Degree of Approximation of Functions in Lipschitz Class with Muckenhoupt Weights by Matrix Means

Uaday Singh and Shailesh Kumar Srivastava

Abstract—In this paper, we investigate the approximation properties of the matrix means of trigonometric Fourier series of $f$ belonging to weighted Lipschitz class $\text{Lip}(\alpha, p, w)$ with Muckenhoupt weights generated by $T \equiv (a_{n,k})$ under relaxed conditions. Our theorem extends some of the previous results pertaining to the degree of approximation of functions in weighted Lipschitz class $\text{Lip}(\alpha, p, w)$ and the ordinary Lipschitz class $\text{Lip}(\alpha, p)$.

Index Terms—Fourier series, Matrix means, Muckenhoupt class, Class $\text{Lip}(\alpha, p, w)$.

I. INTRODUCTION

A MEASURABLE $2\pi$-periodic function $f : [0, 2\pi] \to [0, \infty]$ is said to be a weight function if the set $w^{-1}((0, \infty))$ has the Lebesgue measure zero. We say that $f \in L^p_w[0, 2\pi] := L^p_w$, the weighted Lebesgue space of all measurable $2\pi$-periodic functions if

$$\|f\|_{p,w} = \left(\int_0^{2\pi} |f(x)|^p w(x)\,dx\right)^{1/p} < \infty, \quad 1 \leq p < \infty.$$

Let $1 < p < \infty$. A weight function $w$ belongs to the Muckenhoupt class $A_p$, if

$$\sup_I \left(\frac{1}{|I|} \int_I w(x)\,dx\right) \left(\frac{1}{|I|} \int_I [w(x)]^{-1/(p-1)}\,dx\right)^{p-1} < \infty,$$

where the supremum is taken over all intervals $I$ with length $|I| \leq 2\pi$. The weight functions belonging to $A_p$ class, introduced by Hunt et al. [1], play an important role in different fields of mathematical analysis.

Let $w \in A_p$ and $f \in L^p_w$. The modulus of continuity of the function $f$ is defined by

$$\Omega(f) = \sup_{|h| \leq \delta} \|\Delta_h(f)\|_{p,w}, \quad \delta > 0,$$

where

$$(\Delta_h f)(x) = \frac{1}{h} \int_0^h |f(x + t) - f(x)|\,dt.$$

The existence of the modulus of continuity of $f \in L^p_w$ follows from the boundedness of the Hardy-Littlewood maximal function in the space $L^p_w$ [2]. The modulus of continuity $\Omega(f)$, $p,w$ defined by Ky [3] is non-decreasing, non-negative, continuous function such that

$$\lim_{\delta \to 0} \Omega(f, \delta)_{p,w} = 0$$

and

$$\Omega(f_1 + f_2, \cdot)_{p,w} \leq \Omega(f_1, \cdot)_{p,w} + \Omega(f_2, \cdot)_{p,w}.$$

The modulus of continuity $\Omega(f, \cdot)_{p,w}$ is defined in this way, since the space $L^p_w$ is non-invariant, in general, under the usual shift $f(x) \to f(x+\delta)$. We note that in the case $w \equiv 1$, the modulus of continuity $\Omega(f, \cdot)_w$ and the classical integral modulus of continuity $\omega_p(f, \cdot)$ are equivalent [3]. The weighted Lipschitz class $\text{Lip}(\alpha, p, w)$ for $0 < \alpha \leq 1$ is defined by

$$\text{Lip}(\alpha, p, w) = \{f \in L^p_w : \Omega(f, \cdot)_{p,w} = O(\delta^\alpha), \delta > 0\}.$$

For $w(x) = 1$  for $0 \leq x < 2\pi$ the weighted $\text{Lip}(\alpha, p, w)$ class reduces to well known Lipschitz class $\text{Lip}(\alpha, p)$.

Let $f \in L^p[0, 2\pi]$ ($p \geq 1$) be a $2\pi$-periodic function. Then, for $n \in \mathbb{N} \cup \{0\}$ we write

$$s_n(f; x) = a_0/2 + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^n u_k(f; x), \quad u_n(f; x) = s_0(f; x) = a_0/2,$$

the $(n+1)$th partial sum of Fourier series of $f$ at point $x$, which is a trigonometric polynomial of order (or degree) $n$.

The matrix means of the Fourier series of $f$ are defined by the sequence to sequence transformation

$$\tau_n(f; x) = \tau_n(f) = \sum_{k=0}^n a_{n,k} s_k(f; x), \quad n \in \mathbb{N} \cup \{0\},$$

(1)

where $T \equiv (a_{n,k})$ is a lower triangular matrix with non-negative entries such that $a_{n,-1} = 0$, $A_{n,k} = \sum_{r=k}^n a_{n,r}$ and $A_{n,0} = 1$ $\forall n \geq 0$. The Fourier series of $f$ is said to be $T$-summable to $s(x)$, if $\tau_n(f; x) \to s(x)$ as $n \to \infty$. If for every convergent sequence $\{a_n\}$, $\lim_{n \to \infty} a_n = s$ implies $\lim_{n \to \infty} \tau_n = s$, then matrix $T$ is said to be regular. In particular, if

$$a_{n,k} = \begin{cases} p_{n-k}/P_n, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

and

$$a_{n,k} = \begin{cases} p_k/P_n, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

(2)

where $P_n(\equiv \sum_{k=0}^n p_k \neq 0) \to \infty$ as $n \to \infty$ and $P_{-1} = 0 = p_{-1}$, then the summability matrix $T$ reduces to Nörlund and Riesz matrices, respectively, and $\tau_n(f; x)$ in (1) defines corresponding means $N_n(f; x)$ and $R_n(f; x)$. In case of $a_{n,k} = 1/(n+1)$ for $0 \leq k \leq n$ and $a_{n,k} = 0$ for $k > n$, the $\tau_n(f; x)$ reduces to Cesàro means of order one, denoted by $\sigma_n(f; x)$.

A positive sequence $a = \{a_{n,k}\}$ is called almost monotonically decreasing with respect to $k$, if there exists a constant $K = K(a)$, depending on the sequence $a$ only,
such that $a_{n,p} \leq Ka_{n,m}$ for all $p \geq m$ and we write that $\alpha \in AMDS$. Similarly $\alpha = \{a_{n,k}\}$ is called almost monotonically increasing with respect to $k$, if $a_{n,p} \leq Ka_{n,m}$ for all $p \leq m$ and we write that $\alpha \in AMIS$ [4, 5]. We note that every monotone sequence is an almost monotone sequence.

We also write $\Delta_ka_{n,k} = a_{n,k} - a_{n,k+1}$ and $[x]$, the greatest integer contained in $x$.

The Fourier series and trigonometric polynomials play an important role in various scientific and engineering fields, e.g., Lo and Hui [6] use the Fourier series expansion in a very nice way. Based upon the Fourier series expansion, they propose a simple and easy-to-use approach for computing accurate estimates of Black-Scholes double barrier option prices with time-dependent parameters.

Chandra [7] has studied the approximation properties of the means $N_n(f;x)$ and $R_n(f;x)$ in $Lip(\alpha,p)$, $1 \leq p < \infty$, $0 < \alpha < 1$ with monotonicity on the means generating sequence $\{p_n\}$ and proved $\|N_n(f;x) - f(x)\|_p = O(n^{-\alpha}) = \|R_n(f;x) - f(x)\|_p$, $n = 1, 2, 3, ...$. Mittal et al. [8] generalized the paper of Chandra [7] partially, and extended its Theorem 1 and Theorem 2 (ii) to matrix means with $\sum_{k=0}^{\infty}a_{n,k} - 1 = O(n^{-\alpha})$. On the other hand, Leindler [4] has relaxed the condition of monotonicity on $\{p_n\}$ and proved some of the results of Chandra [7] for almost monotone weights $\{p_n\}$. Mittal et al. [5] extended the results of Leindler [4] to matrix means with almost monotone sequence $\{a_{n,k}\}$ and row sums 1. Following [9], [10] and [11], recently Guven [12] has extended some of the results of Chandra [7] in another direction. He has extended Lipschitz class $Lip(\alpha,p)$ to $Lip(\alpha,p,w)$, and proved the weighted version of the Theorem 1 and Theorem 2 of Chandra [7] for $1 \leq p < \infty$, $0 < \alpha \leq 1$ i.e., for $f \in Lip(\alpha,p)$ and monotone $\{p_n\}$ he proved

$$\|N_n(f;x) - f(x)\|_{p,w} = O(n^{-\alpha}) = \|R_n(f;x) - f(x)\|_{p,w}, \ n = 1, 2, 3, ... \ .$$

Following Mittal et al. [5], very recently, Singh and Sonker [13] have studied the degree of approximation of periodic functions in generalized Hölder metric space through matrix means of Fourier series, where matrix $T \equiv (a_{n,k})$ has almost monotone rows. On the other hand, Guven [14, Theorem 1 and Theorem 2] has given the weighted version of [4] and some of the results of [8] by assuming $T \equiv (a_{n,k})$ almost monotone with $\sum_{k=0}^{\infty}a_{n,k} - 1 = O(n^{-\alpha})$. We note that condition $\sum_{k=0}^{\infty}a_{n,k} - 1 = O(n^{-\alpha})$ was not used by Leindler [4] as $\sum_{k=0}^{\infty}a_{n,k} = 1$ for Nörlund matrix. It also appears that the author in [14] have followed many conditions and calculations as given in [5] and [13] without citing them.

Apart from this Nigam and Sharma [15] applied the concept of $(C,1)(E,q)$ summability method and establish a new theorem on degree of approximation of a function $f \in Lip(\ell(t),r)$ class. Very recently, the authors in [16] obtained the degree of approximation of functions conjugate to the function belonging to weighted Lipschitz class $W(L^p,\ell(t))$ and find the error of approximation free from $p$ and sharper that the earlier obtained results in this direction.

II. MAIN RESULT

In the present paper, we continue the work of [5] and [13] and prove weighted version of the theorem of [5] for $p > 1$ which extend the result of Leindler [4] to weighted version as well as matrix version for $p > 1$. Our theorem also extends theorems of Guven [12] to matrix means $\tau_n(f;x)$ under the relaxed conditions of monotonicity and replaces the two theorems of Guven [14] by a single theorem for $\sum_{k=0}^{\infty}a_{n,k} = 1$. More precisely, we prove:

**Theorem 1.** Let $f \in Lip(\alpha,p,w)$, $p > 1$, $w \in A_p$ and let $T \equiv (a_{n,k})$ be an infinite lower triangular regular matrix and satisfies one of the following conditions:

(i) $0 < \alpha < 1$, $\{a_{n,k}\} \in AMIS$ in $k$,

(ii) $0 < \alpha < 1$, $\{a_{n,k}\} \in AMIS$ in $k$ and $(n+1)a_{n,0} = O(1),$

(iii) $\alpha = 1$ and $\sum_{k=0}^{n-1}(n-k)|\Delta_ka_{n,k}| = O(1),$

(iv) $\alpha = 1$, $\sum_{k=0}^{n-1}|\Delta_ka_{n,k}| = O(a_{n,0})$ with $(n+1)a_{n,0} = O(1),$

(v) $0 < \alpha \leq 1$, $\sum_{k=0}^{n-1}|\Delta_k(a_{n,0}-a_{n,k+1})| = O\left(\frac{1}{n+1}\right)$.

Then $\|f(x) - \tau_n(f;x)\|_{p,w} = O((n+1)^{-\alpha}), n = 0, 1, 2,...$ (3)

III. LEMMAS

To prove our theorem, we need the following lemmas.

**Lemma 1** ([12]). Let $1 < p < \infty$, $w \in A_p$ and $0 \leq \alpha \leq 1$.

Then the estimate

$$\|f(x) - s_n(f;x)\|_{p,w} = O((n+1)^{-\alpha}), n = 0, 1, 2,... \quad (4)$$

holds for every $f \in Lip(\alpha,p,w)$.

**Lemma 2** ([12]). Let $1 < p < \infty$ and $w \in A_p$. Then, for $f \in Lip(1,p,w)$ the estimate

$$\|s_n(f;x) - \sigma_n(f;x)\|_{p,w} = O((n+1)^{-1}), n = 0, 1, 2,... \quad (5)$$

holds.

**Lemma 3** ([5]). Let either $\{a_{n,k}\} \in AMIS$ or $\{a_{n,k}\} \in AMDS$ with $(n+1)a_{n,0} = O(1),$ Then, for $0 < \alpha < 1,$

$$\sum_{k=0}^{n}(k+1)^{-\alpha}a_{n,k} = O((n+1)^{-\alpha}).$$

**Proof.** Let $r = [n/2]$ and $\{a_{n,k}\} \in AMIS$, then

$$\sum_{k=0}^{k} (k+1)^{-\alpha}a_{n,k} \leq Ka_{n,r} \sum_{k=0}^{r} (k+1)^{-\alpha} + (r+1)^{-\alpha} \sum_{k=r+1}^{n} a_{n,k} \leq K(r+1)^{-\alpha} + (r+1)^{-\alpha} \sum_{k=0}^{n} a_{n,k} \leq K(r+1)^{-\alpha} a_{n,r+1} \leq A_n, 0 \quad (6)$$

in view of $(r+1)a_{n,r} \leq (n-r+1)a_{n,r+1} \leq K(a_{n,r+1} + \ldots + a_{n,n}) \leq A_n, 0$ and $(r+1)^{-\alpha} = O((n+1)^{-\alpha}).$

If $\{a_{n,k}\} \in AMDS$ and $(n+1)a_{n,0} = O(1),$ then

$$\sum_{k=0}^{n}(k+1)^{-\alpha}a_{n,k} \leq Ka_{n,0} \sum_{k=0}^{n}(k+1)^{-\alpha} = O((n+1)^{-\alpha}).$$

This completes the proof of Lemma 3.

(Advance online publication: 29 November 2013)
IV. PROOF OF THEOREM 1

We prove the cases (i) and (ii) together by using Lemma 1 and Lemma 3. Since
\[
\tau_n(f; x) - f(x) = \sum_{k=0}^{n} \alpha_{n,k}(s_k(f; x) - f(x)),
\]
\[
\|\tau_n(f; x) - f(x)\|_{p,w} \leq \sum_{k=0}^{n} \alpha_{n,k} \|s_k(f; x) - f(x)\|_{p,w}
= \sum_{k=0}^{n} \alpha_{n,k} O(k + 1)^{-\alpha}
= O((n + 1)^{-\alpha}).
\]

Next, we consider the case (iv). By Abel’s transformation and \(\alpha_{n,n+1} = 0\),
\[
\tau_n(f; x) = \sum_{k=0}^{n} \alpha_{n,k} s_k(f; x)
= \sum_{k=0}^{n} \alpha_{n,k} \left( \sum_{i=0}^{k} u_i(f; x) \right)
= \sum_{k=0}^{n} \alpha_{n,k} u_k(f; x),
\]
and thus
\[
s_n(f; x) - \tau_n(f; x) = \sum_{k=0}^{n} \left( 1 - \alpha_{n,k} \right) u_k(f; x)
= \sum_{k=1}^{n} k^{-1} (A_{n,0} - A_{n,k}) u_k(f; x).
\]

Hence, again by Abel’s transformation and \(A_{n,n+1} = 0\), we get
\[
s_n(f; x) - \tau_n(f; x) = \sum_{k=0}^{n} \left( \Delta_k k^{-1}(A_{n,0} - A_{n,k}) \right) \times
\sum_{i=1}^{k} u_i(f; x) + (n + 1)^{-1} \sum_{k=1}^{n} u_k(f; x).
\]

Therefore,
\[
\|s_n(f; x) - \tau_n(f; x)\|_{p,w}
\leq \sum_{k=1}^{n} \|\Delta_k k^{-1}(A_{n,0} - A_{n,k})\| \left\| \sum_{i=1}^{k} i u_i(f; x) \right\|_{p,w}
+ (n + 1)^{-1} \left\| \sum_{k=1}^{n} u_k(f; x) \right\|_{p,w}.
\]

Also
\[
s_n(f; x) - \sigma_n(f; x)
= (n + 1)^{-1} \sum_{k=0}^{n} ((n + 1) u_k(f; x) - s_k(f; x))
= (n + 1)^{-1} \sum_{k=0}^{n} k u_k(f; x),
\]
which implies
\[
\left\| \sum_{k=1}^{n} u_k(f; x) \right\|_{p,w} = (n + 1) \|\sigma_n(f; x) - s_n(f; x)\|_{p,w}
= O(1),
\]
in view of Lemma 2. Using (7) in (6), we get
\[
\|s_n(f; x) - \tau_n(f; x)\|_{p,w} \leq \sum_{k=1}^{n} \|\Delta_k k^{-1}(A_{n,0} - A_{n,k})\|
+ (n + 1)^{-1}.
\]

Now,
\[
\Delta_k k^{-1} (A_{n,0} - A_{n,k})
= \frac{A_{n,0} - A_{n,k}}{k}
= \frac{A_{n,0} - A_{n,k+1}}{k + 1}
= k^{-1}(k + 1)^{-1} (A_{n,0} - A_{n,k} - k a_{n,k})
= k^{-1}(k + 1)^{-1} (\sum_{i=0}^{k-1} a_{n,i} - k a_{n,k}).
\]
Next, we shall verify by induction that,
\[
\sum_{i=0}^{k-1} a_{n,i} - k a_{n,k} \leq \sum_{i=1}^{k} i |a_{n,i-1} - a_{n,i}|.
\]

For \(k = 1\), we have
\[
\sum_{i=0}^{k-1} a_{n,i} - k a_{n,k} = |a_{n,0} - a_{n,1}| = |a_{n,0} - a_{n,1}|,
\]
i.e., (10) is true for \(k = 1\). Let us assume that (10) is true for \(k = m\), then for \(k = m + 1\),
\[
\sum_{i=0}^{m} a_{n,i} - (m + 1) a_{n,m+1}
= \sum_{i=0}^{m-1} a_{n,i} + a_{n,m} + a_{n,m+1}
+ m a_{n,m} - (m + 1) a_{n,m+1}
\leq \sum_{i=0}^{m-1} a_{n,i} + a_{n,m} + (m + 1) a_{n,m+1}
= \sum_{i=1}^{m+1} i |a_{n,i-1} - a_{n,i} + (m + 1) a_{n,m} - a_{n,m+1}|
= \sum_{i=1}^{m+1} i |a_{n,i-1} - a_{n,i}|
\]
Thus (10) is true for \(k = m + 1\), hence (10) is true for \(1 \leq k \leq n\).

Using (9) and (10), we get
\[
\sum_{k=1}^{n} \|\Delta_k k^{-1}(A_{n,0} - A_{n,k})\|
\leq \sum_{k=1}^{n} k^{-1}(k + 1)^{-1} \sum_{i=1}^{k} i |a_{n,i-1} - a_{n,i}|
\leq \sum_{k=1}^{n} i |a_{n,i-1} - a_{n,i}| \sum_{i=k+1}^{\infty} k^{-1}(k + 1)^{-1}
= \sum_{i=1}^{n} |a_{n,i-1} - a_{n,i}| \sum_{k=0}^{n-1} \|\Delta_k a_{n,k} - k a_{n,k} - k a_{n,k}||
= O(a_{n,n}) = O((n + 1)^{-1}).
\]

Combining (8) and (11), we get
\[
\|s_n(f; x) - \tau_n(f; x)\|_{p,w} = O((n + 1)^{-1}).
\]

Using Lemma 1 and (2), we have for \(\alpha = 1\)
\[
\|f(x) - \tau_n(f; x)\|_{p,w} \leq \|f(x) - s_n(f; x)\|_{p,w}
+ \|s_n(f; x) - \tau_n(f; x)\|_{p,w} = O((n + 1)^{-1}).
\]

Herein the case (iv) is proved.

For the proof of case (iii), we first verify that the condition
\[
\sum_{k=0}^{n-1} (n - k) \|\Delta_k a_{n,k}\| = O(1),
\]
implies that
\[
\sum_{k=1}^{n} \|\Delta_k k^{-1}(A_{n,0} - A_{n,k})\| = O((n + 1)^{-1}).
\]

From (9), we can write
\[
\sum_{k=1}^{n} \|\Delta_k k^{-1}(A_{n,0} - A_{n,k})\|
\leq \sum_{k=1}^{n} k^{-1}(k + 1)^{-1} \sum_{i=1}^{k} i |a_{n,i-1} - a_{n,i}|
= \sum_{k=1}^{n} \Delta(k^{-1}) \sum_{i=1}^{k} i |a_{n,i-1} - a_{n,i}|.
\]
By Abel’s transformation, we have
\[
\sum_{k=1}^{n} |\Delta k A_{n,n,k}|^{1} \leq \sum_{k=1}^{n} |\Delta k A_{n,n,k}|^{1} \leq \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) |a_{n,k-1} - a_{n,k}|
\]
\[
= \sum_{k=1}^{n} \left( \frac{n-k+1}{n+1} \right) |a_{n,k-1} - a_{n,k}|
\]
\[
= \sum_{n=0}^{n} \left( \frac{n-k}{n+1} \right) |a_{n,k+1} - a_{n,k}|
\]
\[
\leq \frac{1}{n+1} \sum_{k=0}^{n-1} (n-k) |\Delta k a_{n,k}| = O((n+1)^{-1}),
\]
which verifies (13). Combining (8), (13) and Lemma 2, we get (3) for \( \alpha = 1 \).

Finally, we prove the case (v). Using Lemma 1 and Abel’s transform
\[
\| \tau_{n}(f;x) - f(x) \| \leq \sum_{k=0}^{n} a_{n,k} \| s_{k}(f;x) - f(x) \|_{p,w}
\]
\[
= O\left( \sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,k} \right)
\]
\[
= O\left( \sum_{k=0}^{n} \Delta k (k+1)^{-\alpha}(k+1)^{-\alpha} \left( \sum_{i=0}^{k} a_{n,i} \right) \right)
\]
\[
+ O\left( \sum_{k=0}^{n} \Delta k (k+1)^{-\alpha} A_{n,0} - A_{n,k+1} \right)) \leq (n+1)^{-\alpha},
\]
where by Abel’s transformation
\[
\sum_{k=0}^{n} (k+1)^{-\alpha} A_{n,0} - A_{n,k+1}
\]
\[
= \sum_{k=0}^{n} \Delta k \left( \frac{A_{n,0} - A_{n,k+1}}{k+1} \right) \sum_{i=0}^{k} (i+1)^{-\alpha} A_{n,0} - A_{n,n+1} \sum_{i=0}^{n} (i+1)^{-\alpha}
\]
\[
\leq \sum_{k=0}^{n} \Delta k \left( \frac{A_{n,0} - A_{n,k+1}}{k+1} \right) (k+1)^{-\alpha} A_{n,0} - A_{n,k+1} \leq (n+1)^{-\alpha},
\]
in view of \( A_{n,n+1} = 0 \) and condition (v) of Theorem 1. Collecting (14) and (15), we get (3). Thus proof of Theorem 1 is complete.

V. COROLLARIES

In order to justify the significance of our result, we prove that the following results are the particular cases of Theorem 1 for \( p > 1 \). We also drive an analogous result of Theorem 1 for monotone \( \{a_{n,k}\} \).

1) If we take \( a_{n,k} = p_{n-k}/P_{n} \) for \( k \leq n \) and \( a_{n,k} = 0 \) for \( k > n \), then conditions (i) to (iv) of Theorem 1 reduce to conditions (i) to (iv) of Theorem 1 of Leindler [4, p. 131], respectively, and \( \tau_{n}(f;x) \) means reduces to \( N_{n}(f;x) \) means. Further, we note that \( Lip(\alpha,p,1) \equiv Lip(\alpha,p) \), \( p > 1 \). Thus our theorem generalizes Theorem 1 of [4], except for the case \( p = 1 \) in two directions.

2) Since \( Lip(\alpha,p,1) \equiv Lip(\alpha,p) \), \( p > 1 \) and conditions of Theorem 1 of Mittal et al. [5, p. 4485] for \( p > 1 \) are included in the conditions (i) to (iv) of Theorem 1, our theorem includes weighted version of Theorem 1 of [5] for \( p > 1 \).

3) Since every monotone sequence is almost monotone, the conditions (i) and (ii) of Theorem 1 are satisfied in case of monotonic \( \{a_{n,k}\} \). Further, every sequence \( \{a_{n,k}\} \) non-decreasing with respect to \( k \) always satisfies condition (iii) of Theorem 1, e. g.,
\[
\sum_{k=0}^{n-1} (n-k) |\Delta k a_{n,k}| = \sum_{k=0}^{n-1} (n-k)(a_{n,k+1} - a_{n,k})
\]
\[
= A_{n,0} - (n+1)a_{n,0} = O(1).
\]

If \( \{a_{n,k}\} \) is non-increasing with respect to \( k \), then (iv) of Theorem 1 is also true, e. g.,
\[
\sum_{k=0}^{n-1} |\Delta k a_{n,k}| = \sum_{k=0}^{n-1} (a_{n,k} - a_{n,k+1})
\]
\[
= a_{n,0} - a_{n,n} \leq a_{n,0}.
\]

Thus, we have the following analogous result of Theorem 1 for monotone \( \{a_{n,k}\} \):

Theorem 2. Let \( f \in Lip(\alpha,p,w) \), \( p > 1 \), \( w \in A_{n} \) and let \( T \equiv (a_{n,k}) \) be an infinite regular triangular matrix and satisfies one of the following conditions:

(i) \( \{a_{n,k}\} \) is non-decreasing in \( k \),

(ii) \( \{a_{n,k}\} \) is non-increasing in \( k \) and \( (n+1)a_{n,0} = O(1) \).

Then (3) holds.

4) If we take \( a_{n,k} = p_{n-k}/P_{n} \) for \( k \leq n \) and \( a_{n,k} = 0 \) for \( k > n \), then Theorem 2 reduces to Theorem 1 of Guven [12, p. 101].

5) Finally, if we take \( a_{n,k} = p_{n-k}/P_{n} \) for \( k \leq n \) and \( a_{n,k} = 0 \) for \( k > n \), then \( \tau_{n}(f;x) \) means reduces to \( R_{n}(f;x) \) means; and
\[
A_{n,0} - A_{n,k+1} = \sum_{i=0}^{n} a_{n,i} - \sum_{i=k+1}^{n} a_{n,i}
\]
\[
= \left( \sum_{i=0}^{n} p_{n} - \sum_{i=k+1}^{n} p_{n} \right) / P_{n}
\]
\[
= \sum_{i=0}^{k} p_{i} / P_{n} = P_{k} / P_{n},
\]

so that
\[
\Delta k \left( A_{n,0} - A_{n,k+1} \right) = \frac{A_{n,0} - A_{n,k+1}}{k+1} + \frac{A_{n,0} - A_{n,k+1}}{k+2}
\]
\[
= \frac{1}{P_{n}} \left( \frac{P_{k+1}}{k+2} - \frac{P_{k}}{k+1} \right),
\]
i. e., condition (v) of Theorem 1 reduces to condition (3) of Guven [12, Theorem 2]. Thus Theorem 1 under condition (v) extends Theorem 2 of Guven [12] to matrix means.

6) If we take \( a_{n,k} = A_{n}^{p-1}/A_{n}^{2} \) for \( k \leq n \) and \( a_{n,k} = 0 \) for \( k > n \), where

(Advance online publication: 29 November 2013)
\[ A_0^β = 1, \; A_k^β = \frac{β(β + 1)\ldots(β + k)}{k!}, \; k ≥ 1, \]

then matrix means \( τ_n(f; x) \) reduces to Cesàro means of order \( β > 0 \) denoted by \( σ_n^β(f; x) \) and defined as

\[ σ_n^β(f; x) = \frac{1}{A_n} \sum_{k=0}^{n} A_n^{-1} s_k(f; x). \]

Hence, Corollary 3 of Guven [12, p. 102] can also be derived from Theorem 1.

7) In the light of remark of Guven [12, p. 102], we note that Theorems 1 and 2 also hold in reflexive weighted Orlicz spaces \( L_w^M \), which are discussed in [10] in detail.

VI. CONCLUSION

Theorems of this paper are an attempt to formulate the problem of approximation of \( f \in Lip(α,p,w) \), \( p > 1 \) through trigonometric polynomials generated by the summability means of the Fourier series of \( f \) in a simpler manner. The case for \( p = 1 \) is still an open problem which can be addressed by making certain modifications in the definition of Muckenhoupt class \( A_p \).

REFERENCES


