Accelerating Monte Carlo Method for Pricing Multi-asset Options under Stochastic Volatility Models

Kun Du, Guo Liu, and Guiding Gu

Abstract—**In this paper we investigate the control variate Monte Carlo method for pricing some multi-asset options with the stochastic volatility model. First several multi-asset options are priced analytically, which volatilities are the deterministic functions of the time, by using the risk-neutral pricing formula. Then we derive the explicit representation of control variate under the Hull-White stochastic volatility model. Numerical experiment results suggest that our proposed method performs good in terms of improving the efficiency of variance reduction by obtaining more accurate prices of multi-asset options. This idea can be extended easily for pricing other financial instruments with stochastic volatility models.**

Index Terms—**Multi-asset options pricing, Stochastic volatility, Monte Carlo method, control variates.**

I. Introduction

FINANCE derivatives being an important part of modern financial market are constructed and traded by financial institutions. Since these finance derivatives or securities are so complex, pricing these securities poses a challenging task. Practitioners usually collect the data happened in today market, update the database and calculate prices of the instruments for trading tomorrow. Because of huge trading volumes, the error resulting from inaccurate pricing may cause big loss, and good pricing algorithms are necessary to be devised. Therefore, we provide a class of efficiency control variates for pricing multi-asset options to accelerate the Monte Carlo method.

Multi-asset option is an exotic option whose payoff depends on the overall performance of more than one underlying asset. It can be divided into three categories: rainbow options, basket options and quanto options. What we discuss like exchange options, spread options, chooser options and max-call options are rainbow options. We will consider basket options and basic quanto options. Jiang [14] introduced concepts and properties of these options in detail, which volatilities are constant. And the stochastic volatility model was first proposed by Hull and White [13] for improving the assumption about the distribution of the stock price in the Black-Scholes model [2]. There are many literatures having been done on the stochastic volatility models, like Scott [21], Stein and Stein [23], Ball and Roma [1], and Heston [12]. Fouque et al. [9] summarized the application of the stochastic volatility models in financial instruments.

Two main methods are proposed for pricing multi-asset options: the analytic approximation approach and the fast Fourier transform(FFT for short) method. The former approach uses a new pricing model which has closed form solution as an approximation value. The advantage of this method is the fast computation, but with the big error which will not be disappeared just not as in the Monte Carlo method when the number of paths goes to infinite. Datey, Gauthier and Simonato [7], Borovkova, Permana and Weide [3], Milevsky and Posner [20], Li, Deng and Zhou [17] focused on this method. The latter method solves low-dimensional problem very well, but can not be applied to the problems whose dimensions over three, which Monte Carlo method is well used. In the other way, we must get the closed form of underlying assets union characteristic functions in advance when using the FFT method. In fact, these functions are not easy to be obtained unless under some very special models(like affine models). Carr and Madan [5], Carr and Wu [6] exploited the FFT method successfully. Also, Monte Carlo method is applied to finance very widely in practice, like Zheng, et al. [24], Boughmoura and Trabelsi [4] and Grzybowski [11], because of its simple exercise, but not much for multi-asset options pricing. The main reason maybe that there is no good method for speed up simulation for Monte Carlo method. So we present a class of control variates for multi-asset options pricing with Monte Carlo method, which provides good variance reduction efficiency.

How to price multi-asset options with stochastic volatility models? Since there is no closed formula for options value, we mainly focus on the control variate Monte Carlo method for multi-asset options pricing because it works more efficiently than the ordinary Monte Carlo method. We derive the control variate by choosing a deterministic function which is equal to some order moments of the Hull-White stochastic volatility. We can obtain the number of control variates just as the number of the order moments. Numerical results show that these control variates improve the Monte Carlo method efficiently by reducing variance hundreds of times or ten thousands of times. Theoretically, we can use multi control variates to get much faster computation. As a byproduct, we get analytical approximation values for several multi-asset options.

The rest of the paper is organized as follows. First we give some introductions on the control variate method and its application procedure in pricing financial instruments under the stochastic volatility model. Then we consider rainbow options, basket options and quanto options with

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the control variate separately, under the Hull-White model, which numerical results are also provided. At the end, we summary the paper and extend the idea to other financial derivatives.

II. Control Variate Monte Carlo Method and its Application in Valuating Financial Securities

The control variate method is a variance reduction technique used most widely in the Monte Carlo method. It exploits information about the error of the estimators known quantities to reduce the error of the estimator unknown quantity. We will introduce this method in detail as follows. Suppose we want to calculate an expectation $E[V]$, and $E[X]$ is a known value. Then we can generate (X_i, V_i), $i = 1, ..., n$, which are independent and identical distribution samples from the population (X, V). Then for some real number b, we derive $V_i(b) = V_i - b(X_i - E[X])$, and its sample average value $\overline{V}(b) = \overline{V} - b(\overline{X} - E[X])$.

Glasserman [10] proved $\overline{V}(b)$ is an unbiased and consistent estimator of the expectation $E[V]$. The variance of $V(b)$ is

$$Var(V(b)) = \sigma_V^2 - 2b\sigma_X\sigma_V\rho_{XV} + b^2\sigma_X^2. \tag{1}$$

To make $Var(V(b))$ achieve the minima, we choose

$$b^* = \frac{\sigma_V}{\sigma_X}\rho_{XV},$$

and derive the minimum variance estimator by replacing b in (1), which divides the variance of the ordinary Monte Carlo method, i.e.

$$\frac{Var(\overline{V}(b^*))}{Var(\overline{V})} = 1 - \rho_{XV}^2. \tag{2}$$

(2) suggests that the larger the coefficient is, the larger the variance reduction ratio is. Then there are two basic principles for choosing the good control variate: the conditional expectation of the control variate should have closed forms; the correlation between the control variate and the random variates that we want the expectation should be strong. In practice, we always obtain b^* by the estimator $\hat{b}$ based on samples,

$$\hat{b} = \frac{\sum_{i=1}^n (X_i - \overline{X})(V_i - \overline{V})}{\sum_{i=1}^n (X_i - \overline{X})^2}.$$

Just as Glasserman [10] said, we can derive most advantage from the control variate even the approximation $\hat{b}$ used. Strictly, we only need samples V_i when we do not use the control variate, but samples (X_i, V_i) with the control variate method. Then we can ignore the time increasing for the control variate simulation and just consider the variance reduction ratios as in the paper of Ma and Xu [18].

Suppose we consider the problem in the probability space $(\Omega,\ \{\mathcal{F}_t\}_{t\geq 0},\ \mathbf{P})$, where Ω is the sample space, $\{\mathcal{F}_t\}_{t\geq 0}$ is the σ-algebra based on Ω, and $\mathbf{P}$ is the risk-neutral probability measure. In this paper all stochastic processes are presented under the risk-neutral measure $\mathbf{P}$. Suppose the underlying asset is the stock, whose price follows the geometric Brownian Motion

$$dS_t = rS_tdt + \sigma_tS_tdW_{1t}, \tag{3}$$

where r is the risk-free interest rate and supposed to be a constant, W_{1t} is the standard Brownian Motion under $\mathbf{P}$, and σ_t is the volatility of the stock yield, which is a diffusion process, whose diffusion part is the standard Brownian Motion W_{2t} satisfying $Cov(dW_{1t}, dW_{2t}) = \rho dt$. Then we can denote $W_{2t} = \rho W_{1t} + \sqrt{(1-\rho^2)}B_t$, where B_t is the standard Brownian Motion independent with W_{1t}. Let $\mathcal{F}_t$ be the filtering generated by (W_{1t}, B_t). Then we know S_t and σ_t are $\mathcal{F}_t$ adapted, where σ_t is integrable, that is

$$E[\int_0^t \sigma_s^2 ds] < \infty.$$

Suppose the financial derivative with the underlying asset S_t has the payoff function $G(S)$ at the expiration time T, where S is the stock price at the time T. If there are several(two or more than two) underlying assets(like multi-asset), S is a vector. If the derivative is path-dependent, then we need anther variate J(like Asian options, J is the average of the underlying asset prices) to derive the payoff function $G(S, J)$. The risk-neutral pricing formula suggests that the security price at the time t is

$$V_t = E[e^{-r(T-t)}G(S, J)|\mathcal{F}_t]. \tag{4}$$

When calculating the estimator of V_t by Monte Carlo method, we need to simulate path S_t from the time t to T. Denote S^i and J^i as the i-th simulation path S and J respectively. Then we can derive the estimator of the discounted V_t by N simulation paths

$$\overline{V}_t = \frac{e^{-r(T-t)}}{N}\sum_{i=1}^N G(S^i, J^i). \tag{5}$$

When the variance of $G(S, J)$ is large, it suggests that the error will be large between the estimator $\overline{V}_t$ and the true value V_t. However, if a random variable $H(\widetilde{S}, \widetilde{J})$ can be found, whose conditional expectation can be calculated easily, and with high correlation with $G(S, J)$, we can use $H(\widetilde{S}, \widetilde{J})$ as a control variate for $G(S, J)$ to reduce the estimator variance, and make the estimator more accurate.

There are two methods to make $H(\widetilde{S}, \widetilde{J})$ and $G(S, J)$ with high correlation: the first one is to make the function H and the payoff function G closer, just as the method used in Kemma and Vorst [16]; the second method is to make the variable $\widetilde{S}$ and the underlying asset S closer. Here we consider the control variate for options pricing under the stochastic volatility model. So the point is to find a good random variable $\widetilde{S}$ as close as possible to the underlying asset S, and satisfying the conditional expectation of $G(\widetilde{S}, \widetilde{J})$ can be easily calculated.

The conditional expectation in (4) cannot derive a closed form for the stochastic volatility of S_t, which remind me we can find $\widetilde{S}_t$ with non-random volatility satisfying the conditional expectation $G(\widetilde{S}, \widetilde{J})$ obtained easily. However this cannot guarantee a good control variate unless the correlation between $G(\widetilde{S}, \widetilde{J})$ and $G(S, J)$ is strong. Suppose $\widetilde{S}_t$ satisfies the following stochastic differential equation

$$d\widetilde{S}_t = r\widetilde{S}_tdt + \sigma(t)\widetilde{S}_tdW_{1t}, \tag{6}$$

where the volatility function $\sigma(t)$ is a deterministic function rather a random one. We can tell the difference between (3) and (6), the volatility functions (here we suppose $S_0 = \widetilde{S}_0$). So $G(\widetilde{S}, \widetilde{J})$ may be a good control variate for $G(S, J)$. The point is how to choose $\sigma(t)$ satisfying close relation with σ_t.

Our method is adapting the m-order moment condition, that is

$$\sigma^m(t) = E[\sigma_t^m]. \tag{7}$$

Then the random variable σ_t^m varies from the center $\sigma^m(t)$, which makes $\sigma(t)$ and σ_t have strong correlation, by the law of large numbers.

III. Control Variate Monte Carlo Method for Multi-asset Options Pricing under the Stochastic Volatility Model

The control variate Monte Carlo method is introduced simply in the last section. Here we will present an explicit method choosing control variate for multi-asset options with the stochastic volatility model to improve the efficiency of the Monte Carlo method. There are two advantages of the method we provide: we just need the moment conditions of the stochastic volatility for determining the control variate, and the numerical results show the high efficiency of the variance reduction. The control variate previously proposed in Fouque and Han [8] need the distribution function of the volatility process or the variance process, which is not easily obtained for most of the models. Thus, our proposed method is better than those methods used before. First we model the underlying asset processes(stock price, stock price index etc.), and present some lemmas which are not special for some explicit stochastic volatility models.

Jiang [14] introduced multi-asset options carefully, which volatilities are constant. Here we will consider the control variate method for pricing some multi-asset options with the stochastic volatility model, and give the numerical experiments for the Hull-White model especially. Of course, our method can also be used for pricing financial derivatives with the multi-factor stochastic volatility model, or even more complex models.

A. Exchange Options and Spread Options

Suppose the underlying assets(two stocks for simplicity) prices satisfy geometric Brownian Motions

$$dS_{it} = rS_{it}dt + \sigma_{it}S_{it}dW_{it}, \tag{8}$$

where $i = 1, 2$ (in the rest of the paper for convenience we ignore $i = 1, 2$ where i appears), r is the constant risk-free interest rate. $\sigma_{it} = \sigma_i\sqrt{Y_t}$ are stochastic volatilities of stock prices, σ_i are constant, and

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t. \tag{9}$$

The combination of (8) and (9) is the Hull-White stochastic volatility model. The method in this paper is also useful to the models represented by the stochastic differential equations without explicit relation. Let W_{it} and B_t be standard Browmian Motions, and $dW_{1t}dW_{2t} = \rho dt$, $dW_{1t}dB_t = \rho_1 dt$, $dW_{2t}dB_t = \rho_2 dt$.

Denote $V_e|_t$ as the exchange option value at the time t. The payoff function is

$$V_e|_{t=T} = (S_{2T} - S_{1T})^+. \tag{10}$$

The spread option payoff function is similar to that of the exchange option, as follows

$$V_e|_{t=T} = (S_{2T} - S_{1T} - K)^+, \tag{11}$$

where $K > 0$ is the strike price. Suppose σ_{it}, $i = 1, 2$, is square integrable, that is $E[\int_0^t \sigma_{is}^2 ds] < \infty$. We can derive the exchange option price by the risk-neutral price formula and (10)

$$V_e|_{t=0} = E[e^{-rT}V_e|_{t=T}] = e^{-rT}E[(S_{2T} - S_{1T})^+]. \tag{12}$$

Similarly, the price of the spread option is

$$\begin{aligned} V_e|_{t=0} &= E[e^{-rT}V_e|_{t=T}] \\ &= e^{-rT}E[(S_{2T} - S_{1T} - K)^+]. \end{aligned} \tag{13}$$

The exchange option is first studied by Margrabe [19], who gave a closed form value for the exchange option with the constant volatility model, and there is no analytical solution for the exchange option with the stochastic volatility model. However there is no closed form value even for the spread options with constant volatility. Then we price them with Monte Carlo method, and improve the convergence rate and the accurate through the control variate induced by (7).

Replacing σ_{it} in (8) with the deterministic functions $\sigma_i(t)$, we derive the auxiliary processes

$$dS_i(t) = rS_i(t)dt + \sigma_i(t)S_i(t)dW_i(t). \tag{14}$$

With the underlying assets prices as in (14), the price of the exchange option is

$$\begin{aligned} X_e|_{t=0} &= E[e^{-rT}X_e|_{t=T}] \\ &= e^{-rT}E[(S_2(T) - S_1(T))^+]. \end{aligned} \tag{15}$$

We will obtain a solution analytically for (15) in the following theorem.

Theorem 1 Suppose stochastic volatilities σ_{it} in (8) are replaced by deterministic square-integrable volatilities $\sigma_i(t)$, there is an analytic solution for the exchange option with underlying assets (14),

$$X_e|_{t=0} = S_2(0)N(d_2) - S_1(0)N(d_1),$$

where $N(\cdot)$ is the standard normal distribution function,

$$d_1 = \frac{\log(\frac{S_2(0)}{S_1(0)}) - \frac{1}{2}\sigma^2(T)}{\sigma(T)}, d_2 = \frac{\log(\frac{S_2(0)}{S_1(0)}) + \frac{1}{2}\sigma^2(T)}{\sigma(T)},$$

and $\sigma^2(T) = \int_0^T [\sigma_1^2(s) - 2\rho\sigma_1(s)\sigma_2(s) + \sigma_2^2(s)]ds$.

From Theorem 1, we can use X_e as the control variate for V_e. In order to derive better variance reduction efficiency we need the correlation between X_e and V_e stronger. The difference between them is replacing the stochastic volatilities σ_{it} by the time-varying functions $\sigma_i(t)$. Then the stronger the relation between $\sigma_i(t)$ and σ_{it} is, the stronger the relation between X_e and V_e is. Then we can choose $\sigma_i(t)$ as in the formula (7),

$$[\sigma_i(t)]^m = E[\sigma_{it}^m] = E[\sigma_i^m Y_t^{\frac{m}{2}}] = \sigma_i^m E[Y_t^{\frac{m}{2}}].$$

It is easy to know from (9) that $Y_t = Y_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$, and

$$\sigma_i(t) = \sigma_i\sqrt{Y_0}e^{\frac{1}{2}[\mu + \frac{1}{4}(m-2)\sigma^2]t}. \tag{16}$$

Then we derive deterministic volatilities whose m-th order moments equal to those of the stochastic volatilities respectively.

Numerical experiments will be exploited to show the variance reduction ratio of the control variate mentioned above. For simplicity, we delete the inferior characters e of

X_e and V_e. As done by Glasserman [10] and Ma and Xu [18], we use the standard deviation reduction ratio instead of the variance reduction ratio, which is defined by (2) as

$$R = \sqrt{\frac{1}{1-\rho_{XV}^2}}, \tag{17}$$

where ρ_{XV} is the correlation between X and V. Usually we cannot derive accurate ρ_{XV} because there is no accurate value $E[V]$. Then we calculate $\hat{\rho}_{XV}$ by samples to obtain the estimator $\hat{R}$ for R. Theoretically the larger $\hat{R}$ is, the larger variance reduction ratio is. We provide the algorithm steps in Algorithm 1.

Algorithm 1: 1) Divide $[0,\ T]$ into n even intervals with mesh size $\triangle t = T/n = t_{k+1} - t_k$, $0 = t_1 < t_2 < ... < t_n = T$.

2) After replacing σ_t by $\sigma_i(t)$ in (3), we have $S(t_{k+1})$ as

$$\begin{aligned} S_i^j(t_{k+1}) = \quad & S_i^j(t_k)\cdot\exp\left\{r\triangle t - \frac{1}{2}\int_{t_k}^{t_{k+1}}\sigma_i^2(s)ds\right. \\ & \left. + \int_{t_k}^{t_{k+1}}\sigma_i(s)dW_{is}\right\}. \end{aligned} \tag{18}$$

Because $\sigma_i(t)$ are non-random, we have $\int_{t_k}^{t_{k+1}}\sigma_i(s)dW_{1t} \sim N(0,\ \int_{t_k}^{t_{k+1}}\sigma_i^2(s)ds)$(Shreve [22], Example 4.7.3). Then we can simulate paths

$$\begin{aligned} S_i^j(t_{k+1}) = \quad & S_i^j(t_k)\exp\left\{r\triangle t - \frac{1}{2}\int_{t_k}^{t_{k+1}}\sigma_i^2(s)ds\right. \\ & \left. + \sqrt{\int_{t_k}^{t_{k+1}}\sigma_i^j(s)ds Z_k^{i,j}}\right\}, \end{aligned} \tag{19}$$

where $Z_k^{i,j}$ are the standard normal random variables, and $cov(Z_k^{1,j}, Z_k^{2,j}) = \rho$, j means the j-th simulation path, $S_i^j(t_0) = S_{i0}$.

3) From (15), we know that the control variate price in the j-th simulation path is

$$X_j = [S_2^j(T) - S_1^j(T)]^+. \tag{20}$$

4) When volatilities are σ_{it}, we derive $S_{i,t_{k+1}}$ from S_{i,t_k} by

$$\begin{aligned} S_{i,t_{k+1}} = \quad & S_{i,t_k}\exp\left\{r\triangle t - \frac{1}{2}\int_{t_k}^{t_{k+1}}\sigma_{is}^2 ds\right. \\ & \left. + \int_{t_k}^{t_{k+1}}\sigma_{is}dW_{is}\right\}. \end{aligned} \tag{21}$$

The approximation in the j-th simulation path is

$$\begin{aligned} S_{i,t_{k+1}}^j = \quad & S_{i,t_k}^j\exp\left\{[r\triangle t - \frac{1}{2}(\sigma_{it_k}^j)^2]\triangle t\right. \\ & \left. + \sigma_{it_k}^j\sqrt{\triangle t}Z_k^{i,j}\right\}, \end{aligned} \tag{22}$$

where $S_{it_0}^j = S_{i0}$, $Z_k^{i,j}$ are the same as in (2). $\sigma_{it_k}^j = \sigma_i\sqrt{Y_{t_k}^j}$, where $Y_{t_k}^j$ satisfies the following equation

$$Y_{t_{k+1}}^j = Y_{t_k}^j\exp[(\mu - \frac{1}{2}\sigma^2)\triangle t + \sigma\sqrt{\triangle t}U_k^j], \tag{23}$$

where U_k^j is the standard normal distribution variable and $cov(U_k^j, Z_k^{i,j}) = \rho_i$.

5) From (10), we have the exchange option price in the j-th simulation path

$$V_j = (S_{2T}^j - S_{1T}^j)^+. \tag{24}$$

6) Suppose there are p simulation paths,

$$\overline{X}_p = \frac{1}{p}\sum_{j=1}^{p}X_j,$$

$\overline{V}_p = \frac{1}{p}\sum_{j=1}^{p}V_j$, we have the estimator

$$\hat{\rho}_{XV} = \frac{\sum_{j=1}^{p}(X_j - \overline{X}_p)(V_j - \overline{V}_p)}{\sqrt{\sum_{j=1}^{p}(X_j - \overline{X}_p)^2}\sqrt{\sum_{j=1}^{p}(V_j - \overline{V}_p)^2}}.$$

Then from (17) we derive the estimator $\hat{R}$ for R

$$\hat{R} = \sqrt{\frac{1}{1-\hat{\rho}_{XV}^2}}.$$

Experiment 1:

In this experiment we use the exchange option price X_e under the time-varying volatility model as the control variate for the exchange option price V_e under the stochastic volatility model. The numerical results are shown in Table 1, where $\hat{R}$ represent the standard derivation reduction ratio with different m-th order moment, MC is the option price estimator by the ordinary Monte Carlo method, and MC+CV is the option price estimator by the control variate Monte Carlo method referring to Theorem 1. Following Carr and Madan [5], and Ma and Xu [18], we set the parameters $p = 5000$, $n = 100$, $r = 0.05$, $Y_0 = 0.04$, $\rho = 0.5$, $\rho_1 = 0.25$, $\rho_2 = -0.5$. Let m vary. When $m = 2 - \frac{4\mu}{\sigma^2}$, $\sigma_i(t) = \sigma_i\sqrt{Y_0}$ are constant.

In the first row of Table I m means that the time-varying volatility is obtained by m-th order moment of the stochastic volatility, as in (16). The data in Table I show us that:

1) The standard derivation reduction ratios vary as m changes. Except in the last column, the variance reduction performance keep in a good level in the rest columns. The reason is that the weak correlation between the constant volatility and the stochastic volatility, and the time-varying volatility varies as the stochastic volatility which their strong correlation is promised by (17) in the rest of the columns. The largest standard derivation reduction ratio is about 300, which means the variance reduction ratio is about 90000, and the price error reduce about 300 times as the control variate method used, or equally, to achieve the same accurate, we can minus about 900000 times paths by using the control variate method. So the numerical results suggest that our control variate method is effective in options pricing by the Monte Carlo method.

2) Comparing the data in the last two rows, we can find that option prices obtained by the control variate Monte Carlo method are close to $X_e|_{t=0}$ given by Theorem 1, so $X_e|_{t=0}$ can be used as an analytic approximation solution to the option price. The standard derivation reduction ratios in the last column is smaller than that of the rest columns, then we don't use these prices as approximation prices to $V_e|_{t=0}$.

Experiment 2:

In this experiment the exchange option price X_e is used as the control variate for the spread option under the stochastic volatility model. The algorithm for calculating $\hat{R}$ (24) should

TABLE I
STANDARD DEVIATION REDUCTION RATIOS FOR EXCHANGE OPTIONS PRICING BY THE CONTROL VARIATE METHOD

	$m=-50$	$m=0$	$m=1$	$m=2$	$m=100$	$m=2-4\mu/\sigma^2$
$\hat{R}$	303.85	299.74	287.31	296.18	278.76	110.16
MC	4.57	4.62	4.67	4.55	4.49	4.49
MC+CV	4.20	4.20	4.20	4.20	4.20	4.21
$X_e\|_{t=0}$	4.20	4.20	4.20	4.20	4.20	4.19

be changed with $V_j = (S^j_{2T} - S^j_{1T} - K)^+$. We choose different K, m, and keep other parameters unchanged. The numerical results are shown in Table II.

The data in Table II are the standard deviation reduction ratios of the control variate Monte Carlo method to that of the ordinary Monte Carlo method, which squared are the variance reduction ratios. They suggest that the efficiency of using X_e as the control variate for V_S is worse than that using X_e as the control variate. The reason is that the difference between X_e and V_e is just only the volatility, however, the difference between X_e and V_S is the different payoff functions (10) and (11), besides of the volatility. And when K becomes larger, the option change from in-the-money to out-of-money, and the distance of the structures in (10) and (11) becomes larger. So we need new control variates. The reason why the standard deviation reduction ratio is small is the big difference between their payoff functions, especially for the large strike price K. Then we propose an instrumental option V_{ai}, which has the payoff function at the time T

$$V_{ai}|_{t=T} = (S_{2T} - S_{1T} - \frac{e^{-rT}K}{S_1(0)} S_{1T})^+. \tag{25}$$

The risk-neutral pricing formula suggest that

$$E[\frac{e^{-rT}K}{S_1(0)} S_{1T}] = K,$$

which can be treated as $\frac{e^{-rT}K}{S_1(0)} S_{1T} \approx K$. Then the structures of (25) and (11) are closer than that of (10) and (11). The option price for the instrumental option is

$$\begin{aligned} V_{ai}|_{t=0} &= E[e^{-rT}(V_{ai}|_{t=T})] \\ &= e^{-rT}E[(S_{2T} - S_{1T} - \frac{e^{-rT}K}{S_1(0)} S_{1T})^+]. \end{aligned} \tag{26}$$

The instrumental option price (26) has no analytical solution because of the stochastic volatility. If replacing the stochastic volatility with the time-varying one, we can have the instrumental option price with the condition (14) as follows

$$\begin{aligned} X_{ai}|_{t=0} &= E[e^{-rT}(X_{ai}|_{t=T})] \\ &= e^{-rT}E[(S_2(T) - \widetilde{K} S_1(T))^+], \end{aligned} \tag{27}$$

where $\widetilde{K} = 1 + \frac{e^{-rT}K}{S_1(0)}$. We will give the solution in the following theorem.

Theorem 2 Suppose stochastic volatilities σ_{it} in (8) are replaced by deterministic square-integrable volatilities $\sigma_i(t)$, there is an analytic solution for the instrumental option price (27) with underlying assets (14),

$$X_{ai}|_{t=0} = S_2(0)N(d_2) - S_1(0)N(d_1), \tag{28}$$

where $N(\cdot)$ is the standard normal distribution function,

$$\begin{aligned} d_1 &= \frac{a(T) - \log(\widetilde{K})}{\sigma(T)}, d_2 = d_1 + \sigma(T), \\ \sigma^2(T) &= \int_0^T [\sigma_1^2(s) - 2\rho\sigma_1(s)\sigma_2(s) + \sigma_2^2(s)]ds, \\ a(T) &= \log(\frac{S_1(0)}{S_2(0)}) - \frac{1}{2}\sigma^2(T). \end{aligned}$$

Theorem 2 suggests that we can choose X_{ai} as the control variate for V_S, and derive $\sigma_i(t)$ by (16) to achieve larger variance reduction ratio.

Experiment 3:

In this experiment we use the instrumental option X_{ai} with the time-varying volatility model as the control variate for the spread option V_S with the stochastic volatility model. The algorithm is similar to Algorithm 1, where we need replace (20) with $X_j = (S_2^j(T) - \widetilde{K} S_1^j(T))^+$, and (24) with $V_j = (S^j_{2T} - S^j_{1T} - K)^+$. Let K and m vary and keep the other parameters unchanged. The numerical results are shown in Table III.

Comparing the data in Table II and Table III, we know that the variance reduction ratios of the new control variate X_{ai} are much better than that of X_e, because X_{ai} is much closer to V_S. Even for out-of-the-money, the variance reduction ratio of the new control variate is about 6400 times. Totally, the variance reduction efficiency become low for the out-of-the-money option, which is caused by the simulation paths with zeros payoff. We provide two methods to solve this problem:

1) Enlarge the probability of non-zeros payoff simulation paths by important sampling, which can be found in Glasserman [10].

2) Change out-of-the-money option with in-the-money option by the call-put parity. We know that

$$\begin{aligned} [(S_{2T} - S_{1T}) - K]^+ &= [K - (S_{2T} - S_{1T})]^+ \\ &\quad + [(S_{2T} - S_{1T}) - K]. \end{aligned}$$

Discounting both sides of above equations and taking the expectations, we derive the following equation by the risk-neutral pricing formula

$$\begin{aligned} V_S|_{t=0} &= E[e^{-rT}[(S_{2T} - S_{1T}) - K]^+] \\ &= E[e^{-rT}[K - (S_{2T} - S_{1T})]^+] \\ &\quad + [(S_{2T} - S_{1T}) - e^{-rT}K]. \end{aligned}$$

Then calculating the spread call option price has been changed into pricing the put option

$$V_P|_{t=0} = E[e^{-rT}[K - (S_{2T} - S_{1T})]^+].$$

If V_S is out of the money, V_p must be in the money. Then we change out-of-the-money options pricing into the in-the-money options pricing. Just as the way of constructing the

TABLE II
STANDARD DEVIATION REDUCTION RATIOS FOR SPREAD OPTIONS WITH EXCHANGE OPTIONS AS CONTROL VARIATES

	$m=-50$	$m=0$	$m=1$	$m=2$	$m=100$	$m=2-4\mu/\sigma^2$
$K=0.1$	129.25	131.59	129.11	132.65	132.49	72.45
$K=1$	12.98	12.90	12.91	12.98	12.94	12.37
$K=4$	3.09	3.09	3.10	3.09	3.11	3.13
$K=4.5$	2.75	2.81	2.76	2.76	2.77	2.76

TABLE III
STANDARD DEVIATION REDUCTION RATIOS FOR EXCHANGE OPTIONS WITH EXCHANGE OPTIONS AS CONTROL VARIATES

	$m=-50$	$m=0$	$m=1$	$m=2$	$m=100$	$m=2-4\mu/\sigma^2$
$K=0.1$	298.99	296.07	291.62	302.64	283.91	110.34
$K=1$	241.35	243.40	235.38	235.58	240.29	111.62
$K=4$	90.69	86.61	91.27	89.72	89.25	69.64
$K=4.5$	77.44	77.63	78.29	76.96	81.70	62.70

control variate for V_S, we choose the control variate for V_p, which is in the money, and reduce the variance a lot.

In practice, the strike price of the option is chosen around the initial price of the underlying assets. Then we can provide good control variate for the accurate price. The way of the standard deviation reduction ratio changed as m varies is similar to that in Table I, so as the reasons.

B. Chooser Options and Max-call Options

Suppose there are two underlying assets for chooser options and max-call options. Of course, the method we provide is useful to the options with more than two underlying assets. Let the underlying assets and volatilities satisfy (8) and (9). The payoff function for the chooser option is

$$V_{bo}|_{t=T} = \max\{S_{1T}, S_{2T}\}. \tag{29}$$

The max-call option's payoff function is

$$V_m|_{t=T} = (\max\{S_{1T}, S_{2T}\} - K)^+, \tag{30}$$

where the strike price $K > 0$. Suppose σ_{it} is square integrable, that is $E[\int_0^t \sigma_{is}^2 ds] < \infty$. By the risk-neutral pricing formula and (28), we can obtain the chooser option price

$$V_{bo}|_{t=0} = E[e^{-rT} V_{bo}|_{t=T}] = e^{-rT} E[\max\{S_{1T}, S_{2T}\}]. \tag{31}$$

Similarly, the price of the max-call option is

$$\begin{aligned} V_m|_{t=0} &= E[e^{-rT} V_m|_{t=T}] \\ &= e^{-rT} E[(\max\{S_{1T}, S_{2T}\} - K)^+]. \end{aligned} \tag{32}$$

Johnson [15] priced the max-call option with any number of the underlying assets under the constant volatility analytically. However, there is no closed form for that options with the stochastic volatility model (30) and (31). We use the time-varying volatility function $\sigma_i(t)$ instead of σ_{it} in (8), then we can derive the chooser option's value under the auxiliary process (14)

$$\begin{aligned} V_{bo}|_{t=0} &= E[e^{-rT} V_{bo}|_{t=T}] \\ &= e^{-rT} E[\max\{S_1(T), S_2(T)\}], \end{aligned} \tag{33}$$

and the price of the max-call option is

$$\begin{aligned} V_m|_{t=0} &= E[e^{-rT} V_m|_{t=T}] \\ &= e^{-rT} E[(\max\{S_1(T), S_2(T)\} - K)^+]. \end{aligned} \tag{34}$$

We present the solution to (32) analytically when $\sigma_i(t)$ satisfy some conditions.

Theorem 3 Suppose stochastic volatilities σ_{it} in (8) are replaced by square-integrable volatilities $\sigma_i(t)$, there is an analytical solution for the chooser option with underlying assets (14),

$$X_e|_{t=0} = S_2(0)N(d_2) + S_1 N(-d_1),$$

where $N(\cdot)$ is the standard normal distribution function,

$$\begin{aligned} d_1 &= \frac{\log(\frac{S_2(0)}{S_1(0)}) - \frac{1}{2}\sigma^2(T)}{\sigma(T)}, \\ d_2 &= \frac{\log(\frac{S_2(0)}{S_1(0)}) + \frac{1}{2}\sigma^2(T)}{\sigma(T)}, \\ \sigma^2(T) &= \int_0^T [\sigma_1^2(s) - 2\rho\sigma_1(s)\sigma_2(s) + \sigma_2^2(s)] ds. \end{aligned}$$

Experiment 4:

In this experiment we use the chooser option price X_{bo} with the time-varying volatility as the control variate of that option V_{bo} with the stochastic volatility model. The algorithm is similar to Algorithm 1 but replacing (20) with $X_j = \max\{S_2^j(T), S_1^j(T)\}$, and (24) with $V_j = \max\{S_{2T}^j, S_{2T}^j\}$, and the time-varying volatility $\sigma_i(t)$ chosen by (16). The other parameters are the same as in Algorithm 1. The numerical results are shown in Table IV.

The data in Table IV show that the efficiency of the standard deviation reduction when m varies, which is the same as that in Table I, and so are the reasons. The best standard deviation reduction ratio is about 280 times, which suggests that the control variate we provide improve the Monte Carlo method effectively. The numbers in the last two rows show that the option price estimator given by the control variate Monte Carlo method is close to $X_{bo}|_{t=0}$ by Theorem 2. Then we can use $X_{bo}|_{t=0}$ as an analytic approximation solution to the option price.

For the max-call option, we deduce the closed form solution to the option price (33) with $\sigma_i(t)$ in the following theorem.

Theorem 4 Suppose stochastic volatilities σ_{it} in (8) are replaced by square-integrable volatilities $\sigma_i(t)$, there is an analytical solution for the max-call option price (33) with

TABLE IV
STANDARD DEVIATION REDUCTION RATIOS FOR CHOOSER OPTIONS BY THE CONTROL VARIATE METHOD

	$m=-50$	$m=0$	$m=1$	$m=2$	$m=100$	$m=2-4\mu/\sigma^2$
$\hat{R}$	278.22	275.28	278.44	268.80	254.07	94.23
MC	100.65	100.60	100.57	100.58	100.62	100.60
MC+CV	100.20	100.20	100.20	100.20	100.20	100.21
$X_e\|_{t=0}$	100.20	100.20	100.20	100.20	100.20	100.19

underlying assets (14),

$$\begin{aligned} X_m|_{t=0} \quad = \quad & S_1(0)N_2[d_1(S_1(0),K,b_1^2(T));\\ & d_1^{'}(S_1(0),S_2(0),b_{12}^2(T));\rho_{112}(T)]\\ & +S_2(0)N_2[d_1(S_2(0),K,b_2^2(T));\\ & d_1^{'}(S_2(0),S_1(0),b_{12}^2(T));\rho_{212}(T))]\\ & -e^{-rT}K\{1-N_2[-d_2(S_1(0),K,b_1^2(T));\\ & -d_2(S_2(0),K,b_2^2(T));\rho]\}, \end{aligned}$$

where $N(\cdot,\cdot,\cdot)$ is the bivariate standard normal distribution function.

Experiment 5:

In this experiment we use the max-call option price X_m with the time-varying volatility as the control variate for that option V_m with the stochastic volatility model. The algorithm for calculating $\hat{R}$ is similar to Algorithm 1, which we just need replace (20) with $X_j=(\max\{S_2^j(T),S_1^j(T)\}-K)^+$, and (24) with $V_j=(\max\{S_{2T}^j,S_{1T}^j\}-K)^+$. We change b and m, and keep other parameters unchanged. The numerical results are shown in Table V.

The standard deviation reduction efficiency for out-of-the-money option is worse than that for in-the-money option, which the reasons are the same as that in Table III, so are the methods for better efficiency. The efficiency of the standard deviation reduction when m varies is the same as that in the experiments above.

C. Basket Options

The payoff of the basket option at the expiration time is

$$V_{Ab}=(\sum_{i=1}^{N}\alpha_i S_{iT}-K)^+, \tag{35}$$

where $\sum_i^N\alpha_1=1$, $\alpha_i\geq 0$. The basket option based on the arithmetic average does't have a closed form price even with the constant volatility. Then we construct the basket option based on the geometric average as an instrumental option, whose payoff function at the time T is

$$V_{Gb}=(\prod_{i=1}^{N}S_{iT}^{\alpha_i}-K)^+. \tag{36}$$

Suppose these underlying stocks prices satisfy the following stochastic differential equations

$$dS_i(t)=rS_i(t)dt+\sigma_i^{'}(t)S_i(t)dW_t,$$

where $W_t=(W_{1t},W_{2t},...,W_{Mt})^{'}$ are M-dimensional Brownian Motions, $\sigma_i(t)=(\sigma_{i1}(t),\sigma_{i2}(t),...,\sigma_{iM}(t))'$ are volatility vectors. The basket option price (36) based on the geometric average with the volatility $\sigma_{ij}(t)$ satisfying some conditions has a closed form value.

Theorem 5 Suppose stochastic volatilities σ_{it} in (8) are replaced by square-integrable volatilities $\sigma_i(t)$, there is an analytical solution for the basket option based on the geometric average,

$$X_{Gb}|_{t=0}=e^{\frac{1}{2}\sigma^2(t)-rT+a(T)}N(d_2)-Ke^{-rT}N(d_1),$$

where $N(\cdot)$ is the standard normal distribution function,

$$\begin{aligned} a(T)&=rT+\sum_{i=1}^{N}\alpha_i S_i(0)-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{M}\int_0^T\alpha_i\sigma_{ij}(s)ds,\\ \sigma^2(T)&=\sum_{j=1}^{M}\int_0^T[\sum_{i=1}^{N}\alpha_i\sigma_{ij}(s)]^2ds,\\ d_1&=\frac{a(T)-\log K}{\sigma(T)},d_2=d_1+\sigma(T). \end{aligned}$$

Theorem 5 suggests that the basket option based on the geometric average with the time-varying volatility can be used as the control variate for that option based on the arithmetic average with the stochastic volatility model. For simplicity, we just consider that the underlying assets are two stocks, which prices are driven by two-dimensional Brownian Motions, that is stocks prices satisfy (8) and (9), and time-varying volatilities are determined by (16).

Experiment 6:

In this experiment the basket option price X_{Gb} based on the geometric average with the time-varying volatility is used as the control variate for that option price V_{Ab} based on the arithmetic average with the stochastic volatility. The algorithm used to calculate $\hat{R}$ is similar to Algorithm 1, which need replace (20) with $X_j=(S_1^j(T)^{\alpha_1}S_2^j(T)^{\alpha_2}-K)^+$, and (24) with $V_j=(\alpha_1S_{1T}^j+\alpha_2S_{2T}^j-K)^+$. Let K, m vary, $\alpha_2=0.8$, and other parameters be used as before. The numerical results are shown in Table VI.

The standard deviation reduction ratios change when K and m vary, where the way they change is the same as that in Table III and V, so are the reasons and the methods used to improve the efficiency.

D. Quanto Options

Quanto option is a contract written when someone invest money in foreign securities. Usually, its risk depends on the volatility of the securities prices and the change of the foreign currency rate. For example, a China investor buys a European call option which strike price is K(dollar). There are two underlying assets here: American stock price S_{2t}(dollar) and the dollar rate(the units of RMB per unit of dollar) S_{1t}. Then the payoff function of the option at the expiration time is

$$V_q|_{t=T}=S_{1T}(S_{2T}-K)^+ \tag{37}$$

TABLE V
STANDARD DEVIATION REDUCTION RATIOS FOR MAX-CALL OPTIONS BY THE CONTROL VARIATE METHOD

	$m=-50$	$m=0$	$m=1$	$m=2$	$m=100$	$m=2-4\mu/\sigma^2$
$K=90$	260.53	272.09	281.39	272.91	269.40	96.68
$K=95$	274.18	274.39	277.13	275.18	261.68	91.38
$K=100$	275.01	277.93	284.52	270.26	270.98	90.86
$K=105$	243.25	248.58	236.86	231.30	235.63	99.05
$K=110$	138.68	147.68	141.53	129.78	139.87	48.48

TABLE VI
STANDARD DEVIATION REDUCTION RATIOS FOR BASKET OPTIONS BASED ON ARITHMETIC AVERAGE WITH BASKET OPTIONS BASED ON GEOMETRIC AVERAGE AS CONTROL VARIATES

	$m=-50$	$m=0$	$m=1$	$m=2$	$m=100$	$m=2-4\mu/\sigma^2$
$K=96$	90.95	90.00	89.14	84.76	86.83	72.80
$K=96^{0.2}\times 100^{0.8}$	93.10	92.48	92.94	89.88	91.05	68.06
$K=100$	89.33	95.42	94.54	89.80	90.60	67.67
$K=105$	55.75	55.13	56.46	55.16	55.49	49.41

Suppose that S_{it} satisfy (8), and their volatilities satisfy (9). Then by the risk-neutral pricing formula, we can derive the option price at the time 0

$$V_q|_{t=0} = E[e^{-rT}V_q|_{t=T}] = e^{-rT}E[S_{1T}(S_{2T}-K)^+]. \quad (38)$$

There is no analytical solution to (38) under the stochastic volatility model. However we can derive a closed form price for that option when the volatility σ_{it} replaced by $\sigma_i(t)$, which is promised by Theorem 6.

Theorem 6 Suppose stochastic volatilities σ_{it} in (8) are replaced by square-integrable volatilities $\sigma_i(t)$, there is an analytical solution for the quanto option price (38) with auxiliary processes (14),

$$X_q|_{t=0} = S_1(0)e^{\frac{1}{2}\sigma^2(t)+a(T)}N(d_2) - S_1(0)KN(d_1),$$

where $N(\cdot)$ is the standard normal distribution function,

$$\begin{aligned} a(T) &= \log S_2(0) + rT - \frac{1}{2}\int_0^T \sigma_2^2(s)ds \\ &\quad +\rho\int_0^T \sigma_1(s)\sigma_2(s)ds, \\ \sigma^2(T) &= \int_0^T \sigma_2(s)ds, \\ d_1 &= \frac{a(T)-\log K}{\sigma(t)}, d_2 = d_1+\sigma(t). \end{aligned}$$

Theorem 6 suggests that we can use the quanto option price X_q with the time-varying volatility as the control variate for that option price V_q with the stochastic volatility model, where $\sigma_i(t)$ are determined by (16).

Experiment 7:

In this experiment the quanto option price X_q with the time-varying volatility is used as the control variate for that option price V_q with the stochastic volatility model. The algorithm used to derive $\hat{R}$ is similar to Algorithm 1, where we need replace (20) with $X_j = S_1^j(T)(S_2^j(T)-K)^+$, and (24) with $V_j = S_{1T}^j(S_{2T}^j - K)^+$. Let K, m vary, $S_1(0) = 6.5$, and other parameters unchange. The numerical results are shown in Table VII.

The standard deviation reduction ratios change when K and m vary, where the way is the same as that in Table III, V and VI, so are the reasons. We don't repeat them. Except the last column, the standard deviation reduction ratios all over 250, that is about 60000 times variance reduction ratios, which suggests that the control variate we use improve the efficiency of the Monte Carlo method.

IV. CONCLUSION

In this paper, we have investigated the control variate Monte Carlo method for pricing several multi-asset options under the stochastic volatility model. First we have priced these multi-asset options analytically with the time-varying non-random function volatility, which is utilized to derive a class of control variates for pricing corresponding multi-asset options under the stochastic volatility model to improve the pricing efficiency. The time-varying functions have been obtained by the m-th order moment of the stochastic volatility model. Numerical results shown in Tables I, IV and VII have suggested that our control variate performs good for exchange options, chooser options and quanto options. As for spread options, max-call options and basket options, the data in Tables III, V and VI have shown that we achieved significant improvement on the efficiency of the variance reduction when the option is in-the-money, but our method performed not good when the option is out-of-the-money, which can be solved by the call-put parity. With respect to exchange options, chooser options, basket options and quanto options, when it is required to calculate the price more quickly with less accurate, Theorems 1, 2, 3 and 4 can be exploited for deriving the prices as an analytical approximation prices to the options with the stochastic volatility model. The numerical experiments have shown that they are very close to the estimator values obtained by the Monte Carlo method.

The idea we proposed can be extended to other stochastic volatility models, such as the volatility satisfying Ornstein-Uhlenbeck process or the square root diffusion process, by replacing (13) with the expectation of the new volatility model. However, the expectation can be obtained without the closed form solution or the transform density, which is convenient for choosing the time-varying deterministic volatility. The idea can also be applied to price other financial derivatives under the stochastic volatility model, such as Asian options, barrier options, variance swaps, etc.

TABLE VII
STANDARD DEVIATION REDUCTION RATIOS FOR QUANTO OPTIONS BY THE CONTROL VARIATE METHOD

	$m = -50$	$m = 0$	$m = 1$	$m = 2$	$m = 100$	$m = 2 - 4\mu/\sigma^2$
$K = 95$	277.19	285.92	276.10	289.99	290.55	122.71
$K = 100$	318.26	301.50	312.54	322.29	311.27	113.63
$K = 105$	248.99	250.63	256.62	259.43	253.38	110.23

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