Continuous Time Mean-Variance Portfolio Selection Problem with Stochastic Salary and Strategic Consumption Planning for a Defined Contribution Pension Scheme

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Abstract—**This paper examines a continuous-time mean-variance portfolio selection problem with stochastic salary and strategic consumption planning for a constant relative risk averse (CRRA) pension plan member (PPM) in the accumulation phase of a defined contribution (DC) pension plan. It was assumed that the flow of contributions made by the PPM are invested into a market that is characterized by a cash account, an index bond and a stock. Due to the increasing risk of inflation rate and diminishing value of pension benefits, the need for managing such risk has becomes imperative. In this paper, index bond is traded and used to protect the investment against inflation risks. The aim of this paper are to determine the optimal variational Merton portfolios, optimal variational consumption plan for a life-cycle of a PPM and to maximize the expected final value of wealth and simultaneously minimize its variance and consumption risk. Efficient frontier for the three classes of assets that will enable PPMs to decide their own value of wealth and risk in their investment profile at retirement was obtained. The optimal consumption overtime and final consumption of the PPM are established. The variational portfolio processes for the three classes of assets were established. Some numerical results are also consider in this paper.**

Index Terms—**mean-variance, optimal portfolio, stochastic salary, defined contribution, strategic consumption planning, efficient frontier.**

AMS Subject Classifications. 91B28, 91B30, 91B70, 93E20.

I. INTRODUCTION

This paper consider a continuous-time mean variance portfolio selection problem with stochastic salary and strategic life consumption planning for a defined contribution pension plan. The optimal portfolios, expected value of wealth of a PPM, optimal consumption plan of a life-cycle of a PPM and efficient frontier of the three classes of assets were established. The contributions of the PPM are invested into a market that is composed of cash account, an index bond and a stock.

In a related literature, Jensen and Sϕrensen [17] studied the effect of a minimum interest rate guarantee constraint. It was studied through the wealth equivalent in case of un-constraints. They shown numerically that the guarantees may induce a significant utility loss for relative risk tolerant investors. Deelstra *et al* ([12], [13] and [14]) studied optimal design of the minimum guarantee in a defined contribution pension scheme. They studied the investment in the financial market by ensuring that the pension fund optimizes its retribution as a part of the surplus. Brawne *et al* [5] modeled and analyzed the *ex ante* liquidity premium demanded by the holder of an "illiquid annuity". The annuity was an insurance product that is similar to a pension scheme that involve both accumulation and "decumulation" phase. They computed the yield required to offset the utility welfare loss, which was induced by the inability to re-balance and maintain an optimal portfolio when holding an annuity. Cairns *et al* [7] developed a pension plan accumulation programmed designed to delivered a pension fund at retirement which is closely related to salary received by PPM prior to retirement. Cairns *et al* [7] considered the optimal dynamic asset allocation policy for a defined contribution (DC) pension scheme by taking into consideration the stochastic features of the PPM's lifetime salary progression as well as the stochastic properties of the assets held in his accumulating pension scheme. They emphasized that salary risk was not fully hedgeable by using existing financial assets. They further emphasized that wage-indexed bonds could be suitable to hedged productivity and inflation shocks, but such assets are not widely traded. They referred to the optimal dynamic asset allocation strategy stochastic life-styling. They compared it against various static and deterministic lifestyle strategies in order to obtained the costs of adopting suboptimal strategies.

Cairns *et al* [8] considered the solution technique of Cairns *et al* [7] and made used of the present value of future contribution premiums into the scheme, see also (Boulier *et al* [4], Deelstra *et al* [11], Korn and Krekel

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[19] and Blake *et al* [3]. Deterministic life-styling designed to protect the pension fund from a catastrophic fall in the stock market just prior to retirement can be found in Cairns *et al* [7], Blake *et al* [3] and Cajueiro and Yoneyama [9]. Haberman and Vigna [15] and Cairns *et al* ([6], [7]) analyzed the occupational DC pension fund, where the contribution rate was a fixed percentage of the salary. For a constant flow of contributions, see Hϕjgaard and Vigna [16]. For stochastic cash inflows, see Maurer *et al* [21], Battocchio [2], Zhang *et al* [29], Zhang *et al* [28], Korn and Kruse [20]. Maurer *et al* [21] modeled inflation index that involves inflation uncertainty. They considered multi-decade investment horizons. Battocchio and Menoncin [1] considered a stochastic dynamic programming approach to model a DC pension fund in a complete financial market with stochastic investment opportunities and two background risks: salary risk and inflation risk. They gave a closed form solution to the asset allocation problem and analyzed the behavior of the optimal portfolio with respect to salary and inflation. Zhang *et al* [28] considered the optimal management and inflation protection strategy for defined contribution pension plans using Martingale approach. They derived an analytical expression for the optimal strategy and expresses it in terms of observable market variables. Dai *et al* [10] studied a continuous-time Markowitz's mean-variance portfolio selection problem involving propositional transaction costs. They established a critical length of time which depends on the stock excess return as well as the transaction fees but independent of the investment target and stock volatility. Nwozo and Nkeki [25] considered optimal portfolio and strategic consumption planning ina life-cycle of a PPM in a DC pension scheme. They found that investment in the risky assets should be gradually transferred to riskless asset prior to retirement date. Nkeki [24] and Nkeki and Nwozo [26] studied the variational form of classical portfolio strategy and expected wealth for a pension plan member. They assumed that the growth rate of salary is a linear function of time and that the cash inflow is stochastic. Nkeki and Nwozo [27] studied the optimal portfolio strategies with stochastic cash flows and expected optimal terminal wealth under inflation protection for a certain investment company (IC) who trades in a complete diffusion models, receives a stochastic cash inflows and pays a stochastic outflows to its holder. They found that as the market evolved parts of the index bond and stock portfolio values should be transferred to cash account. This, to a great extent will protect the IC from catastrophic fall in the stock market. They also found that the portfolio processes involved inter-temporal hedging terms that offset any shock to both the stochastic cash inflows and cash outflows. Josa-Fombellida and Rincon-Zapatero [18] considered simultaneous minimization of risks problem, and maximization of the terminal value of expected funds assets in a defined benefit pension plan. They considered risks associated with the solvency, the variance of the final funds level, and the contribution risk, in the form of a running cost that was related to deviations from the evolution of the stochastic normal cost. They found the efficient frontier. They shown that the optimal portfolio depends linearly on the supplementary cost of the fund, plus an additional term due to the stochastic evolution of the benefits.

The aims of this paper are to study optimal variational Merton portfolio, (for Merton portfolio, see Merton ([22], [23])) variational consumption plan for a life-cycle of a PPM and market efficiency test and efficient frontier of the three classes of assets for a DC pension scheme. This paper also aim at maximizing the expected final value of wealth and simultaneously minimize the variance of expected final value of wealth of a PPM and consumption risk.

The remainder of this paper is organized as follows. In section II, we presents the financial market models. The expected value of a PPM's discounted future contribution is presented in section III. The value of a PPM's wealth process is presented in section IV. Section V presents the optimization problem. The solution of the resulting Hamilton-Jacobi-Bellman equation is presented in section VI. In section VII, we present optimal portfolio and consumption planning strategies of the PPM and some numerical examples and discussions. In section VIII, we present the efficient frontier of the PPM's value of wealth. Finally, section XI concludes the paper.

II. THE FINANCIAL MODELS

In this section, we describe the financial markets where by the flow of wealth operates. In this paper, the sign $'$ denotes transpose. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\mathbf{F}(\mathcal{F}) = \{\mathcal{F}_t : t \in [0, T]\}$, where $\mathcal{F}_t = \sigma(W(s) : 0 \leq s \leq t)$, where the process $W(t) = (W^I(t), W^S(t))'$, $0 \leq t \leq T$ is a 2-dimensional Brownian motion, defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}(\mathcal{F}), \mathbf{P})$, where $\mathbf{P}$ is the real world probability measure and $\sigma \in \mathbb{R}^2$ and $\sigma_Z \in \mathbb{R}^2$ are the volatility vectors of stock and volatility of the index bond with respect to changes in $W^I(t)$ and $W^S(t)$, respectively, referred to as the coefficients of the market and are progressively measurable with respect to the filtration $\mathcal{F}$. The fund manager manages the fund in the planning interval $[0, T]$ by means of a portfolio that is composed by two risky assets; index bond Z and stock S, which are correlated geometric Brownian motions, generated by $W(t)$ and a cash account B, as proposed by Merton (1971). The evolution of these assets are given by the equations:

$$dB(t) = r(t)B(t)dt, B(0) = 1. \tag{1}$$

$$dS(t) = S(t)(\mu(t)dt + \sigma(t)dW(t)), S(0) = s_0 \in \mathbb{R}_+, \tag{2}$$

$$\begin{aligned} dZ(t, Q(t)) = Z(t, Q(t))((r(t) + \sigma_I(t)\theta_I(t))dt \\ +\sigma_Z(t)dW(t)), Z(0, Q(0)) = z_0 \in \mathbb{R}_+, \end{aligned} \tag{3}$$

where $\sigma_Z(t) = (\sigma_I(t), 0)$, $S(t)$ is stock price process at time t, $Z(t, Q(t))$ is the price of index bond, where $Q(t)$

inflation index at time t, $\mu(t) \in \mathbb{R}_+$ is the appreciation rate for stock, $\sigma(t) = (\xi\sigma_S(t), \sqrt{1-\xi^2}\sigma_S(t))$, $r(t) \in \mathbb{R}_+$ is the nominal interest rate, $\theta_I(t) \in \mathbb{R}$ is the price of inflation risk, $B(t)$ is the price process of the cash account at time t, $Q(t)$ is the inflation index at time t and has the dynamics: $dQ(t) = \bar{q}(t)Q(t)dt + \sigma_I(t)Q(t)dW^I(t)$, where $\bar{q}(t)$ is the expected rate of inflation, which is the difference between nominal interest rate, $r(t)$ and real interest rate $\tilde{r}(t) \in \mathbb{R}_+$ (i.e. $\bar{q}(t) = r(t) - \tilde{r}(t) + \sigma_I(t)\theta_I(t)$). It is assumed in this paper that $\mu(t) > r(t)$, so that the fund manager has incentives to invest with risk. We suppose that there exists correlation $\xi \in (-1, 1)$ between $W^I(t)$ and $W^S(t)$.

The proportion of fund invested in time t in index bond is denoted by $\Delta^I(t)$ and stock is denoted by $\Delta^S(t)$. The remainder, $\Delta_0(t) = 1 - \Delta^I(t) - \Delta^S(t)$ is invested in cash account at time t. We assume in this paper, that borrowing and short selling is allowed. Suppose $\{\Delta(t) : t \geq 0\}$ with $\Delta(t) = (\Delta^I(t), \Delta^S(t))$, is a control process adapted to filtration $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{F}_t-$measurable, Markovian and stationary, satisfying

$$E\int_0^T \Delta(t)\Delta(t)'dt < \infty, \tag{4}$$

where E is the expectation operator. The intermediate consumption process is tolerable at a nonnegative rate $C(t) \in \mathbb{R}_+$ at time $t \leq T$. Then, $C(t)$ is also adapted process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, satisfying

$$E\int_0^T C^2(t)dt < \infty. \tag{5}$$

Then, the volatility matrix

$$\Sigma(t) := \begin{pmatrix} \sigma_I(t) & 0 \\ \xi\sigma_S(t) & \sqrt{1-\xi^2}\sigma_S(t) \end{pmatrix} \tag{6}$$

corresponding to the two risky assets and satisfies $det(\Sigma(t)) = \sigma_S(t)\sigma_I(t)\sqrt{1-\xi^2} \neq 0$. Therefore, the market is complete and there exists a unique market price of risks θ satisfying

$$\theta(t) := \begin{pmatrix} \theta_I(t) \\ \theta_S(t) \end{pmatrix} = \begin{pmatrix} \theta_I(t) \\ \dfrac{\mu(t) - r(t) - \theta_I(t)\xi\sigma_S(t)}{\sigma_S(t)\sqrt{(1-\xi^2)}} \end{pmatrix}, \tag{7}$$

where $\theta_S(t)$ is the market price of stock risks. We assume in this paper that the salary process $Y(t)$ at time t of the PPM is governed the by the dynamics

$$dY(t) = Y(t)(\beta(t)dt + \sigma_Y(t)dW(t)), Y(0) = y_0 \in \mathbb{R}_+, \tag{8}$$

where $\beta(t) \in \mathbb{R}_+$ is the expected growth rate of the salary of the PPM and $\sigma_Y(t) = (\sigma_{Y_1}(t), \sigma_{Y_2}(t))$ is the volatility of a PPM's salary. $\sigma_{Y_1}(t) \in \mathbb{R}$ is the volatility caused by the source of inflation, $W^I(t)$ and $\sigma_{Y_2}(t) \in \mathbb{R}$ is the volatility caused by the source of uncertainty arising from the stock market, $W^S(t)$.

Remark 1. *If the pension PPM's salary is deterministic, then (8) becomes* $dY(t) = \beta(t)Y(t)dt$.

In this paper, we assume that $r(t)$, $\mu(t)$, $\sigma(t)$, $\sigma_I(t)$, $\theta_I(t)$, $\theta_S(t)$, $\sigma_Z(t)$, $\bar{q}(t)$, $\sigma_S(t)$, $\beta(t)$, $\sigma_Y(t)$ are constant in time. The process $\Lambda(t)$ referred to as the stochastic discount factor (which adjusts for nominal interest rate and market price of risks for stock and index bond) is assumed to satisfy

$$\Lambda(t) = B(t)^{-1}\Theta(t), \tag{9}$$

where

$$\Theta(t) = e^{-\theta' W(t) - \frac{1}{2}\|\theta\|^2 t}, 0 \leq t \leq T. \tag{10}$$

The PPM starts at time $t \in [0, T]$ with initial wealth $x_0 \in \mathbb{R}_+$. The current wealth $X(t), t \leq s \leq T$, satisfies the budget constraint

$$\begin{aligned} &dX(t) = [X(t)(r + \Delta(t)\lambda) + cY(t) - C(t)]dt + \\ &X(t)(\Sigma'\Delta'(t))'dW(t), X(0) = x_0 \in \mathbb{R}_+, \end{aligned} \tag{11}$$

where $\lambda = (\sigma_I\theta_I, \mu - r)'$.

III. THE EXPECTED VALUE OF PPM's DISCOUNTED FUTURE CONTRIBUTION (EVPPMDFC)

In this section, we determine expected value of PPM's future contribution and consumption process.

Definition 1. *The EVPPMDFC is defined as*

$$\Phi(t) = E_t\left(\int_t^T \frac{\Lambda(u)}{\Lambda(t)} cY(t)du\right) \tag{12}$$

where, $E_t = E(\cdot|\mathcal{F}_t)$ *is the conditional expectation with respect to the Brownian filtration* $\{\mathcal{F}_t\}_{t\geq 0}$.

Theorem 1. *Suppose* $\Phi(t)$ *is the EVPPMDFC, then*

$$\Phi(t) = \frac{cY(t)\left(\exp((\beta - r - \sigma_Y\theta)(T-t)) - 1\right)}{\beta - r - \sigma_Y\theta}. \tag{13}$$

Proof: See Nkeki and Nwozo [26].

Lemma 1. *Suppose that Theorem 1 holds, then*

$$d\Phi(t) = \Phi(t)\left((r + \sigma_Y\theta)dt + \sigma_Y dW(t)\right) - cY(t)dt. \tag{14}$$

Proof: See Nkeki and Nwozo [26].

At $t = 0$, we obtain the present value of PPM's discounted future contribution to be

$$\Phi(0) = \Phi_0 = \frac{cy_0\left(\exp((\beta - r - \sigma_Y\theta)T) - 1\right)}{\beta - r - \sigma_Y\theta}. \tag{15}$$

See Nkeki and Nwozo [26] for details.

III. THE VALUE OF PPM's WEALTH

In this section, we consider the value of the PPM's wealth and obtain the dynamics of the value of the wealth at time t.

Definition 2. *The value of a PPM's life-time consumption process, $\Psi(t)$ is defined as*

$$\Psi(t) = E_t\left[\int_t^\infty \frac{\Lambda(u)}{\Lambda(t)}C(u)du\right], t \geq 0.$$

Definition 3. *Let $V(t)$ be a value process at time t. We defined $V(t)$ as*

$$V(t) := X(t) + \Phi(t), \tag{16}$$

where, $X(t)$ satisfy (11) and $\Phi(t)$ satisfy (14).

Proposition 1. *Let $V(t)$ satisfy (16), $X(t)$ satisfy (11) and $\Phi(t)$ satisfy (14), then*

$$\begin{aligned} dV(t) = [&X(t)(r + \Delta(t)\lambda) + \Phi(t)(r + \sigma_Y\theta) \\ &-C(t)]dt + (\Phi(t)\sigma_Y' + X(t)\Sigma'\Delta'(t))'dW(t), \end{aligned} \tag{17}$$

with $V(0) = v_0 \in \mathbb{R}_+$ such that $X(0) = x_0 \in \mathbb{R}_+$ and $\Phi(0) = \Phi_0 \in \mathbb{R}_+$.
Proof: Finding the differential of both sides of (16) and then substitute in (11) and (14), the result follows.

IV. THE OPTIMIZATION PROBLEM

The objective of the fund manager is to maximize the expected value of final wealth, $V(T)$ and to minimize the variance of the final wealth $Var(V(T))$, and the consumption level, $C(t)$ at time interval $[0, T]$. The PPM expected utility of wealth is given by

$$\begin{aligned} &J(v, t; \Delta, C) = \\ &E\left(\int_0^T e^{-\rho t}U(C(t))dt + U(V(T))|X(t) = x, \Phi(t) = \Phi\right), \end{aligned} \tag{18}$$

for $(v; t) = (x, \Phi; t) \in \mathcal{H} = \mathbb{R}_+ \times \mathbb{R} \times [0, T]$, with the processes X and Φ solving, respectively, (11) and (14), and V solve (17). Here $\rho \in \mathbb{R}_+$ denotes the PPM's consumption preference discount rate. The value function which represents the maximal expected utility of PPM's wealth is defined as

$$U(v; t) = \sup_{(\Delta, C) = \mathcal{K}_{v_0, c_0}} J(x, \Phi, t; \Delta, C), \tag{19}$$

subject to (17). Here $\mathcal{K}_{v_0,c_0}$ is a set of measurable processes (Δ, C), where Δ satisfies (4), C satisfies (5) and such that (17) admit a unique solution $\mathcal{F}_t-$measurable adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We now introduce the following differential operator:

$$\mathcal{L} = \frac{1}{2}\Phi^2\sigma_Y\sigma_Y'\frac{\partial^2}{\partial\Phi^2} + \Phi(r + \sigma_Y\theta)\frac{\partial}{\partial\Phi}. \tag{20}$$

Proposition 2. *The value function U is a solution of the Hamilton-Jacobi-Bellman (HJB) equation*

$$\begin{aligned} &U_t + \max_\Delta[x(r + \Delta(t)\lambda)U_x \\ &+\frac{1}{2}x^2\Sigma\Delta(t)\Sigma'\Delta'(t)U_{xx} + x\Phi\Sigma\Delta(t)\sigma_Y'U_{x\Phi}] \\ &+\max_C(-C(t)U_x + U(C(t))e^{-\rho t}) + \mathcal{L}U = 0, \end{aligned} \tag{21}$$

and

$$U(v, T) = \frac{v^\gamma}{\gamma}, \gamma > 0$$

with $\mathcal{L}$ as defined in (20).

The optimal variational portfolio in the risky assets and optimal consumption are obtained as

$$\Delta'^*(t) = \frac{-(\Sigma\Sigma')^{-1}(\lambda U_x + \Phi\Sigma\sigma_Y'U_{x\Phi})}{xU_{xx}}, \tag{22}$$

$$C^*(t) = I(U_xe^{\rho t}), \text{where } I = \left(\frac{\partial U(C(t))}{\partial C(t)}\right)^{-1}. \tag{23}$$

Substituting (22) and (23) into (21), we obtain the following

$$\begin{aligned} &U_t + rxU_x + \Phi(r + \sigma_Y\theta)U_\Phi - I(U_xe^{\rho t})U_x \\ &+U(I(U_xe^{\rho t}))e^{-\rho t} - \frac{1}{2}(\Sigma'M\lambda)'\Sigma'M\lambda\frac{U_x^2}{U_{xx}} \\ &+\Phi(\Sigma'M\lambda)'\Sigma'M\Sigma\sigma_Y'\frac{U_xU_{x\Phi}}{U_{xx}} \\ &+\frac{1}{2}\Phi^2(\Sigma'M\Sigma\sigma_Y')'(\Sigma'M\Sigma\sigma_Y')\frac{U_{x\Phi}^2}{U_{xx}} \\ &-2\Phi(\Sigma'M\lambda)'\sigma_Y'\frac{U_xU_{x\Phi}}{U_{xx}} \\ &-\Phi^2(\Sigma'M\Sigma\sigma_Y')'\sigma_Y'\frac{U_{x\Phi}^2}{U_{xx}} + \frac{1}{2}\Phi^2\sigma_Y\sigma_Y'U_{\Phi\Phi} = 0. \end{aligned} \tag{24}$$

V. THE SOLUTION TO THE HJB EQUATION

In this section, we consider and provide the solution to the HJB equation (24).

Proposition 3. *The solution to the HJB equation (24) is of the form*

$$\begin{cases} U(t, v) = \left(\dfrac{(X^*(t) + \Phi(t))^\gamma}{\gamma} - \dfrac{C(t)^\gamma}{\gamma}\right)(A(t)B(t))^\gamma \\ U(T, v) = \dfrac{X^*(T)^\gamma}{\gamma} - \dfrac{C(T)^\gamma}{\gamma} \end{cases} \tag{25}$$

with

$$\begin{cases} A(t) = e^{(r + \frac{1}{2(\gamma-1)}(\Sigma M\lambda)'\Sigma M\lambda)(T-t)}, \\ A(T) = 1, \end{cases} \tag{26}$$

$$\begin{cases} B(t) = \left(1 - \dfrac{e^{\frac{-\rho T}{\gamma-1}}(e^{\frac{-(r\gamma-\rho)(T-t)}{\gamma-1}} - 1)(\gamma - 1)}{r\gamma - \rho}\right)^{\frac{1-\gamma}{\gamma}}, \\ B(T) = 1. \end{cases} \tag{27}$$

Proof: We start by finding the following partial derivatives:

$$U_t = (X^*(t)+\Phi(t))^\gamma(A(t)B(t))^{\gamma-1}(\dot{A}(t)B(t)+A(t)\dot{B}(t)), \tag{28}$$

$$U_x = (X^*(t) + \Phi(t))^{\gamma-1}(A(t)B(t))^\gamma, \tag{29}$$

$$U_{xx} = (X^*(t) + \Phi(t))^{\gamma-2}(A(t)B(t))^\gamma, \tag{30}$$

$$U_\Phi = (X^*(t) + \Phi(t))^{\gamma-1}(A(t)B(t))^\gamma, \tag{31}$$

$$U_{\Phi\Phi} = (\gamma - 1)(X^*(t) + \Phi(t))^{\gamma-2}(A(t)B(t))^\gamma, \tag{32}$$

$$U_{x\Phi} = (\gamma - 1)(X^*(t) + \Phi(t))^{\gamma-2}(A(t)B(t))^\gamma. \tag{33}$$

Substituting (28)-(33) into (24), we have

$$\begin{aligned}&(X^*(t)+\Phi(t))^{\gamma}(A(t)B(t))^{\gamma-1}(\dot{A}(t)B(t)\\&+A(t)\dot{B}(t))+rX^*(t)(X^*(t)\\&+\Phi(t))^{\gamma-1}(A(t)B(t))^{\gamma}+r\Phi(t)(X^*(t)\\&+\Phi(t))^{\gamma-1}(A(t)B(t))^{\gamma}\\&-\frac{1}{2}(\Sigma'M\lambda)'\Sigma'M\lambda(X^*(t)+\Phi(t))^{\gamma}(A(t)B(t))^{\gamma}\\&-I[(X^*(t)+\Phi(t))^{\gamma-1}(A(t)B(t))^{\gamma}e^{\rho t})](X^*(t)\\&+\Phi(t))^{\gamma-1}(A(t)B(t))^{\gamma}+U(I[(X^*(t)\\&+\Phi(t))^{\gamma-1}(A(t)B(t))^{\gamma}e^{\rho t}])e^{-\rho t}=0.\end{aligned} \quad (34)$$

From (34), we obtain the following ordinary differential equations (ODEs), (35) and (36):

$$\dot{A}(t)+A(t)(r+\frac{1}{2(\gamma-1)}(\Sigma M\lambda)'\Sigma M\lambda)=0, A(T)=1. \quad (35)$$

$$\dot{B}(t)-\frac{\gamma-1}{\gamma}B(t)^{\frac{2\gamma-1}{\gamma-1}}e^{\frac{\rho t+r\gamma(T-t)}{\gamma-1}}=0, B(T)=1. \quad (36)$$

Solving the ODEs (35) and (36), we have

$$\begin{cases} A(t)=e^{(r+\frac{1}{2(\gamma-1)}(\Sigma'M\lambda)'\Sigma'M\lambda)(T-t)},\\ A(T)=1.\end{cases} \quad (37)$$

$$\begin{cases} B(t)=\left(1-\frac{e^{\frac{-\rho T}{\gamma-1}}(e^{\frac{-(r\gamma-\rho)(T-t)}{\gamma-1}}-1)(\gamma-1)}{r\gamma-\rho}\right)^{\frac{1-\gamma}{\gamma}},\\ B(T)=1.\end{cases} \quad (38)$$

Proposition 3 gives the expected utility of optimal value of wealth that will accrued to the PPM at time t.

VI. OPTIMAL PORTFOLIO AND CONSUMPTION PLAN

We present the optimal portfolio and optimal consumption of a PPM at time t.

Proposition 4. *Let $X^*(t)$ be the optimal wealth process of a CRRA PPM solving (11), $\Phi(t)$ be the discounted value of PPM's contributions at time t satisfying (14), then*
(i) The optimal investment, $\Delta^*(t)$ at time t is given by

$$\Delta^{'*}(t)=\frac{-(\Sigma\Sigma')^{-1}\lambda(X^*(t)+\Phi(t))}{(\gamma-1)X^*(t)}-(\Sigma\Sigma')^{-1}\Sigma\sigma_Y'\frac{\Phi(t)}{X^*(t)}. \quad (39)$$

(ii) The optimal consumption process of a PPM at time t is given by

$$C^*(t)=\frac{\gamma(r\gamma-\rho)(X^*(t)+\Phi(t))f(t)}{\gamma(r\gamma-\rho)-\gamma(\gamma-1)(e^{\frac{-r\gamma(T-t)-\rho t}{\gamma-1}}-e^{\frac{-\rho T}{\gamma-1}})}, \quad (40)$$

where $f(t)=e^{\frac{1}{\gamma-1}(\rho t+(r+\frac{\gamma}{2(\gamma-1)}(\Sigma'M\lambda)'\Sigma'M\lambda)(T-t))}$.

Proof: Using the partial derivatives (29), (30) and (33) on (22) and (23), we have

$$\Delta^{'*}(t)=\frac{-(\Sigma\Sigma')^{-1}\lambda(X^*(t)+\Phi(t))}{(\gamma-1)X^*(t)}-(\Sigma\Sigma')^{-1}\Sigma\sigma_Y'\frac{\Phi(t)}{X^*(t)}. \quad (41)$$

$$\begin{aligned}\Delta_0^*(t)&=1-\mathbf{I}\Delta^{'*}(t)=1+\mathbf{I}((\Sigma\Sigma')^{-1}\Sigma\sigma_Y')\frac{\Phi(t)}{X^*(t)}\\&+\frac{\mathbf{I}(\Sigma\Sigma')^{-1}\lambda(X^*(t)+\Phi(t))}{(\gamma-1)X^*(t)},\end{aligned} \quad (42)$$

where, $\mathbf{I}=(1,1)$.

$$\begin{aligned}C^*(t)&=I(U_xe^{\rho t})\\&=I((X^*(t)+\Phi(t))^{\gamma-1}(A(t)B(t))^{\gamma}e^{\rho t})\\&=[(X^*(t)+\Phi(t))^{\gamma-1}(A(t)B(t))^{\gamma}e^{\rho t}]^{\frac{1}{\gamma-1}}\\&=(X^*(t)+\Phi(t))(A(t)B(t))^{\frac{\gamma}{\gamma-1}}e^{\frac{\rho t}{\gamma-1}}\\&=\frac{\gamma(r\gamma-\rho)(X^*(t)+\Phi(t))f(t)}{\gamma(r\gamma-\rho)-\gamma(\gamma-1)(e^{\frac{-r\gamma(T-t)-\rho t}{\gamma-1}}-e^{\frac{-\rho T}{\gamma-1}})}.\end{aligned}$$

Intuitively, the expected growth rate (GR) of PPM optimal consumption is equals

$$GR=\frac{1}{\gamma-1}\left(r-\rho+\frac{1}{2(\gamma-1)}(M\lambda)'\lambda\right), \gamma>1. \quad (43)$$

This is referred to as the Eulers equation for the intertemporal maximization under uncertainty. The coefficient $\frac{1}{\gamma-1}$ is referred to as the elasticity of substitution of consumption in macroeconomics. The positive term $(M\lambda)'\lambda$ captures the uncertainty of the financial markets. When the market become risky, it induces the PPM not make more contributions into the pension scheme. From (43), we have that for a fixed, γ, if $\rho>r+\frac{1}{2(\gamma-1)}(M\lambda)'\lambda$ the growth rate of the expected consumption is strictly negative, it is strictly nonnegative if $\rho<r+\frac{1}{2(\gamma-1)}(M\lambda)'\lambda$ and constant if $\rho=r+\frac{1}{2(\gamma-1)}(M\lambda)'\lambda$. Intuitively, as the discount rate captures the PPMs preference over time, if ρ is less than $r+\frac{1}{2(\gamma-1)}(M\lambda)'\lambda$, it implies that PPM will like to consume more since the markets are risky to invest in. If ρ is greater than $r+\frac{1}{2(\gamma-1)}(M\lambda)'\lambda$, it implies that PPM will like to consume less and invest more into the pension scheme. Finally, if ρ is equal to $r+\frac{1}{2(\gamma-1)}(M\lambda)'\lambda$, then PPM will be at the critical position to determine whether to consume more and invest less or to invest more and to consume less.

At time $t=0$, we have the optimal initial value of the portfolio in the risky assets and optimal initial consumption of the PPM as follows:

$$\Delta^{'*}(0)=\frac{-(\Sigma\Sigma')^{-1}\lambda(x_0+\Phi_0)}{(\gamma-1)x_0}-(\Sigma\Sigma')^{-1}\Sigma\sigma_Y'\frac{\Phi_0}{x_0}. \quad (44)$$

$$\begin{aligned}\Delta_0^*(0)&=1-\mathbf{I}\Delta^{'*}(0)=1+\mathbf{I}((\Sigma\Sigma')^{-1}\Sigma\sigma_Y')\frac{\Phi_0}{x_0}\\&+\frac{\mathbf{I}(\Sigma\Sigma')^{-1}\lambda(x_0+\Phi_0)}{(\gamma-1)x_0}.\end{aligned} \quad (45)$$

$$C^*(0) = \frac{\gamma(r\gamma-\rho)(x_0+\Phi_0)e^{\frac{1}{\gamma-1}(r+\frac{\gamma}{2(\gamma-1)}(\Sigma' M\lambda)'\Sigma' M\lambda)T)}}{\gamma(r\gamma-\rho)-\gamma(\gamma-1)(e^{\frac{-r\gamma T}{\gamma-1}}-e^{\frac{-\rho T}{\gamma-1}})}. \tag{46}$$

At time $t = T$, we have

$$C^*(T) = X^*(T)e^{\frac{\rho T}{\gamma-1}}. \tag{47}$$

(47) gives the optimal terminal consumption of the PPM. Observe that the terminal consumption of the PPM depend on the optimal final wealth, consumption preference factor ρ, retirement date, T and coefficient of PPM's risk preference, γ. We can re-express (47) as $X^*(T) = C^*(T)e^{\frac{-\rho T}{\gamma-1}}$. Observe that

$$\lim_{T\to\infty} X^*(T) = \lim_{T\to\infty} C^*(T)e^{\frac{-\rho T}{\gamma-1}} = 0.$$

This shows that the accumulated wealth will terminate after a long run of consumption of the wealth by a PPM after retirement. Proposition 5 show the optimal value of PPM's wealth dynamics and second degree of Ito process of the optimal value of PPM's wealth.

VII. SOME NUMERICAL EXAMPLES AND DISCUSSIONS

All the figures are obtained (except the optimal consumption where the value of ρ is varied) by setting $\rho = 0.1$, $\sigma_I = 0.2$, $r = 0.04$, $\theta_I = 0.125$, $\mu = 0.09$, $\sigma_S = 0.3$, $\xi = 0.3$, $T = 20$, $\gamma = 0.3$, $\sigma_Y = (0.18, 0.24)$, $\beta = 0.0292$, $c = 0.15$ and $y_0 = 0.9$. Figure 1 shows the portfolio value of a PPM invested in index bond under stochastic salary over time. Figure 2 gives the portfolio value with stochastic salary of a PPM invested in stock at time t. Figure 3 shows the portfolio value with stochastic salary invested in cash account at time t. Observe from figure 1 to figure 3 that the portfolios are made up of several shocks. We referred to these as the variational Merton portfolios. The portfolio value in stock remain negative over time and the portfolio values index bond and cash account remain nonnegative over time. This implies that the portfolio value in stock should be withdraw and invested the fund in index bond and put the remaining in cash account at time t. Figure 4 shows the optimal variational consumption of the PPM at time t. We observe that as wealth increases, consumption increases stochastically over time. Figure 5 shows the portfolio value of a PPM in index bond given that the salary of the PPM is deterministic. It was found that the portfolio value in index bond remain nonnegative over time. Similarly, in figure 6 under deterministic salary process of the PPM, the portfolio value in stock is also nonnegative over time. Figure 7 shows the portfolio value in cash account at time t. We found that the portfolio value in cash account remain negative over time. This shows that the fund in cash account should be shorten and invested the amount in stock and index bond at time t. Interestingly, in the case of portfolio value under stochastic salary, the result

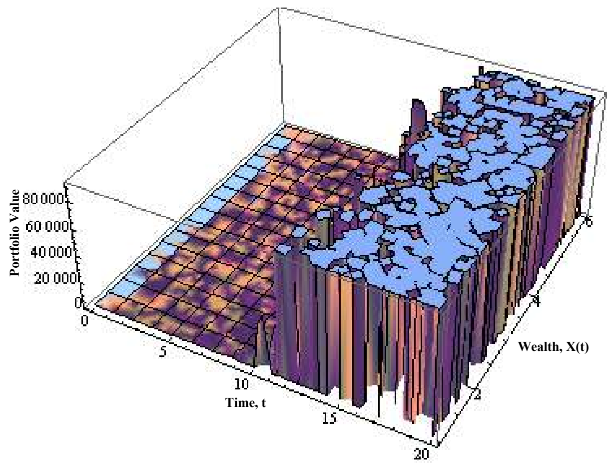

Figure 1: Portfolio value in index bond under stochastic salary

shows that the fund invested in stock should be shorten to finance index bond and put the remaining in cash account. Now, under deterministic salary case, the cash account is to be shorten to finance index bond and stock. We therefore conclude that under stochastic salary case the amount of fund invested should be gradual transferred to the riskless assets as the retirement date approaches. In the case of deterministic salary case, the investment should remain in stock and index bond. Observe in both cases, index bond remain favourite. We therefore conclude that index bond is a suitable asset to invest in by a PPM.

Under a deterministic salary case: in figure 8, we have the optimal wealth-consumption over time for a PPM, given that PPM consumption preference discount rate, ρ is zero. Figure 9 shows the optimal wealth-consumption over time for a PPM, given that $\rho = 0.01$, figure 10 shows optimal wealth-consumption over time, given that $\rho = 0.1$ and figure 11 when $\rho = 0.5$. From figure 8 to figure 11, we observe that as ρ increases and all other parameters remain fixed, consumption decreases drastically, and vice versa. The economic implication of this, is that as the investor (i.e., PPM) taste to consume reduces, contributions will increase, thereby the expected wealth of the PPM will increase and more wealth to consume after retirement period. A critical observation from (43), shows that PPM will only be encouraged to make more contributions in to the scheme only if the financial markets are not under bearish condition (i.e., the markets are booming). It implies that more contributions will be made into the scheme under booming market conditions, which is expected. If we allow $\rho \longrightarrow +\infty$ and $\gamma > 1$, then we found that $C^*(t) \longrightarrow 0$. If we allow $\rho \longrightarrow -\infty$ and $\gamma < 1$, then we found that $C^*(t) \longrightarrow 0$. We therefore conclude that consumption is zero only when $\rho \longrightarrow +\infty$ and $\gamma > 1$ or $\rho \longrightarrow -\infty$ and $\gamma < 1$.

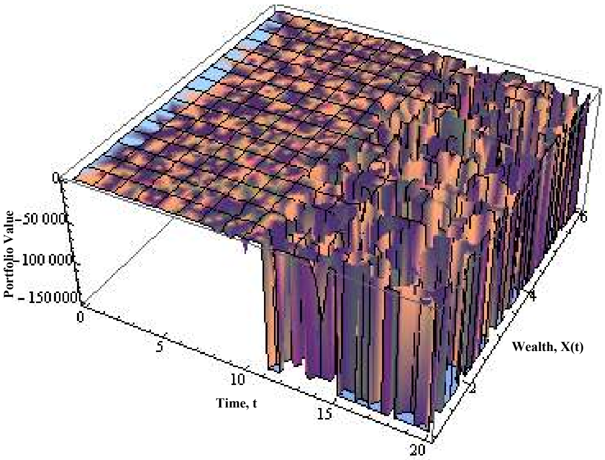

Figure 2: Portfolio value in stock under stochastic salary

Figure 5: Portfolio value in index bond

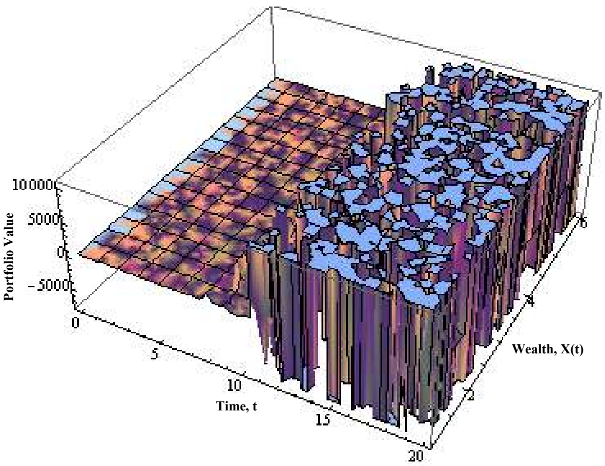

Figure 3: Portfolio value in cash account under stochastic salary

Figure 6: Portfolio value in stock

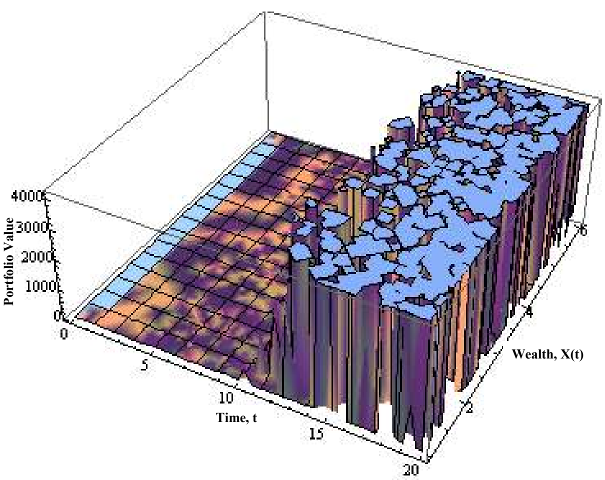

Figure 4: Optimal variational consumption of a PPM for $\rho = 0.1$

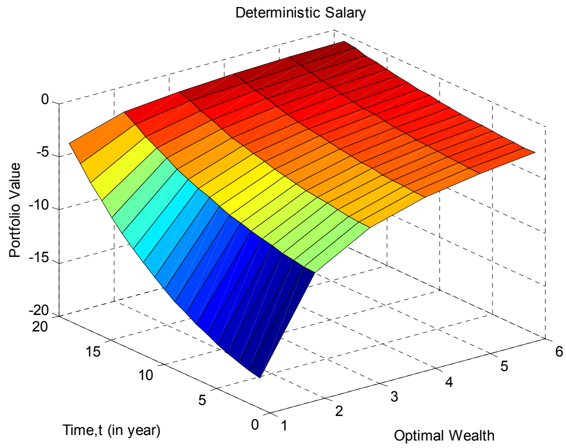

Figure 7: Portfolio value in cash account

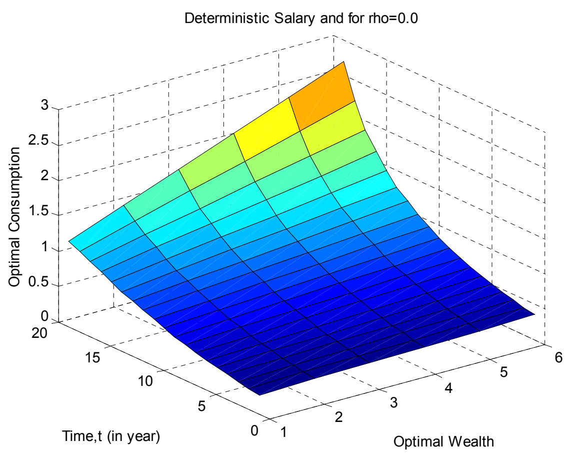

Figure 8: Optimal consumption of the PPM given that $\rho = 0.0$

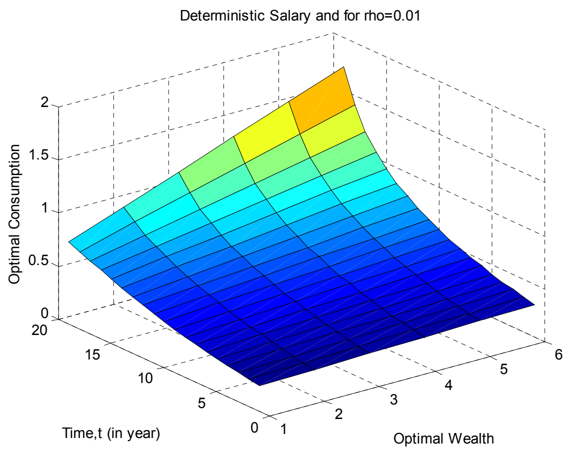

Figure 9: Optimal consumption of the PPM given that $\rho = 0.01$

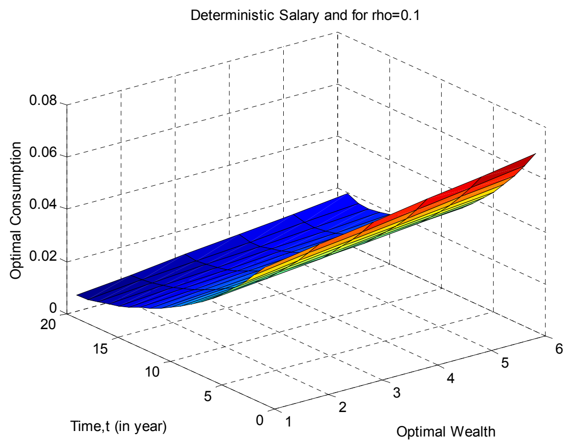

Figure 10: Optimal consumption of the PPM given that $\rho = 0.1$

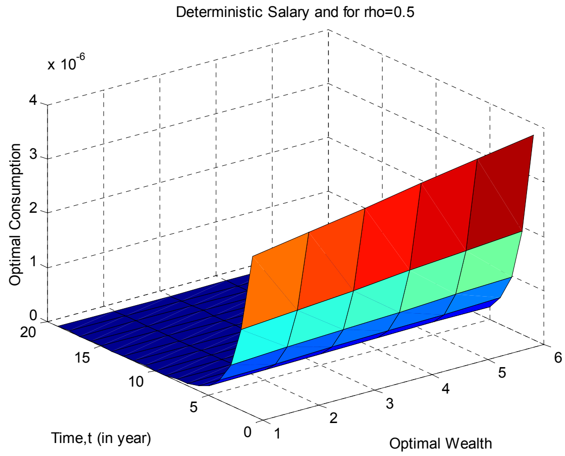

Figure 11: Optimal consumption of the PPM given that $\rho = 0.5$

Table I

Initial investment in the risky assets, $\Delta^{*I}(0)$, $\Delta^{*S}(0)$

γ	Δ^{*I} $T=1$	Δ^{*S} $T=1$	Δ^{*I} $T=2$	Δ^{*S} $T=2$	Δ^{*I} $T=5$	Δ^{*S} $T=5$	Δ^{*I} $T=10$	Δ^{*S} $T=10$	Δ^{*I} $T=20$	Δ^{*S} $T=20$
0.1	0.6676	0.6315	0.6424	0.5877	0.5762	0.4724	0.4911	0.3241	0.3858	0.1407
0.2	0.7657	0.7274	0.7512	0.6940	0.7129	0.6059	0.6638	0.4928	0.6030	0.3528
0.3	0.8919	0.8507	0.8911	0.8306	0.8887	0.7776	0.8858	0.7096	0.8821	0.6255
0.4	1.0601	1.0150	1.0775	1.0127	1.1231	1.0066	1.1818	0.9988	1.2544	0.9891
0.5	1.2957	1.2451	1.3386	1.2677	1.4513	1.3272	1.5962	1.4036	1.7755	1.4981
0.6	1.6491	1.5903	1.7301	1.6502	1.9436	1.8080	2.2178	2.0108	2.5572	2.2617
0.7	2.2379	2.1655	2.3827	2.2876	2.7640	2.6094	3.2538	3.0230	3.8600	3.5343
0.8	3.4157	3.3159	3.6879	3.5626	4.4048	4.2122	5.3259	5.0467	6.4657	6.0795
0.9	6.9490	6.7672	7.6035	7.3873	9.3274	9.0205	11.5420	11.1186	14.2827	13.7152

Table II

Initial optimal consumption, C_0^*

ρ	$T=1$	$T=2$	$T=3$	$T=4$	$T=5$	$T=6$	$T=10$	$T=12$	$T=15$	$T=20$
0.00	0.5420	0.3828	0.2993	0.2466	0.2095	0.1817	0.1144	0.0939	0.0716	0.0472
0.01	0.5400	0.3791	0.2945	0.2410	0.2034	0.1751	0.1072	0.0869	0.0648	0.0413
0.02	0.5381	0.3755	0.2897	0.2354	0.1973	0.1687	0.1003	0.0801	0.0585	0.0359
0.03	0.5361	0.3718	0.2850	0.2299	0.1913	0.1624	0.0937	0.0737	0.0535	0.0310
0.04	0.5342	0.3682	0.2802	0.2245	0.1854	0.1562	0.0873	0.0676	0.0470	0.0265
0.05	0.5322	0.3645	0.2755	0.2191	0.1796	0.1501	0.0812	0.0618	0.0418	0.0226
0.06	0.5306	0.3609	0.2708	0.2138	0.1738	0.1441	0.0754	0.0564	0.0371	0.0191
0.07	0.5283	0.3573	0.2662	0.2085	0.1682	0.1383	0.0699	0.0513	0.0328	0.0160
0.08	0.5263	0.3536	0.2616	0.2033	0.1626	0.1326	0.0647	0.0465	0.0289	0.0134
0.09	0.5244	0.3500	0.2570	0.1981	0.1572	0.1271	0.0597	0.0421	0.0253	0.0111
0.10	0.5224	0.3464	0.2524	0.1930	0.1519	0.1217	0.0555	0.0380	0.0221	0.0091

Table III

Optimal terminal consumption, $C^*(T)$ for $\rho = 0.1$

$X^*(T)$	$T=1$	$T=2$	$T=3$	$T=4$	$T=5$	$T=6$	$T=10$	$T=12$	$T=15$	$T=20$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.8669	0.7515	0.6514	0.5647	0.4895	0.4244	0.2397	0.1801	0.1173	0.0574
2	1.7338	1.5030	1.3029	1.1294	0.9791	0.8487	0.4793	0.3602	0.2346	0.1149
3	2.6006	2.2544	1.9543	1.6942	1.4686	1.2731	0.7190	0.5403	0.3520	0.1723
4	3.4675	3.0059	2.6058	2.2589	1.9582	1.6975	0.9586	0.7204	0.4693	0.2297
8	6.9350	6.0118	5.2115	4.5177	3.9163	3.3950	1.9172	1.4407	0.9386	0.4595
10	8.6688	7.5148	6.5144	5.6472	4.8954	4.2437	2.3965	1.8009	1.1732	0.5743

Table IV

Optimal terminal consumption, $C^*(T)$ for $\rho = 0.3$

$X^*(T)$	$T=1$	$T=2$	$T=3$	$T=4$	$T=5$	$T=6$	$T=10$	$T=12$	$T=15$	$T=20$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.6514	0.4244	0.2765	0.1801	0.1173	0.0764	0.0138	0.0058	0.0016	0.0002
2	1.3029	0.8487	0.5529	0.3602	0.2846	0.1529	0.2753	0.0117	0.0032	0.0004
3	1.9543	1.2731	0.8294	0.5403	0.3520	0.2293	0.0413	0.0175	0.0048	0.0006
4	2.6058	1.6975	1.1058	0.7204	0.4693	0.3057	0.0551	0.0234	0.0065	0.0008
8	5.2115	3.3950	2.2116	1.4407	0.9386	0.6114	0.1101	0.0467	0.0129	0.0015
10	6.5144	4.2437	2.7645	1.8009	1.1732	0.7643	0.1376	0.0584	0.0161	0.0019

Table V

Optimal terminal consumption, $C^*(T)$ for $\rho = 0.5$

$X^*(T)$	$T=1$	$T=2$	$T=3$	$T=4$	$T=5$	$T=6$	$T=10$	$T=12$	$T=15$	$T=20$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.4895	0.2397	0.1173	0.0574	0.0281	0.0138	0.0008	0.0002	0.0000	0.0000
2	0.9791	0.4793	0.2346	0.1149	0.0562	0.0275	0.0016	0.0004	0.0000	0.0000
3	1.4686	0.7190	0.3520	0.1723	0.0843	0.0413	0.0024	0.0006	0.0000	0.0000
4	1.9582	0.9586	0.4693	0.2297	0.1125	0.0551	0.0032	0.0008	0.0000	0.0000
8	3.9163	1.9172	0.9386	0.4595	0.2249	0.1101	0.0063	0.0015	0.0002	0.0000
10	4.8954	2.3965	1.1732	0.5743	0.2812	0.1376	0.0079	0.0019	0.0002	0.0000

Table VI
Initial consumption, C_0^* with different values of γ

γ	$T=1$	$T=2$	$T=3$	$T=4$	$T=5$	$T=6$	$T=10$	$T=12$	$T=15$	$T=20$
0.10	0.5270	0.3556	0.2642	0.2063	0.1659	0.1359	0.0674	0.0488	0.0307	0.0144
0.20	0.5246	0.3510	0.2584	0.1998	0.1590	0.1289	0.06123	0.0434	0.0263	0.0116
0.30	0.5224	0.3464	0.2524	0.1930	0.1519	0.1217	0.0550	0.0380	0.0221	0.0091
0.40	0.5210	0.3424	0.2468	0.1865	0.1449	0.1147	0.0491	0.0330	0.0184	0.0070
0.50	0.5221	0.3408	0.2432	0.1818	0.1397	0.1091	0.0443	0.0289	0.0154	0.0054
0.60	0.5306	0.3470	0.2471	0.1839	0.1405	0.1091	0.0429	0.0274	0.0141	0.0047
0.70	0.5640	0.3824	0.2803	0.2140	0.1674	0.1329	0.0565	0.0374	0.0202	0.0071
0.80	0.7242	0.5977	0.5254	0.4783	0.4444	0.4181	0.3434	0.3127	0.2699	0.2060
0.81	0.7653	0.6614	0.6076	0.5774	0.5599	0.5494	0.5327	0.5263	0.5126	0.4780
0.83	0.8859	0.8679	0.8987	0.9610	1.0475	1.1546	1.7686	2.1890	2.9820	4.8490
0.85	1.1032	1.3090	1.6312	2.0950	2.7394	3.6184	11.3281	19.9654	46.0985	180.111
0.87	1.5560	2.5087	4.2149	7.2825	12.7925	22.6709	227.334	713.828	3909.88	64315.2

Table I shows the total amount of initial investment proportion in the risky assets, $\Delta^{*I}(0)$ and $\Delta^{*S}(0)$, chosen to maximize wealth to the prescribed levels at difference values of γ. The investment in cash account is $\Delta_0^*(0)$, that can be obtained from table I. As expected, the risky investment increases with higher value of γ. Table II shows the total expected optimal value of the initial consumption when the portfolio comprises the cash account and two risky assets at difference values of ρ. We observed that as ρ increases, the optimal initial consumption levels reduces. Table III shows the optimal terminal consumption for $\rho = 0.1$ as optimal wealth increases. We observed that the as the optimal terminal wealth increases, the optimal terminal consumption levels increases alongside. Table IV shows the optimal terminal consumption for $\rho = 0.3$ as optimal wealth increases. Furthermore, as the optimal terminal wealth increases, the optimal terminal consumption levels increases as well. Table V shows the optimal terminal consumption for $\rho = 0.5$ as optimal wealth increases. We observed that as time increases, wealth level increases and consumption level reduces to zero. This shows that as the PPM get older and older, consumption will terminate. Table VI shows the initial optimal consumption at varying value of γ. We observed that as γ increases from 0.10 to 0.40, consumption level decreases. From 0.50 and above consumption level increases. We also observed that consumption level becomes much more sensitive to γ, when γ raises form 0.8 upward.

VIII. THE EFFICIENT FRONTIER

We now consider the efficient frontier of the PPM portfolio in mean-standard deviation.

Proposition 5. *Let $V^*(t)$ be the optimal value of wealth process of a CRRA PPM solving (17), then*
(i) the dynamic of the value of a PPM's wealth is

$$\begin{aligned} dV^*(t) &= V^*(t)\left(\left(r - \frac{(M\lambda)'\lambda}{\gamma-1}\right) - \phi(t)\right)dt \\ &-V^*(t)\left(\frac{\Sigma' M\lambda}{\gamma-1}\right)' dW(t), \end{aligned} \tag{48}$$

(ii) the second moment of the dynamic of the value of wealth of a PPM is

$$\begin{aligned} dV^{*2}(t) &= V^{*2}(t)(2(r - \frac{(M\lambda)'\lambda}{\gamma-1} - \phi(t)) \\ &+\frac{(\Sigma' M\lambda)'\Sigma' M\lambda}{(\gamma-1)^2})dt - 2V^{*2}(t)\left(\frac{\Sigma' M\lambda}{\gamma-1}\right)' dW(t), \end{aligned} \tag{49}$$

where

$$\phi(t) = \frac{\gamma(r\gamma-\rho)e^{\frac{1}{\gamma-1}(\rho t + (r + \frac{\gamma}{2(\gamma-1)}(\Sigma' M\lambda)'\Sigma' M\lambda)(T-t))}}{\gamma(r\gamma-\rho) - \gamma(\gamma-1)(e^{\frac{-r\gamma(T-t)-\rho t}{\gamma-1}} - e^{\frac{-\rho T}{\gamma-1}})}.$$

Proof: Substituting (39) and (40) into (17), we have

$$\begin{aligned} dV^*(t) &= \left(\left(r - \frac{(M\lambda)'\lambda}{\gamma-1}\right)V^*(t) - C^*(t)\right)dt \\ &-V^*(t)\left(\frac{\Sigma M\lambda}{\gamma-1}\right)' dW(t). \end{aligned} \tag{50}$$

Simplifying (50), we have

$$\begin{aligned} dV^*(t) &= V^*(t)[(r - \frac{(M\lambda)'\lambda}{\gamma-1}) \\ &-\frac{\gamma(r\gamma-\rho)e^{\frac{1}{\gamma-1}(\rho t + (r + \frac{\gamma}{2(\gamma-1)}(\Sigma' M\lambda)'\Sigma' M\lambda)(T-t))}}{\gamma(r\gamma-\rho) - \gamma(\gamma-1)(e^{\frac{-r\gamma(T-t)-\rho t}{\gamma-1}} - e^{\frac{-\rho T}{\gamma-1}})})]dt \\ &-V^*(t)\left(\frac{\Sigma' M\lambda}{\gamma-1}\right)' dW(t). \end{aligned} \tag{51}$$

Therefore,

$$\begin{aligned} dV^*(t) &= V^*(t)\left(\left(r - \frac{(M\lambda)'\lambda}{\gamma-1}\right) - \phi(t)\right)dt \\ &-V^*(t)\left(\frac{\Sigma' M\lambda}{\gamma-1}\right)' dW(t), \end{aligned} \tag{52}$$

where

$$\phi(t) = \frac{\gamma(r\gamma-\rho)e^{\frac{1}{\gamma-1}(\rho t + (r + \frac{\gamma}{2(\gamma-1)}(\Sigma' M\lambda)'\Sigma' M\lambda)(T-t))}}{\gamma(r\gamma-\rho) - \gamma(\gamma-1)(e^{\frac{-r\gamma(T-t)-\rho t}{\gamma-1}} - e^{\frac{-\rho T}{\gamma-1}})}.$$

Applying Ito lemma on (52), we have

$$\begin{aligned} dV^{*2}(t) &= V^{*2}(t)[2(r - \frac{(M\lambda)'\lambda}{\gamma-1} - \phi(t)) + \\ &\frac{(\Sigma' M\lambda)'\Sigma' M\lambda}{(\gamma-1)^2}]dt - 2V^{*2}(t)\left(\frac{\Sigma' M\lambda}{\gamma-1}\right)' dW(t). \end{aligned} \tag{53}$$

Taking the mathematical expectation of (52) and (53), we have the following ODEs which are the first and second moments of the value of PPM's wealth at time t:

$$\begin{cases} dE(V^*(t)) = E(V^*(t))\left(\left(r - \frac{(M\lambda)'\lambda}{\gamma-1}\right) - \phi(t)\right)dt, \\ E(V^*(0)) = v_0, \end{cases} \tag{54}$$

$$\begin{cases} dE(V^{*2}(t)) = E(V^{*2}(t))[2(r - \frac{(M\lambda)'\lambda}{\gamma-1} - \phi(t)) \\ +\frac{(\Sigma' M\lambda)'\Sigma' M\lambda}{(\gamma-1)^2}]dt, E(V^{*2}(0)) = v_0^2. \end{cases} \tag{55}$$

Solving the ODEs (54) and (55), we have

$$E(V^*(t)) = v_0 e^{(r - \frac{(M\lambda)'\lambda}{\gamma-1})t - \int_0^t \phi(u)du}, \tag{56}$$

$$\begin{aligned} &E(V^{*2}(t)) = \\ &v_0^2 e^{(2(r - \frac{(M\lambda)'\lambda}{\gamma-1}) + \frac{(\Sigma' M\lambda)'\Sigma' M\lambda}{(\gamma-1)^2})t - 2\int_0^t \phi(u)du}. \end{aligned} \tag{57}$$

At time $t = T$, we have

$$E(V^*(T)) = v_0 e^{(r - \frac{(M\lambda)'\lambda}{\gamma - 1})T - \int_0^T \phi(u)du}, \quad (58)$$

$$E(V^{*2}(T)) = v_0^2 \times e^{(2(r - \frac{(M\lambda)'\lambda}{\gamma-1}) + \frac{(\Sigma' M\lambda)'\Sigma' M\lambda}{(\gamma-1)^2})T - 2\int_0^T \phi(u)du}. \quad (59)$$

The variance of the expected value of final wealth of the PPM is obtained as

$$Var(V^*(T)) = v_0^2 \left(e^{(\frac{(\Sigma' M\lambda)'\Sigma' M\lambda}{(\gamma-1)^2})T} - 1 \right) \times e^{(2(r - \frac{(M\lambda)'\lambda}{\gamma-1})T - 2\int_0^T \phi(u)du}. \quad (60)$$

Simplifying (60), we have

$$\sigma(V^*(T)) = v_0 \sqrt{e^{(\frac{(\Sigma' M\lambda)'\Sigma' M\lambda}{(\gamma-1)^2})T} - 1} \times e^{(r - \frac{(M\lambda)'\lambda}{\gamma-1})T - \int_0^T \phi(u)du}. \quad (61)$$

We now express (61) in terms of expected value of final wealth of the PPM as follows:

$$\sigma(V^*(T)) = E(V^*(T)) \sqrt{e^{(\frac{(\Sigma' M\lambda)'\Sigma' M\lambda}{(\gamma-1)^2})T} - 1}. \quad (62)$$

Therefore, the efficient frontier of a PPM's wealth in mean-standard deviation is

$$E(V^*(T)) = \frac{\sigma(V^*(T))}{\sqrt{e^{(\frac{(\Sigma' M\lambda)'\Sigma' M\lambda}{(\gamma-1)^2})T} - 1}}. \quad (63)$$

Figure 12 shows the efficient frontier of the three classes of assets in mean-standard deviation approach. It shows that to have 6 million expected value of wealth, the investor stand the risk of losing 1.9 million. We observe that from (63), we can write

$$v_0 = \frac{\sigma(V^*(T))}{g(T) e^{(\frac{(M\lambda)'\lambda}{\gamma-1} - r)T - \int_0^T \phi(u)du}}, \quad (64)$$

where $g(T) = \sqrt{e^{(\frac{(\Sigma' M\lambda)'\Sigma' M\lambda}{(\gamma-1)^2})T} - 1}$.
It implies that

$$x_0 = \frac{\sigma(V^*(T))}{g(T) e^{(\frac{(M\lambda)'\lambda}{\gamma-1} - r)T - \int_0^T \phi(u)du}} - \Phi_0. \quad (65)$$

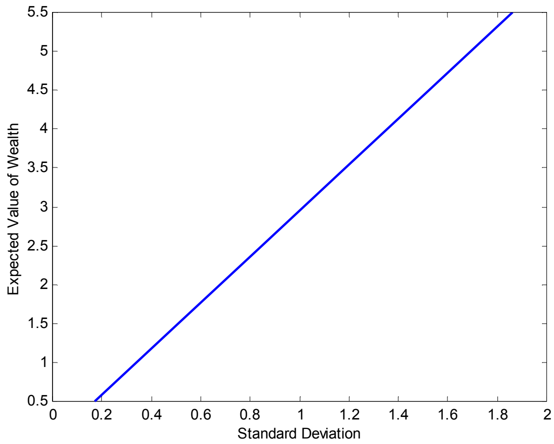

Figure 12: The efficient frontier

We observe from (64) that the initial value of wealth, v_0 can be expressed in terms of the standard deviation of a PPM final value of wealth, the sharpe ratio, $(M\lambda)'\lambda$, coefficient of risk averse utility function γ, consumption rate, ϕ and the retirement date, T. From (65). we found that the initial wealth, x_0 can be expressed in terms of the standard deviation of a PPM final value of wealth, the sharpe ratio, $(M\lambda)'\lambda$, coefficient of risk averse utility function γ, consumption rate, ϕ, the present value of PPM's future discounted contributions, Φ_0 and the retirement date, T.

Proposition 6. *The optimal consumption of the life-cycle of a PPM is*

$$\Psi(t) = \int_0^\infty e^{-rt} E(C^*(t))dt,$$

with

$$E(C^*(t)) = \frac{\gamma(r\gamma - \rho)E(V^*(t))e^{\frac{1}{\gamma-1}(\rho t + (r + \frac{\gamma}{2(\gamma-1)}(\Sigma' M\lambda)'\Sigma' M\lambda)(T-t))}}{\gamma(r\gamma - \rho) - \gamma(\gamma-1)(e^{\frac{-r\gamma(T-t)-\rho t}{\gamma-1}} - e^{\frac{-\rho T}{\gamma-1}})}. \quad (66)$$

From (66), we have

$$\frac{E(C^*(t))}{E(V^*(t))} = \frac{\gamma(r\gamma - \rho)e^{\frac{1}{\gamma-1}(\rho t + (r + \frac{\gamma}{2(\gamma-1)}(\Sigma' M\lambda)'\Sigma' M\lambda)(T-t))}}{\gamma(r\gamma - \rho) - \gamma(\gamma-1)(e^{\frac{-r\gamma(T-t)-\rho t}{\gamma-1}} - e^{\frac{-\rho T}{\gamma-1}})}, \quad (67)$$

is the tradeoff between expected consumption and expected value of wealth. This implies that at time $t \leq T$, the expected consumption is decreasing in expected value of wealth. It implies that there is a linear relationship between expected value of optimal consumption and expected value of optimal wealth. We then say that expected value of optimal consumption varies directly as expected value of optimal wealth, with time varying proportionality of $\frac{\gamma(r\gamma - \rho)e^{\frac{1}{\gamma-1}(\rho t + (r + \frac{\gamma}{2(\gamma-1)}(\Sigma' M\lambda)'\Sigma' M\lambda)(T-t))}}{\gamma(r\gamma - \rho) - \gamma(\gamma-1)(e^{\frac{-r\gamma(T-t)-\rho t}{\gamma-1}} - e^{\frac{-\rho T}{\gamma-1}})}$.

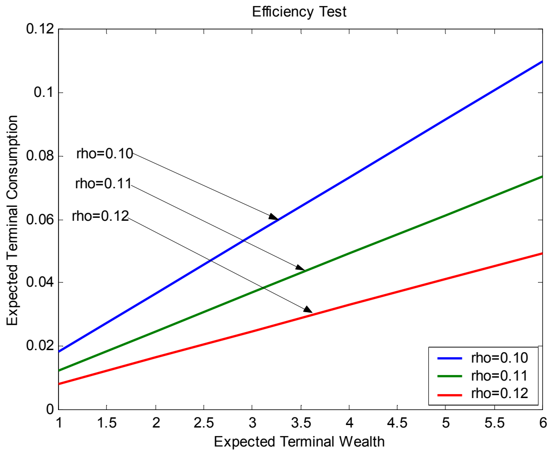

Figure 13: Efficiency Test between Optimal Expected Terminal Consumption and Wealth. This figure was obtained by setting $\gamma = 0.5$ and $T = 20$.

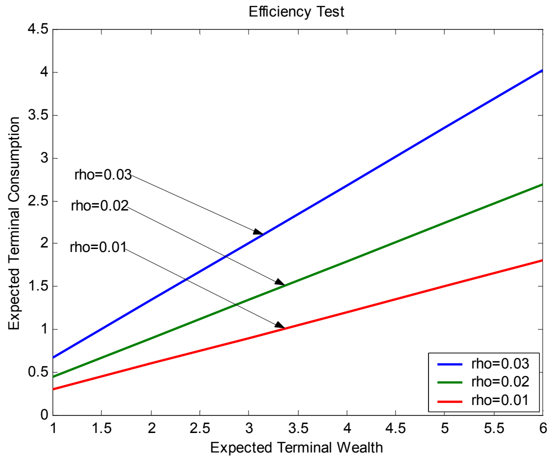

Figure 14: Efficiency Test between Optimal Expected Terminal Consumption and Wealth. This figure was obtained by setting $\gamma = 0.5$ and $T = 20$.

We observe that at $t = T$, (67) becomes

$$E\left(\frac{C^*(T)}{V^*(T)}\right) = \frac{E(C^*(T))}{E(V^*(T))} = e^{\frac{\rho T}{\gamma-1}}. \qquad (68)$$

This shows that $E(C^*(T))$ varies directly as $E(V^*(T))$ with constant of proportionality, $e^{\frac{\rho T}{\gamma-1}}$. This shows the efficiency test between expected terminal consumption, $E(C^*(T))$ and expected value of final wealth, $E(V^*(T))$ with gradient, $e^{\frac{\rho T}{\gamma-1}}$. We observed from (68) that if $\rho = 0$, then $E(C^*(T)) = E(V^*(T))$. Hence, this efficiency test depend on T, ρ and γ. The second moment of (68) is given by

$$E\left(\frac{C^{*2}(T)}{V^{*2}(T)}\right) = \frac{E(C^{*2}(T))}{E(V^{*2}(T))} = e^{\left(\frac{2\rho}{\gamma-1} - \frac{(M\lambda)'\lambda}{(\gamma-1)^2}\right)T}. \qquad (69)$$

The variance is

$$Var\left(\frac{C^*(T)}{V^*(T)}\right) = E\left(\frac{C^{*2}(T)}{V^{*2}(T)}\right) - \left(E\left(\frac{C^*(T)}{V^*(T)}\right)\right)^2$$
$$= e^{\frac{2\rho T}{\gamma-1}}\left(e^{-\frac{(M\lambda)'\lambda}{(\gamma-1)^2}T} - 1\right). \qquad (70)$$

The efficient frontier of the ratio $\dfrac{C^*(T)}{V^*(T)}$ is

$$E\left(\frac{C^*(T)}{V^*(T)}\right) = \frac{1}{\sqrt{e^{-\frac{(M\lambda)'\lambda}{(\gamma-1)^2}T} - 1}}\sigma\left(\frac{C^*(T)}{V^*(T)}\right). \qquad (71)$$

Figure 13 and figure 14 show the efficient test between optimal expected terminal consumption and expected optimal wealth of the PPM at different values of ρ. In figure 13 at a higher value of ρ, we found that expected optimal consumption was at a reduced end. When the value of ρ was reduced as shown in figure 14, expected optimal consumption increases, which confirmed the initial argument in subsection .

XI. CONCLUSION

This paper have studied a continuous-time mean-variance portfolio selection problem with stochastic salary and strategic life-cycle consumption planning in the accumulation phase of a defined contribution pension plan. Index bond was traded and used to protect the investment against inflation risks. Efficient frontier for the three classes of assets that will enable PPMs to decide their own value of wealth and risk in their investment profile at retirement was obtained. The optimal consumption overtime and final consumption of the PPM are established. The variational Merton portfolio processes for the three classes of assets were established. As expected, the numerical example (see table VI) shows that as the investment becomes more risky, PPM will prefer consuming more and invest little or none.

REFERENCES

[1] Battocchio, P. and Menoncin, F. Optimal pension management under stochastic interest rates, wages and inflation. *Insurance: Mathematics and Economics,* 34, 2004, 79-95.

[2] Battocchio, P. Optimal portfolio strategies with stochastic wage income: the case of a defined contribution pension plan. Working paper, IRES, Universite Catholique de Louvain, 2002.

[3] Blake, D., Wright, D. and Zhang, Y. *Optimal funding and investment strategies in defined contribution pension plans under Epstein-Zin utility,* Discussion paper, The pensions Institute, Cass Business School, City University, UK, 2008.

[4] Boulier, J.-F, Huang, S. J. and Taillard, G. Optimal management under stochastic interest rates:

The case of a protected defined contribution pension fund. *Insurance: Mathematics and Economics* 28, 2001, 173-189.

[5] Brawne, S., Milevsky, A. M. and Salisbury, T. S. Asset allocation and the liquidity premium for illiquid annuities. Manuscript, 2003.

[6] Cairns, A.J.G., Blake, D. and Dowd, K. Optimal dynamic asset allocation for defined-contribution pension plans. *In proceedings of the 10th AFIR International Colloquium*, 2000.

[7] Cairns, A.J.G., Blake, D. and Dowd, K. *Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans.* CRIS Discussion paper series, The University of Nottingham, 2003.

[8] Cairns, A.J.G., Blake, D. and Dowd, K. Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans. *Journal of Economic Dynamics and Control* 30, 2004, 843-877.

[9] Cajueiro, D.O. and Yoneyama, T. *Optimal portfolio, Optimal consumption and the Markowitz mean-variance analysis in a switching diffusion market.* vsites.unb.br/face/eco/seminarios/sem0803.pdf, 2003.

[10] Dai, M., Xu, Z. Q. and Zhou, X. Y. Continuous-time Markowitz's model with transaction costs. *SIAM J. Financial Mathematics*, 1, 2010, 96-125.

[11] Deelstra, G., Grasselli, M. and Koehl, P. Optimal investment strategies in a CIR framework. *Journal of Applied Probability* 37, 2000, 936-946.

[12] Deelstra, G., Grasselli, M and Koehl, P. Optimal design of the guarantee for defined contribution funds, Manuscript, 2002.

[13] Deelstra, G., Grasselli, M and Koehl, P. Optimal investment strategies in the presence of a minimum guarantee, Insurance: Mathematics and Economics, 33, 2003, 189-207.

[14] Deelstra, G. Grasselli, M and Koehl, P. Optimal design of the guarantee for defined contribution funds, Journal of Economics Dynamics and Control, 28, 2004, 2239-2260.

[15] Haberman, S. and Vigna, E. Optimal investment strategies and risk measures in defined contribution pension scheme. *Insurance: Mathematics and Economics* 31, 2002, 35-69.

[16] Hϕjgaard, B. and Vigna, E. *Mean-variance portfolio selection and efficient frontier for defined contribution pension schemes*, technical report R-2007-13, Department of Mathematical Sciences, Aalborg University, 2007.

[17] Jensen, B.A. and Sϕrensen, C. Paying for minimum interest guarantees. Who should compensate who? *European financial management* 7, 1999, 183-211.

[18] Josa-Fombellida, R. and, Rincon-Zapatero, J. P. Meanvariance portfolio and contribution selection in stochastic pension funding. *European Journal of Operational Research*, 187 (2008), 120137.

[19] Korn, R. and Krekel, M. Optimal portfolios with fixed consumption or income streams. Working paper, University of Kaiserslautern, 2001.

[20] Korn, R. and Kruse, S. Einfache verfahren zur Bewertung von inflationsgekoppeiten Finanzprodukten. *Bltter der DGVFM, XXVI* 3, 2004, 351-67.

[21] Maurer, R. H., Schlag, C., and Stamos, M. Z. *Optimal life-cycle strategies in the presence of interest rate and inflation risk.* Working paper, Goethe University, Germany, 2008.

[22] Merton, R.C. Lifetime portfolio selection under uncertainty: the continuous-time case. *Review of Economics and Statistics* 51, 1969, 247-257.

[23] Merton, R.C. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic theory* 3, 1971, 373-413.

[24] Nkeki, C. I. (2011) On optimal portfolio management of the accumulation phase of a defined contributory pension scheme, Ph.D thesis, Department of Mathematics, University of Ibadan, Ibadan, Nigeria, 2011.

[25] Nwozo, C. R. and Nkeki, C. I. Optimal portfolio and strategic consumption planning in a life-cycle of a pension plan member in a defined contributory pension scheme, *IAENG International Journal of Applied Mathematics*, 41(4) 2011, 299-309.

[26] Nkeki, C. I. and Nwozo, C. R. Variational form of classical portfolio strategy and expected wealth for a defined contributory pension scheme, *Journal of Mathematical Finance*, 2(1), 2012, 132-139.

[27] Nkeki, C. I. and Nwozo, C. R. Optimal investment under inflation protection and optimal portfolios with stochastic cash flows strategy, *IAENG Journal of Applied Mathematics*, 43(2) 2013, 54-63.

[28] Zhang, A., Korn, R. and Ewald, C. O. Optimal management and inflation protection for defined contribution pension plans, Worker paper, University of St. Andrews, 2007.

[29] Zhang, A., Korn, R. and Christian-Oliver, E. Optimal management and inflation protection for defined contribution pension plan. *MPRA paper no.3300*, 2009.

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