	1110 (2.9)	23.3 (0.09)	0.23 (0.000)	0.12 (0.001)
Ga DRS(A)	1107 (3.1)	20.3 (0.03)	0.30 (0.001)	-
Ga DRS(B)	1107 (3.0)	21.9 (0.06)	0.37 (0.001)	-
Ga DRS(C)	1139 (4.6)	34.1 (0.19)	0.51 (0.001)	-
Ga DRS(D)	1640 (4.3)	295.0 (1.39)	0.78 (0.001)	-
Gu DRS(A)	1107 (0.9)	20.5 (0.07)	0.28 (0.000)	-
Gu DRS(B)	1105 (3.8)	22.0 (0.05)	0.37 (0.000)	-
Gu DRS(C)	1131 (2.8)	34.2 (0.13)	0.52 (0.001)	-
Gu DRS(D)	1610 (2.1)	295.1 (0.48)	0.80 (0.001)	-
Ga SRS(A)	1132 (1.4)	23.5 (0.05)	0.33 (0.000)	0.23 (0.001)
Ga SRS(B)	1184 (1.5)	26.2 (0.05)	0.40 (0.001)	0.29 (0.001)
Ga SRS(C)	1295 (3.6)	44.4 (0.13)	0.53 (0.001)	0.45 (0.001)
Ga SRS(D)	2226 (6.2)	443.6 (1.78)	0.79 (0.001)	0.75 (0.001)
Gu SRS(A)	1142 (1.8)	23.4 (0.09)	0.29 (0.000)	0.16 (0.000)
Gu SRS(B)	1210 (2.0)	26.3 (0.09)	0.38 (0.001)	0.24 (0.001)
Gu SRS(C)	1329 (4.0)	44.6 (0.17)	0.53 (0.001)	0.42 (0.001)
Gu SRS(D)	2247 (6.1)	445.4 (1.32)	0.80 (0.001)	0.75 (0.001)

though the state of the systematic factor is A or B. Thus, SRS is more sensitive to economic change risk than DRS.

C. Sensitivity analyses

Sensitivity analyses are performed to further investigate the effects of economic change risk. The first analysis is on the expectation of economic change risk, the second is on the copula parameters, and the third is on the range of protection.

1) Sensitivity for the expectation of economic change risk: The shape of the distribution of economic change risk, the Kumaraswamy distribution, is determined by two parameters, a and b. The parameters are set to have variance $11/225$ in case C. Table IV shows the expectations of the density functions and the parameters.

TABLE IV
THE EXPECTATIONS OF ECONOMIC CHANGE RISK AND PARAMETERS

Mean	a	b
0.1	1.7218	10.9397
0.2	1.9802	7.0867
0.3	2.2133	5.4958
0.4	2.4447	4.5762
0.5	2.6819	3.9508
0.6	1.7874	1.2282
0.7	1.7455	0.7188
0.8	1.8834	0.4276
0.9	2.4134	0.2244

TABLE V
THE SPREADS OF THE FIRST-TO-DEFAULT SWAP OF THE SENSITIVITY ANALYSIS TO ECONOMIC CHANGE RISK (BP)

Mean	Ga DRS	Gu DRS	Ga SRS	Gu SRS
0.1	1110 (3.0)	1108 (2.8)	1140 (3.3)	1153 (2.3)
0.2	1113 (1.8)	1106 (3.1)	1169 (2.3)	1189 (2.2)
0.3	1111 (2.3)	1108 (2.9)	1190 (2.3)	1217 (2.5)
0.4	1112 (2.5)	1105 (3.0)	1214 (3.2)	1242 (3.7)
0.5	1112 (2.9)	1108 (4.7)	1240 (2.8)	1272 (3.9)
0.6	1225 (3.1)	1213 (3.1)	1462 (4.0)	1490 (3.2)
0.7	1485 (4.9)	1468 (3.8)	1910 (3.9)	1940 (4.3)
0.8	2050 (2.7)	2017 (4.7)	2845 (5.7)	2856 (6.9)
0.9	3592 (11.4)	3511 (7.1)	5354 (21.2)	5310 (9.8)

Tables V, VI, VII, and VIII show the results of the sensitivity analysis. These tables indicate, first, the correlation coefficients are almost the same when the expectation is between 0.1 and 0.5, and apparently increase otherwise. Secondly, as concerning the spreads, in DRS, the spreads are almost the same when the expectation is between 0.1 and 0.5, and obviously increase otherwise. In SRS, the spreads gradually grow when the expectation is between 0.1 and 0.5, and greatly increase otherwise. Thus, the model expresses the fact that credit risk and its dependence become high and strong when the state of the systematic factor is very poor.

The effect of the copula choice is also seen. The difference in the correlation coefficients between two copulas decreases as economic change risk increases. In DRS, the spreads of each copula are not so different when the expectation is between 0.1 and 0.5. However, otherwise the spreads of the Gaussian copula model are higher. In SRS, the spreads of the Gumbel copula model are larger except when expectation is 0.9. This means that the spreads of Gumbel SRS model are the more conservative in realistic situations.

2) Sensitivity to the copula parameters: The sensitivity to the copula parameters is analyzed. Only case C is considered, because in cases A and B the difference between the models with and without economic change risk is small, and in case D the results are similar to those in case C. The copula parameters are set to have the same spread as the Gaussian DR, while the other parameters are set as in Section III. A. The copula parameter of the Gaussian DR, ρ, is changed. Table IX shows the copula parameters, which increase as ρ increases.

Tables X, XI, XII, and XIII show the results of the sensitivity analysis. These tables indicate that if ρ is high, economic

TABLE VI
THE SPREADS OF EACH ASSET OF THE SENSITIVITY ANALYSIS TO ECONOMIC CHANGE RISK (BP)

Mean	Ga DRS	Gu DRS	Ga SRS	Gu SRS
0.1	20.3 (0.08)	20.4 (0.09)	23.6 (0.07)	23.5 (0.08)
0.2	21.1 (0.05)	21.2 (0.06)	24.9 (0.07)	25.0 (0.14)
0.3	22.2 (0.05)	22.3 (0.07)	26.6 (0.11)	26.7 (0.08)
0.4	23.5 (0.08)	23.6 (0.09)	28.7 (0.10)	28.8 (0.12)
0.5	25.4 (0.07)	25.4 (0.20)	31.4 (0.11)	31.6 (0.14)
0.6	61.8 (0.15)	61.7 (0.23)	86.7 (0.37)	86.9 (0.38)
0.7	146.8 (0.73)	147.6 (0.52)	218.8 (0.63)	220.0 (0.73)
0.8	342.9 (0.75)	344.9 (1.58)	529.7 (2.11)	530.2 (1.54)
0.9	966.1 (4.18)	967.6 (2.01)	1537.7 (6.55)	1536.7 (4.54)

TABLE VII
THE CORRELATION COEFFICIENTS BETWEEN DEFAULT TIMES OF SENSITIVITY ANALYSIS TO ECONOMIC CHANGE RISK

Mean	Ga DRS	Gu DRS	Ga SRS	Gu SRS
0.1	0.30 (0.001)	0.29 (0.001)	0.33 (0.001)	0.30 (0.001)
0.2	0.33 (0.001)	0.34 (0.001)	0.37 (0.001)	0.35 (0.001)
0.3	0.37 (0.001)	0.38 (0.001)	0.40 (0.001)	0.39 (0.001)
0.4	0.41 (0.001)	0.42 (0.001)	0.44 (0.001)	0.43 (0.001)
0.5	0.45 (0.001)	0.47 (0.001)	0.47 (0.001)	0.47 (0.001)
0.6	0.57 (0.001)	0.58 (0.001)	0.60 (0.001)	0.59 (0.001)
0.7	0.66 (0.001)	0.66 (0.001)	0.67 (0.001)	0.67 (0.001)
0.8	0.73 (0.001)	0.74 (0.001)	0.74 (0.001)	0.74 (0.000)
0.9	0.80 (0.001)	0.81 (0.001)	0.81 (0.001)	0.81 (0.001)

TABLE VIII
THE CORRELATION COEFFICIENTS BETWEEN DEFAULT TIMES AND RECOVERY RATES OF SENSITIVITY ANALYSIS TO ECONOMIC CHANGE RISK

Mean	Ga DRS	Gu DRS	Ga SRS	Gu SRS
0.1	-	-	0.23 (0.001)	0.16 (0.001)
0.2	-	-	0.26 (0.001)	0.20 (0.001)
0.3	-	-	0.29 (0.001)	0.24 (0.001)
0.4	-	-	0.33 (0.001)	0.29 (0.001)
0.5	-	-	0.37 (0.001)	0.34 (0.001)
0.6	-	-	0.54 (0.001)	0.51 (0.001)
0.7	-	-	0.64 (0.001)	0.62 (0.001)
0.8	-	-	0.72 (0.001)	0.71 (0.001)
0.9	-	-	0.79 (0.001)	0.78 (0.001)

change risk dose not so affect the correlation coefficients of models of the Gaussian copula, while it affects more those of models of the Gumbel copula. This implies that the Gumbel copula models are more sensitive to economic change risk even if the correlation of the reference assets is strong.

The spreads of all models decrease as ρ increases. This means that if the correlation is strong, the variation in default times is small and also that defaults occur in few cases.

In DRS, the spreads of the Gaussian DRS are as big as those of the Gumbel DRS for all values of ρ. In SRS, the spreads of the Gaussian SRS are smaller than those of the Gumbel SRS. Although the gaps between the spreads of the Gaussian SRS and those of the Gumbel SRS change according to ρ, the Gumbel SRS is more conservative regardless of the copula parameters.

TABLE IX
COPULA PARAMETERS OF THE SENSITIVITY ANALYSIS

Ga DR	Gu DR	Ga SR		Gu SR	
0.100	1.079	0.113	0.090	1.049	1.031
0.300	1.253	0.341	0.230	1.143	1.110
0.500	1.490	0.582	0.450	1.286	1.200
0.700	1.870	0.800	0.690	1.560	1.330
0.900	2.715	0.967	0.930	2.310	2.030

TABLE X
THE SPREADS OF THE FIRST-TO-DEFAULT SWAP OF THE SENSITIVITY ANALYSIS TO COPULA PARAMETERS (BP)

ρ	Ga DR	Gu DR	Ga DRS(C)	Gu DRS(C)
0.1	1624 (3.3)	1623 (4.1)	1645 (4.3)	1650 (3.8)
0.3	1112 (3.2)	1109 (3.2)	1137 (3.3)	1132 (2.8)
0.5	764 (3.2)	762 (2.6)	793 (2.0)	789 (3.0)
0.7	504 (2.6)	508 (1.1)	530 (1.7)	533 (1.5)
0.9	272 (1.3)	273 (1.0)	300 (1.2)	300 (1.3)

ρ	Ga SRS	Gu SRS	Ga SRS(C)	Gu SRS(C)
0.1	1626 (2.7)	1627 (3.1)	1934 (6.1)	1960 (3.2)
0.3	1107 (2.9)	1109 (3.4)	1295 (3.3)	1328 (2.9)
0.5	764 (3.5)	765 (2.4)	866 (3.8)	907 (2.8)
0.7	503 (1.8)	505 (1.9)	558 (2.0)	594 (2.7)
0.9	269 (2.0)	272 (1.5)	311 (1.2)	318 (2.1)

TABLE XI
THE SPREADS OF EACH ASSET OF THE SENSITIVITY ANALYSIS TO COPULA PARAMETERS (BP)

ρ	Ga DR	Gu DR	Ga DRS(C)	Gu DRS(C)
0.1	20.6 (0.04)	20.6 (0.05)	34.0 (0.07)	34.0 (0.11)
0.3	20.7 (0.07)	20.9 (0.07)	34.0 (0.17)	34.2 (0.18)
0.5	20.5 (0.10)	21.2 (0.08)	34.0 (0.12)	34.8 (0.18)
0.7	20.6 (0.11)	21.9 (0.08)	34.0 (0.15)	35.4 (0.14)
0.9	20.5 (0.11)	19.6 (0.15)	33.9 (0.11)	31.9 (0.16)

ρ	Ga SRS	Gu SRS	Ga SRS(C)	Gu SRS(C)
0.1	21.7 (0.04)	21.4 (0.07)	43.4 (0.23)	43.3 (0.15)
0.3	23.4 (0.09)	23.3 (0.11)	44.3 (0.10)	44.5 (0.16)
0.5	26.0 (0.10)	24.9 (0.20)	46.4 (0.40)	45.9 (0.22)
0.7	28.6 (0.18)	27.2 (0.17)	48.3 (0.36)	48.2 (0.31)
0.9	30.9 (0.26)	31.2 (0.17)	50.8 (0.31)	51.2 (0.45)

3) Sensitivity to the range of protections: Here we consider sensitivity to the range of the protection of basket CDSs. We focus on case C for the same reason to the former sensitivity analysis. It is assumed that $D_{\text{out}} = D_{\text{in}} + 1$. We deal with kth-to-default swaps where $k = 1, 5, 10, 15, 20$.

Tables XIV, XV, XVI, and XVII show the results of the sensitivity analysis. These tables indicate, first, the range of the protection does not affect the correlation coefficients and the individual spreads. Secondly, the spreads of basket CDSs increase as k decreases and as economic change risk increases. Furthermore, comparing DRS and SRS, the spreads evaluated in SRS are larger for all k.

Comparing the Gaussian copula models and the Gumbel copula models, if k is small then the spreads of the Gaussian

TABLE XII
THE CORRELATION COEFFICIENTS BETWEEN DEFAULT TIMES OF SENSITIVITY ANALYSIS TO COPULA PARAMETERS

ρ	Ga DR	Gu DR	Ga DRS(C)	Gu DRS(C)
0.1	0.08 (0.001)	0.07 (0.000)	0.39 (0.001)	0.40 (0.001)
0.3	0.26 (0.001)	0.22 (0.000)	0.51 (0.001)	0.52 (0.001)
0.5	0.45 (0.001)	0.37 (0.000)	0.63 (0.001)	0.63 (0.001)
0.7	0.66 (0.001)	0.54 (0.001)	0.76 (0.001)	0.75 (0.000)
0.9	0.88 (0.000)	0.74 (0.000)	0.91 (0.000)	0.89 (0.000)

ρ	Ga SRS	Gu SRS	Ga SRS(C)	Gu SRS(C)
0.1	0.09 (0.000)	0.08 (0.000)	0.40 (0.001)	0.40 (0.001)
0.3	0.30 (0.001)	0.23 (0.000)	0.53 (0.001)	0.52 (0.001)
0.5	0.54 (0.001)	0.39 (0.001)	0.68 (0.001)	0.65 (0.001)
0.7	0.77 (0.001)	0.60 (0.001)	0.83 (0.001)	0.78 (0.001)
0.9	0.96 (0.000)	0.89 (0.000)	0.97 (0.000)	0.95 (0.000)

TABLE XIII
THE CORRELATION COEFFICIENTS BETWEEN DEFAULT TIMES AND RECOVERY RATES OF SENSITIVITY ANALYSIS TO COPULA PARAMETERS

ρ	Ga DR	Gu DR	Ga DRS(C)	Gu DRS(C)
0.1	-	-	-	-
0.3	-	-	-	-
0.5	-	-	-	-
0.7	-	-	-	-
0.9	-	-	-	-

ρ	Ga SRS	Gu SRS	Ga SRS(C)	Gu SRS(C)
0.1	0.08 (0.001)	0.03 (0.001)	0.37 (0.001)	0.35 (0.001)
0.3	0.21 (0.001)	0.12 (0.001)	0.45 (0.001)	0.42 (0.001)
0.5	0.41 (0.001)	0.20 (0.001)	0.58 (0.001)	0.48 (0.001)
0.7	0.63 (0.001)	0.30 (0.001)	0.73 (0.001)	0.55 (0.002)
0.9	0.86 (0.000)	0.59 (0.001)	0.90 (0.000)	0.76 (0.001)

TABLE XIV
THE SPREADS OF THE FIRST-TO-DEFAULT SWAP OF THE SENSITIVITY ANALYSIS TO PROTECTIONS (BP)

k	Ga DR	Gu DR	Ga DRS(C)	Gu DRS(C)
1	1108 (2.8)	1105 (2.9)	1135 (2.7)	1133 (2.7)
5	124 (0.7)	94 (0.5)	151 (0.7)	122 (0.7)
10	14 (0.5)	37 (0.4)	43 (0.4)	66 (0.7)
15	1 (0.1)	20 (0.3)	30 (0.3)	49 (0.3)
20	0 (0.0)	9 (0.2)	29 (0.4)	38 (0.3)

k	Ga SRS	Gu SRS	Ga SRS(C)	Gu SRS(C)
1	1111 (3.5)	1109 (2.8)	1296 (3.2)	1330 (3.2)
5	157 (1.0)	114 (0.9)	211 (1.0)	166 (0.9)
10	24 (0.3)	49 (0.6)	68 (0.6)	950 (0.6)
15	2 (0.1)	27 (0.5)	46 (0.5)	72 (0.6)
20	0 (0.0)	13 (0.3)	44 (0.4)	57 (0.5)

copula models are larger than those of the Gumbel copula models; otherwise, the spreads of the Gumbel copula models are larger. This shows that the Gumbel copula model considers more extreme cases. This feature is not changed by economic change risk.

TABLE XV
THE SPREADS OF EACH ASSET OF THE SENSITIVITY ANALYSIS TO PROTECTIONSS (BP)

k	Ga DR	Gu DR	Ga DRS(C)	Gu DRS(C)
1	20.6 (0.05)	20.7 (0.09)	34.0 (0.13)	34.3 (0.18)
5	20.6 (0.05)	20.7 (0.07)	33.9 (0.15)	34.1 (0.17)
10	20.6 (0.10)	20.8 (0.09)	34.0 (0.12)	34.1 (0.15)
15	20.7 (0.07)	20.7 (0.08)	34.0 (0.13)	34.2 (0.13)
20	20.6 (0.08)	20.8 (0.07)	34.0 (0.10)	34.1 (0.09)

k	Ga SRS	Gu SRS	Ga SRS(C)	Gu SRS(C)
1	23.5 (0.08)	23.2 (0.11)	44.4 (0.17)	44.6 (0.19)
5	23.4 (0.09)	23.3 (0.16)	44.3 (0.18)	44.6 (0.17)
10	23.5 (0.06)	23.3 (0.12)	44.5 (0.16)	44.6 (0.11)
15	23.5 (0.08)	23.2 (0.11)	44.5 (0.15)	44.4 (0.13)
20	23.5 (0.07)	23.2 (0.07)	44.5 (0.18)	44.6 (0.16)

TABLE XVI
THE CORRELATION COEFFICIENTS BETWEEN DEFAULT TIMES OF SENSITIVITY ANALYSIS TO PROTECTIONS

k	Ga DR	Gu DR	Ga DRS(C)	Gu DRS(C)
1	0.26 (0.001)	0.22 (0.000)	0.51 (0.002)	0.52
5	0.26 (0.000)	0.22 (0.000)	0.51 (0.001)	0.52 (0.001)
10	0.26 (0.001)	0.22 (0.001)	0.51 (0.001)	0.52 (0.001)
15	0.26 (0.001)	0.22 (0.000)	0.51 (0.001)	0.52 (0.000)
20	0.26 (0.001)	0.22 (0.001)	0.51 (0.001)	0.52 (0.001)

k	Ga SRS	Gu SRS	Ga SRS(C)	Gu SRS(C)
1	0.30 (0.001)	0.23 (0.000)	0.53 (0.001)	0.53 (0.001)
5	0.30 (0.001)	0.23 (0.001)	0.53 (0.001)	0.53 (0.001)
10	0.30 (0.001)	0.23 (0.000)	0.53 (0.001)	0.53 (0.001)
15	0.30 (0.001)	0.23 (0.001)	0.53 (0.001)	0.52 (0.001)
20	0.30 (0.001)	0.23 (0.000)	0.53 (0.001)	0.52 (0.000)

TABLE XVII
THE CORRELATION COEFFICIENTS BETWEEN DEFAULT TIMES AND RECOVERY RATES OF SENSITIVITY ANALYSIS TO PROTECTIONS

k	Ga DR	Gu DR	Ga DRS(C)	Gu DRS(C)
1	-	-	-	-
3	-	-	-	-
5	-	-	-	-
7	-	-	-	-
9	-	-	-	-

k	Ga SRS	Gu SRS	Ga SRS(C)	Gu SRS(C)
1	0.21 (0.001)	0.12 (0.001)	0.45 (0.001)	0.42 (0.001)
5	0.21 (0.001)	0.12 (0.001)	0.45 (0.001)	0.42 (0.001)
10	0.21 (0.001)	0.12 (0.000)	0.45 (0.001)	0.42 (0.001)
15	0.21 (0.001)	0.12 (0.001)	0.45 (0.001)	0.42 (0.001)
20	0.21 (0.000)	0.12 (0.001)	0.45 (0.001)	0.41 (0.001)

D. *Summary of numerical experiments*

The results of the numerical experiments are briefly summarized, focusing on three findings. First, economic change risk enlarges the spread and the strength of the dependence of the models with and without it. This is particularly significant when the economic change risk is high. Second, the correlation coefficients of models of the Gumbel copula are more sensitive to economic change risk than those of models of the Gaussian. Furthermore, the models with stochastic recovery rates result in higher spreads than those with deterministic recovery rates.

The numerical results indicate that the model with economic change risk, stochastic recovery rates, and upper tail dependent copulas has the highest spread, and is the most sensitive to economic change risk. Accordingly, it provides the most conservative price under the assumption of the poor state of future economy.

IV. Conclusion

This paper presents a model for pricing portfolio credit derivatives with nested Archimedean copulas, stochastic recovery rates, and an exogenous systematic factor. The model explains the dependence structures between default probabilities and recovery rates, which are also affected by economic change risk. The advantage of the model is the systematic factor is able to be set optimally. This leads that when we compute prices, we can consider economic state using the proposed model. The model captures the phenomena that if economic condition becomes poor, default times and recovery rates of assets will become worse than predicted. Pricing of basket CDSs is also demonstrated as an example of portfolio credit derivatives, calculating spreads and the correlation coefficients numerically to investigate the effects of the systematic factor.

Our main findings are as follows. First, economic change risk enlarges the strength of the dependence and the spread of the models with and without it. This is particularly significant when the economic change risk is high. Second, the correlation coefficients of models of the Gumbel copula are more sensitive to economic change risk than those of models of the Gaussian. Furthermore, the models with stochastic recovery rates result in higher spreads than those with deterministic recovery rates. Our model with economic change risk, stochastic recovery rates, and upper tail dependent copulas has the highest spread, and is the most sensitive to economic change risk. Thus, it computes the most conservative price under the assumption of the poor state of future economy.

Future research includes estimating the distribution of economic change risk and constructing a model where economic change risk elicits both default time and recovery rate differences. Solving these problems will make the relationships between the systematic factor, default times, and recovery rates more realistic.

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