Chaos Control in A 3D Ratio-dependent Food Chain System

Changjin Xu, Qiming Zhang, and Lin Lu

***Abstract*—This paper is devoted to investigate the problem of controlling chaos in a 3D ratio-dependent food chain system. Time-delayed feedback control method is applied to suppress chaos to unstable equilibria or unstable periodic orbits. By local stability analysis, we theoretically prove the occurrences of Hopf bifurcation. Some numerical simulations are presented to support theoretical predictions. Finally, main conclusions are drawn.**

***Index Terms*—food chain system, chaos, stability, Hopf bifurcation, time-delayed feedback.**

I. Introduction

It is well known that chaotic systems play an important role in many fields such as secure communications, information processing, high-performance circuit design for telecommunications and so on [1]. During the last decade, a great deal of methods have been proposed to control chaos, i.e., to stabilize the chaotic dynamical systems to periodic motion, when chaos is not unwanted or undesirable. Recently, many excellent results were reported by Moon [2], Chen and Dong [3], and Kapitaniak [4]. Moreover, numerous outstanding work on chaos control was presented by EI Naschie [5] and Kapitaniak [6-22,30-33]. In 1998, Varriale and Gomes [23] had investigated the dynamics of the following food web consisting of three species

$$\begin{cases} \dot{x}(t) = rx\left(1 - \dfrac{x}{K}\right) - \dfrac{1}{\eta_1}\dfrac{m_1xy}{a_1y + x}, \\ \dot{y}(t) = \dfrac{m_1xy}{a_1y + x} - d_1y - \dfrac{1}{\eta_2}\dfrac{m_2xy}{a_2z + y}, \\ \dot{z}(t) = \dfrac{m_2xy}{a_2z + y} - d_2z, \end{cases} \quad (1)$$

where x, y, z stand for the population density of prey, predator and top predator, respectively. $\eta_i, m_i, a_i, d_i (i = 1, 2)$ are the yield constants, maximal predator growth rates, half-saturation constants and predator's death rates, respectively. r and K are the prey intrinsic growth rate and carrying capacity, respectively.

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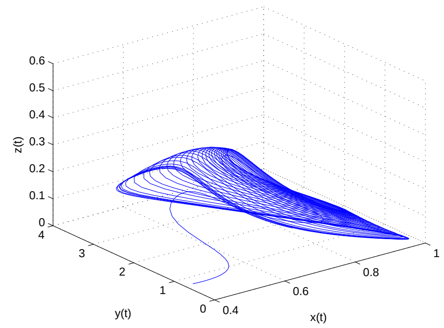

Fig. 1. The new chaotic attractor of system (1.1) with the initial value (0.3433,0.0467,0.0767).

For the sake of simplicity, we make system (1) be non-dimensionality with the following scaling

$$t \to rt, x \to \frac{r}{K}x, y \to \frac{a_1}{K}y, z \to \frac{a_1a_2}{K}z,$$

$$m_1 \to \frac{m_1}{r}, d_1 \to \frac{d_1}{r}, m_2 \to \frac{m_2}{r}, d_2 \to \frac{d_2}{r},$$

then system (1) becomes the form

$$\begin{cases} \dot{x}(t) = rx(1 - x) - \dfrac{c_1xy}{x + y}, \\ \dot{y}(t) = \dfrac{m_1xy}{x + y} - d_1y - \dfrac{c_2xy}{y + z}, \\ \dot{z}(t) = \dfrac{m_2xy}{y + z} - d_2z, \end{cases} \quad (2)$$

where

$$c_1 = \frac{m - 1}{\eta_1a_1r}, c_2 = \frac{m_2}{\eta_2a_2r}.$$

Zhao et al. [29] found that system (2) is chaotic when $m_1 = 10, m_2 = 2, d_1 = d_2 = 1, c_1 = 1, c_2 = 11$, i.e., the following system

$$\begin{cases} \dot{x}(t) = rx(1 - x) - \dfrac{xy}{x + y}, \\ \dot{y}(t) = \dfrac{10xy}{x + y} - y - \dfrac{11xy}{y + z}, \\ \dot{z}(t) = \dfrac{2xy}{y + z} - z \end{cases} \quad (3)$$

is chaotic (see Fig.1).

The aim of this paper is to investigate the dynamics of the 3D ratio-dependent food chain system (3) by considering the effect of delayed feedbacks. By making a detailed analysis on the characteristic equation of linearized system of the model, we theoretically prove that the Hopf bifurcation occur in this model. Numerical results support the theoretical findings.

II. Controlling chaos via feedback control methods

In this section, we shall apply the conventional feedback method to the dynamical system (3). Our aim is to drag the chaotic trajectories to the equilibrium or periodic orbits. Following the idea of Pyragas [24], we add time-delayed force to different equations of system (3). Obviously, system (3) has a unique positive equilibrium $E(x_0, y_0, z_0) = \left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$.

In the following, we consider the dynamical behavior of the positive equilibrium $E(x_0, y_0, z_0)$ of system (3). Now we consider four cases.

Case 1. Delayed feedback on the first equation

In this case, we will investigate the system (3) which the variable x is influenced by the delayed feedback $k(x(t) - x(t-\tau))$, i.e., system (3) takes the form

$$\begin{cases} \dot{x}(t) = rx(1-x) - \dfrac{xy}{x+y} + k(x(t) - x(t-\tau)), \\ \dot{y}(t) = \dfrac{10xy}{x+y} - y - \dfrac{11xy}{y+z}, \\ \dot{z}(t) = \dfrac{2xy}{y+z} - z. \end{cases} \tag{4}$$

The linearized system of Eq.(4) around $E(x_0, y_0, z_0)$ is given by

$$\begin{cases} \dot{x} = (\alpha_1 + k)x + \alpha_2 y - kx(t-\tau), \\ \dot{y} = \alpha_3 x + \alpha_4 y + \alpha_5 z, \\ \dot{z} = \alpha_6 y + \alpha_7 z, \end{cases} \tag{5}$$

where

$$\alpha_1 = 1 - 2x_0 - \frac{y_0^2}{(x_0+y_0)^2}, \alpha_2 = -\frac{x_0^2}{(x_0+y_0)^2},$$

$$\alpha_3 = \frac{10y_0^2}{(x_0+y_0)^2}, \alpha_4 = \frac{10y_0^2}{(x_0+y_0)^2} - 1 - \frac{11z_0^2}{(y_0+z_0)^2},$$

$$\alpha_5 = \frac{11y_0^2}{(y_0+z_0)^2}, \alpha_6 = \frac{2z_0^2}{(y_0+z_0)^2}, \alpha_7 = \frac{2y_0^2}{(y_0+z_0)^2} - 1.$$

The characteristic equation of (5) takes the form

$$\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 + (e_2\lambda^2 + e_1\lambda + e_0)e^{-\lambda\tau} = 0, \tag{6}$$

where

$$\begin{aligned} c_0 &= \alpha_2\alpha_3\alpha_7 - \alpha_4\alpha_7(\alpha_1 + k), \\ c_1 &= (\alpha_1 + k)(\alpha_4 + \alpha_7) + \alpha_4\alpha_7 - \alpha_2\alpha_3 - \alpha_5\alpha_6, \\ c_2 &= -\alpha_1 + k + \alpha_4 + \alpha_7, \\ e_0 &= -k\alpha_4\alpha_7 + \alpha_5\alpha_6(\alpha_1 + k), \\ e_1 &= k(\alpha_4 + \alpha_7) - (\alpha_2\alpha_3 + \alpha_5\alpha_6), e_2 = k. \end{aligned}$$

In the sequel, we will deal with the distribution of roots of the transcendental equation (6).

Lemma 2.1 [25] For the transcendental equation

$$\begin{aligned} & P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}) \\ &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &+ \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right] e^{-\lambda\tau_1} + \cdots \\ &+ \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right] e^{-\lambda\tau_m} = 0, \end{aligned}$$

as $(\tau_1, \tau_2, \tau_3, \cdots, \tau_m)$ vary, the sum of orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.

When $\tau = 0$, (6) has the form

$$\lambda^3 + (c_2 + e_2)\lambda^2 + (c_1 + e_1)\lambda + c_0 + e_0 = 0. \tag{7}$$

It is easy to see that all roots of (7) have a negative real part if the following condition

(H1) $c_2 + e_2 > 0, c_0 + e_0 > 0, (c_2 + e_2)(c_1 + e_1) > c_0 + e_0$

holds. Then the positive equilibrium point $E(x_0, y_0, z_0)$ is locally asymptotically stable when the condition (H1) holds. For $\omega > 0, i\omega$ is a root of (6) if and only if

$$-\omega^3 i - c_2\omega^2 + c_1\omega i + c_0 + (-e_2\omega^2 + e_1\omega i + e_0)e^{-i\omega\tau} = 0.$$

Separating the real and imaginary parts gives

$$\begin{cases} (e_0 - e_2\omega^2)\cos\omega\tau + e_1\omega\sin\omega\tau = c_2\omega^2 - c_0, \\ e_1\omega\cos\omega\tau_1 + (e_0 - e_2\omega^2)\sin\omega\tau = \omega^3 - c_1\omega. \end{cases} \tag{8}$$

It follows from (8) that

$$(e_0 - e_2\omega^2)^2 + (e_1\omega)^2 = (c_2\omega^2 - c_0)^2 + (\omega^3 - c_1\omega)^2,$$

which is equivalent to

$$\omega^6 + p_1\omega^4 + q_1\omega^2 + r_1 = 0, \tag{9}$$

where

$$p_1 = c_2^2 - e_2^2 - 2c_1, q_1 = c_1^2 - 2c_0c_2 - e_1^2 + 2e_0e_2, r_1 = c_0^2 - e_0^2.$$

Denote $z = \omega^2$, then (9) takes the following form

$$z^3 + p_1z^2 + q_1z + r_1 = 0. \tag{10}$$

Let

$$h(z) = z^3 + p_1z^2 + q_1z + r_1. \tag{11}$$

Song et al. [26] obtained the following results on the distribution of roots of Eqs.(6) and (10).

Lemma 2.2. *For the polynomial equation (10),*
(i) If $r_1 < 0$, then Eq.(10) has at least one positive root;
(ii) If $r_1 \geq 0$ and $\Delta_1 = p_1^2 - 3q_1 \leq 0$, then Eq.(10) has no positive roots;
(iii) If $r_1 \geq 0$ and $\Delta_1 = p_1^2 - 3q_1 > 0$, then Eq.(10) has positive if and only if $z_1^* = \frac{-p_1 + \sqrt{\Delta_1}}{3}$ and $h(z_1^*) \leq 0$.

Suppose that Eq.(11) has positive roots. Without loss of generality, we assume that it has three positive roots, denoted by z_1, z_2 and z_3, respectively. Then Eq.(9) has three positive roots

$$\omega_1 = \sqrt{z_1}, \omega_2 = \sqrt{z_2}, \omega_3 = \sqrt{z_3}.$$

By (8), we derive

$$\cos\omega_k\tau = \frac{(c_2\omega_k^2 - c_0)(e_0 - e_2\omega_k^2) - (\omega_k^3 - c_1\omega_k)e_1\omega_k}{(e_0 - e_2\omega_k^2)^2 - (e_1\omega_k)^2}.$$

Thus, if we denote

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left\{ \arccos \left[\frac{(c_2\omega_k^2 - c_0)(e_0 - e_2\omega_k^2) - (\omega_k^3 - c_1\omega_k)e_1\omega_k}{(e_0 - e_2\omega_k^2)^2 - (e_1\omega_k)^2}\right] + 2j\pi \right\}, \tag{12}$$

where $k=1,2,3; j=0,1,2,\cdots$, then $\pm i\omega_k$ are a pair of imaginary roots of Eq.(6) with $\tau_k^{(j)}$. Define

$$\tau_0=\tau_{k0}^{(0)}=\min_{k\in\{1,2,3\}}\{\tau_k^{(0)}\}, \omega_0=\omega_{k0}. \tag{13}$$

The following Lemma 2.3 is taken from Song et al. [26].

Lemma 2.3. *For the third degree exponential polynomial equation (6),we have*
(i) if $r_1\geq 0$ and $\Delta_1=p_1^2-3q_1\leq 0$, then all roots with positive real parts of Eq.(6) has the same sum as those of the polynomial Eq.(8) for all $\tau\geq 0$;
(ii) if either $r_1<0$ or $r_1\geq 0$, $\Delta_1=p_1^2-3q_1>0, z_1^*=\frac{-p_1+\sqrt{\Delta_1}}{3}>0$ and $h(z_1^*)\leq 0$, then all roots with positive real parts of Eq.(6) has the same sum as those of the polynomial Eq.(7) for all $\tau\in[0,\tau_0)$.

Let $\lambda(\tau)=\alpha(\tau)+i\omega(\tau)$ be a root of (6) around $\tau=\tau_0^{(j)}$, and $\alpha(\tau_0^{(j)})=0$ and $\omega(\tau_0^{(j)})=\omega_k$. Differentiating both sides of (6) with respect to τ yields

$$[3\lambda^2+2c_2\lambda+c_1+(2e_2\lambda+e_1-\tau(e_2\lambda^2+e_1\lambda+e_0))e^{-\lambda\tau}]\frac{d\lambda}{d\tau}$$
$$=\lambda e^{-\lambda\tau}(e_2\lambda^2+e_1\lambda+e_0),$$

which gives

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{2\lambda^3+c_2\lambda^2-c_0}{\lambda^2(\lambda^3+c_2\lambda^2+c_1\lambda+c_0)} +\frac{e_2\lambda-e_0}{\lambda^2(e_2\lambda^2+e_1\lambda+e_0)}-\frac{\tau}{\lambda}.$$

Let $\lambda=i\omega_k, \tau=\tau_k^{(j)}$, then we have

$$\left.\left[\frac{d\lambda}{d\tau_1}\right]^{-1}\right|_{\lambda=i\omega_k,\tau=\tau_k^{(j)}} =\frac{-2\omega_k^3 i-c_2\omega_k^2-c_0}{\omega_k^2(c_0-c_2\omega_k^2-i(\omega_k^3-c_1\omega_k))} +\frac{e_2\omega_k^2+e_0}{\omega_k^2(e_0-e_2\omega_k^2+e_1\omega_k i)}-\frac{\tau_k^{(j)}}{i\omega_k}.$$

Then

$$\begin{aligned}\mathrm{Re}&\left\{\left.\left[\frac{d\lambda}{d\tau}\right]^{-1}\right|_{\lambda=i\omega_k,\tau=\tau_k^{(j)}}\right\}\\ &=-\frac{1}{\omega_k^2}\left[\frac{c_0^2-(c_2^2-2c_1)\omega_k^4-2\omega_k^6}{(c_0-c_2\omega_k^2)^2+(\omega_k^3-c_1\omega_k)^2}\right.\\ &\quad\left.-\frac{e_0^2-e_2^2\omega_k^4}{e_1^2\omega_k^2+(e_0-e_2\omega_k^2)^2}\right]\\ &=\frac{2\omega_k^6+p\omega_k^4-r}{\omega_k^2(e_0-e_2\omega_k^2)^2+e_1^2\omega_k^2}\\ &=\frac{3\omega_k^4+2p\omega_k^2+q}{(e_0-e_2\omega_k^2)^2+e_1^2\omega_k^2},\end{aligned}$$

where Re{.} is the real part of $\cdot$. We assume that the following condition holds.

$$\text{(H2) } 3\omega_k^4+2p_1\omega_k^2+q_1\neq 0.$$

According to above analysis and the results of Kuang [27] and Hale [28], we have

Theorem 2.4. If (H1) holds, then the positive equilibrium $E(x_0,y_0,z_0)$ of system (4) is asymptotically stable for $\tau\in[0,\tau_0)$. Under the conditions (H1), if the condition (H2) holds, then system (4) undergoes a Hopf bifurcation at the positive equilibrium $E(x_0,y_0,z_0)$ when $\tau=\tau_0^{(j)}, j=0,1,2,\cdots$.

Case 2. Delayed feedback on the second equation

In this case, we will investigate the system (3) which the variable y is influenced by the delayed feedback$k(y(t)-y(t-\tau))$, i.e., system (3) takes the form

$$\begin{cases}\dot{x}(t)=rx(1-x)-\dfrac{xy}{x+y},\\ \dot{y}(t)=\dfrac{10xy}{x+y}-y-\dfrac{11xy}{y+z}\\ \qquad +k(y(t)-y(t-\tau)),\\ \dot{z}(t)=\dfrac{2xy}{y+z}-z.\end{cases} \tag{14}$$

The linearized system of Eq.(14) around $E(x_0,y_0,z_0)$ is given by

$$\begin{cases}\dot{x}=\alpha_1 x+\alpha_2 y,\\ \dot{y}=\alpha_3 x+(\alpha_4+k)y+\alpha_5 z-ky(t-\tau),\\ \dot{z}=\alpha_6 y+\alpha_7 z.\end{cases} \tag{15}$$

The characteristic equation of (15) takes the form

$$\lambda^3+f_2\lambda^2+f_1\lambda+f_0+(g_2\lambda^2+g_1\lambda+g_0)e^{-\lambda\tau}=0, \tag{16}$$

where

$$\begin{aligned}f_0&=\alpha_2\alpha_3\alpha_7+\alpha_1\alpha_5\alpha_6+\alpha_1\alpha_7(\alpha_4+k),\\ f_1&=(\alpha_4+k)(\alpha_1+\alpha_7)+\alpha_1\alpha_7-\alpha_2\alpha_3-\alpha_5\alpha_6,\\ f_2&=-(\alpha_1+k+\alpha_4+\alpha_7), g_0=k\alpha_1\alpha_7,\\ g_1&=k(\alpha_1+\alpha_7)-(\alpha_2\alpha_3+\alpha_5\alpha_6), g_2=k.\end{aligned}$$

Next, we will analyze the distribution of roots of the transcendental equation (16).

When $\tau=0$, (16) has the form

$$\lambda^3+(f_2+g_2)\lambda^2+(f_1+g_1)\lambda+f_0+g_0=0. \tag{17}$$

It is easy to see that all roots of (17) have a negative real part if the following condition

(H3) $f_2+g_2>0, f_0+g_0>0, (f_2+g_2)(f_1+g_1)>f_0+g_0$

holds. Then the positive equilibrium point $E(x_0,y_0,z_0)$ is locally asymptotically stable when the condition (H3) holds. For $\tilde{\omega}>0, i\tilde{\omega}$ is a root of (16) if and only if

$$-\tilde{\omega}^3 i-f_2\tilde{\omega}^2+f_1\tilde{\omega}i+f_0+(-g_2\tilde{\omega}^2+g_1\tilde{\omega}i+g_0)e^{-i\tilde{\omega}\tau}=0.$$

Separating the real and imaginary parts gives

$$\begin{cases}(g_0-g_2\tilde{\omega}^2)\cos\tilde{\omega}\tau+g_1\tilde{\omega}\sin\tilde{\omega}\tau=f_2\tilde{\omega}^2-f_0,\\ g_1\tilde{\omega}\cos\tilde{\omega}\tau_1+(g_0-g_2\tilde{\omega}^2)\sin\tilde{\omega}\tau=\tilde{\omega}^3-f_1\omega.\end{cases} \tag{18}$$

It follows from (18) that

$$(g_0-g_2\tilde{\omega}^2)^2+(g_1\tilde{\omega})^2=(f_2\tilde{\omega}^2-f_0)^2+(\tilde{\omega}^3-f_1\tilde{\omega})^2,$$

which is equivalent to

$$\tilde{\omega}^6+\tilde{p}_1\tilde{\omega}^4+\tilde{q}_1\tilde{\omega}^2+\tilde{r}_1=0, \tag{19}$$

where

$$\tilde{p}_1=f_2^2-g_2^2-2f_1, \tilde{q}_1=f_1^2-2f_0f_2-g_1^2+2g_0g_2, \tilde{r}_1=f_0^2-g_0^2.$$

Denote $\tilde{z} = \tilde{\omega}^2$, then (19) takes the following form

$$\tilde{z}^3 + \tilde{p}_1\tilde{z}^2 + \tilde{q}_1\tilde{z} + \tilde{r}_1 = 0. \tag{20}$$

Let

$$\tilde{h}(\tilde{z}) = \tilde{z}^3 + \tilde{p}_1\tilde{z}^2 + \tilde{q}_1\tilde{z} + \tilde{r}_1. \tag{21}$$

Song et al. [26] obtained the following results on the distribution of roots of Eqs.(16) and (20).

Lemma 2.5. *For the polynomial equation (20),*
(i) if $\tilde{r}_1 < 0$, then Eq.(20) has at least one positive root;
(ii) if $\tilde{r}_1 \geq 0$ and $\tilde{\Delta}_1 = \tilde{p}_1^2 - 3\tilde{q}_1 \leq 0$, then Eq.(20) has no positive roots;
(iii) if $\tilde{r}_1 \geq 0$ and $\tilde{\Delta}_1 = \tilde{p}_1^2 - 3\tilde{q}_1 > 0$, then Eq.(20) has positive if and only if $\tilde{z}_1^* = \frac{-\tilde{p}_1+\sqrt{\tilde{\Delta}_1}}{3}$ and $\tilde{h}(\tilde{z}_1^*) \leq 0$.

Suppose that Eq.(21) has positive roots. Without loss of generality, we assume that it has three positive roots, denoted by $\tilde{z}_1, \tilde{z}_2$ and $\tilde{z}_3$, respectively. Then Eq.(19) has three positive roots

$$\tilde{\omega}_1 = \sqrt{\tilde{z}_1}, \tilde{\omega}_2 = \sqrt{\tilde{z}_2}, \tilde{\omega}_3 = \sqrt{\tilde{z}_3}.$$

By (18), we derive

$$\cos\omega_k\tau = \frac{(f_2\tilde{\omega}_k^2 - f_0)(g_0 - g_2\tilde{\omega}_k^2) - (\tilde{\omega}_k^3 - f_1\tilde{\omega}_k)g_1\tilde{\omega}_k}{(g_0 - g_2\tilde{\omega}_k^2)^2 - (g_1\tilde{\omega}_k)^2}.$$

Thus, if we denote

$$\tau_k^{(j)} = \frac{1}{\tilde{\omega}_k}\Bigg\{ \arccos \left[\frac{(f_2\tilde{\omega}_k^2 - f_0)(g_0 - g_2\tilde{\omega}_k^2) - (\tilde{\omega}_k^3 - f_1\tilde{\omega}_k)g_1\tilde{\omega}_k}{(g_0 - g_2\tilde{\omega}_k^2)^2 - (g_1\tilde{\omega}_k)^2}\right] + 2j\pi\Bigg\}, \tag{22}$$

where $k = 1, 2, 3; j = 0, 1, 2, \cdots$, then $\pm i\tilde{\omega}_k$ are a pair of imaginary roots of Eq.(6) with $\tau_k^{(j)}$. Define

$$\tau_0 = \tau_{k0}^{(0)} = \min_{k\in\{1,2,3\}}\{\tau_k^{(0)}\}, \tilde{\omega}_0 = \tilde{\omega}_{k0}. \tag{23}$$

The following Lemma 2.6 is taken from Song et al. [26].

Lemma 2.6. *For the third degree exponential polynomial equation (16),we have*
(i) if $\tilde{r}_1 \geq 0$ and $\tilde{\Delta}_1 = \tilde{p}_1^2 - 3\tilde{q}_1 \leq 0$, then all roots with positive real parts of Eq.(16) has the same sum as those of the polynomial Eq.(17) for all $\tau \geq 0$;
(ii) if either $\tilde{r}_1 < 0$ or $\tilde{r}_1 \geq 0$, $\tilde{\Delta}_1 = \tilde{p}_1^2 - 3\tilde{q}_1 > 0, \tilde{z}_1^* = \frac{-\tilde{p}_1+\sqrt{\tilde{\Delta}_1}}{3} > 0$ and $\tilde{h}(\tilde{z}_1^*) \leq 0$, then all roots with positive real parts of Eq.(16) has the same sum as those of the polynomial Eq.(17) for all $\tau \in [0, \tau_0)$.

Let $\lambda(\tau) = \tilde{\alpha}(\tau) + i\tilde{\omega}(\tau)$ be a root of (16) around $\tau = \tau_0^{(j)}$, and $\tilde{\alpha}(\tau_0^{(j)}) = 0$ and $\tilde{\omega}(\tau_0^{(j)}) = \tilde{\omega}_k$. Differentiating both sides of (16) with respect to τ yields

$$[3\lambda^2 + 2f_2\lambda + f_1 + (2g_2\lambda + g_1 - \tau(g_2\lambda^2 + g_1\lambda + g_0))e^{-\lambda\tau}]\frac{d\lambda}{d\tau} = \lambda e^{-\lambda\tau}(g_2\lambda^2 + g_1\lambda + g_0),$$

which gives

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{2\lambda^3 + f_2\lambda^2 - f_0}{\lambda^2(\lambda^3 + f_2\lambda^2 + f_1\lambda + f_0)} + \frac{g_2\lambda - g_0}{\lambda^2(g_2\lambda^2 + g_1\lambda + g_0)} - \frac{\tau}{\lambda}.$$

Let $\lambda = i\tilde{\omega}_k, \tau = \tau_k^{(j)}$, then we have

$$\begin{aligned}\left[\frac{d\lambda}{d\tau}\right]^{-1}\Bigg|_{\lambda=i\tilde{\omega}_k,\tau=\tau_k^{(j)}} &= \frac{-2\tilde{\omega}_k^3 i - f_2\tilde{\omega}_k^2 - f_0}{\tilde{\omega}_k^2(f_0 - f_2\tilde{\omega}_k^2 - i(\omega_k^3 - f_1\tilde{\omega}_k))} \\ &+ \frac{g_2\tilde{\omega}_k^2 + g_0}{\tilde{\omega}_k^2(g_0 - g_2\tilde{\omega}_k^2 + g_1\tilde{\omega}_k i)} - \frac{\tau_k^{(j)}}{i\tilde{\omega}_k}.\end{aligned}$$

Then

$$\begin{aligned}&\mathrm{Re}\left\{\left[\frac{d\lambda}{d\tau}\right]^{-1}\Bigg|_{\lambda=i\tilde{\omega}_k,\tau=\tau_k^{(j)}}\right\} \\ &= -\frac{1}{\tilde{\omega}_k^2}\left[\frac{f_0^2 - (f_2^2 - 2f_1)\tilde{\omega}_k^4 - 2\tilde{\omega}_k^6}{(f_0 - f_2\tilde{\omega}_k^2)^2 + (\tilde{\omega}_k^3 - f_1\tilde{\omega}_k)^2} - \frac{g_0^2 - g_2^2\tilde{\omega}_k^4}{g_1^2\tilde{\omega}_k^2 + (g_0 - g_2\tilde{\omega}_k^2)^2}\right] \\ &= \frac{2\omega_k^6 + p\tilde{\omega}_k^4 - r}{\tilde{\omega}_k^2(g_0 - g_2\tilde{\omega}_k^2)^2 + g_1^2\tilde{\omega}_k^2} \\ &= \frac{3\tilde{\omega}_k^4 + 2p\tilde{\omega}_k^2 + q}{(g_0 - g_2\tilde{\omega}_k^2)^2 + g_1^2\tilde{\omega}_k^2}.\end{aligned}$$

We assume that the following condition holds.

(H4) $3\tilde{\omega}_k^4 + 2p_1\tilde{\omega}_k^2 + q_1 \neq 0.$

According to above analysis and the results of Kuang [27] and Hale [28], we have

Theorem 2.7. If (H3) holds, then the positive equilibrium $E(x_0, y_0, z_0)$ of system (14) is asymptotically stable for $\tau \in [0, \tau_0)$. Under the condition (H3), if the condition (H4) holds, then system (14) undergoes a Hopf bifurcation at the positive equilibrium $E(x_0, y_0, z_0)$ when $\tau = \tau_0^{(j)}, j = 0, 1, 2, \cdots$.

Case 3. Delayed feedback on the third equation

In this case, we will investigate the system (3) which the variable z is influenced by the delayed feedback $k(z(t) - z(t-\tau))$, then system (3) takes the form

$$\begin{cases}\dot{x}(t) = rx(1-x) - \dfrac{xy}{x+y}, \\ \dot{y}(t) = \dfrac{10xy}{x+y} - y - \dfrac{11xy}{y+z}, \\ \dot{z}(t) = \dfrac{2xy}{y+z} - z + k(z(t) - z(t-\tau)).\end{cases} \tag{24}$$

The linearized system of Eq.(24) around $E(x_0, y_0, z_0)$ is given by

$$\begin{cases}\dot{x} = \alpha_1 x + \alpha_2 y, \\ \dot{y} = \alpha_3 x + (\alpha_4 + k)y + \alpha_5 z, \\ \dot{z} = \alpha_6 y + (\alpha_7 + k)z - kz(t-\tau).\end{cases} \tag{25}$$

The characteristic equation of (25) takes the form

$$\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + n_0 e^{-\lambda\tau} = 0, \tag{26}$$

where

$$\begin{aligned} m_0 &= (\alpha_2\alpha_3 - \alpha_1\alpha_4)(\alpha_7 + k) + \alpha_1\alpha_5\alpha_6, \\ m_1 &= \alpha_1\alpha_4 + (\alpha_1 + \alpha_4)(\alpha_7 + k) \\ &\quad -k(\alpha_1 + \alpha_4) + \alpha_5\alpha_6, \\ m_2 &= -(\alpha_1 + \alpha_4 + \alpha_7 + 2k), n_0 = k(\alpha_1\alpha_4 + \alpha_2\alpha_3).\end{aligned}$$

When $\tau = 0$, (26) has the form

$$\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + n_0 = 0. \tag{27}$$

It is easy to see that all roots of (27) have a negative real part if the following condition

$$\text{(H5) } m_2 > 0, m_0 + n_0 > 0, m_1 m_2 > m_0 + n_0$$

is satisfied. Then the positive equilibrium point $E(x_0, y_0, z_0)$ is locally asymptotically stable when the condition (H5) holds.

For $\bar{\omega} > 0, i\bar{\omega}$ is a root of (26) if and only if

$$-\bar{\omega}^3 i - m_2\bar{\omega}^2 + m_1\bar{\omega}i + m_0 + n_0 e^{-\bar{\omega}\tau i} = 0.$$

Separating the real and imaginary parts gives

$$\begin{cases} n_0 \cos\bar{\omega}\tau = m_2\bar{\omega}^2 - m_0, \\ n_0 \sin\bar{\omega}\tau = m_1\bar{\omega} - \bar{\omega}^3, \end{cases} \tag{28}$$

which is equivalent to

$$\bar{\omega}^6 + \theta_2\bar{\omega}^4 + \theta_1\bar{\omega}^2 + \theta_0 = 0, \tag{29}$$

where

$$\theta_0 = m_0^2 - n_0^2, \theta_1 = m_1^2 - 2m_0 m_2, \theta_2 = m_2^2 - 2m_1.$$

Denote $\bar{z} = \bar{\omega}^2$, then (29) takes the following form

$$\bar{z}^3 + \theta_2\bar{z}^2 + \theta_1\bar{z} + \theta_0 = 0. \tag{30}$$

Let

$$\bar{h}(\bar{z}) = \bar{z}^3 + \theta_2\bar{z}^2 + \theta_1\bar{z} + \theta_0. \tag{31}$$

Since $\theta_0 < 0$ and $\lim_{\bar{z}\to+\infty} \bar{h}(\bar{z}) = +\infty$, then (31) has at least one positive root.

Without loss of generality, we assume that (31) has three positive roots, denoted by $\bar{z}_1, \bar{z}_2, \bar{z}_3$, respectively. Then Eq.(29) has three positive roots

$$\bar{\omega}_1 = \sqrt{\bar{z}_1}, \bar{\omega}_2 = \sqrt{\bar{z}_2}, \bar{\omega}_3 = \sqrt{\bar{z}_3}.$$

If we denote

$$\tau_k^{(j)} = \frac{1}{\bar{\omega}_k}\left[\arccos\left(\frac{m_2\bar{\omega}^2 - m_0}{n_0}\right) + 2j\pi\right], \tag{32}$$

where $k = 1, 2, 3; j = 0, 1, 2, \cdots$, then $\pm i\bar{\omega}_k$ are a pair of imaginary roots of Eq.(26) with $\tau_k^{(j)}$. Define

$$\tau_0 = \tau_{k0}^{(0)} = \min_{k\in\{1,2,3\}}\{\tau_k^{(0)}\}, \bar{\omega}_0 = \bar{\omega}_{k0}. \tag{33}$$

Let $\lambda(\tau) = \bar{\alpha}(\tau) + i\bar{\omega}(\tau)$ be a root of (26) around $\tau = \tau_0^{(j)}$, and $\bar{\alpha}(\tau_0^{(j)}) = 0$ and $\bar{\omega}(\tau_0^{(j)}) = \bar{\omega}_k$. Differentiating both sides of (26) with respect to τ yields

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{(3\lambda^2 + 2m_2\lambda + m_1)e^{\lambda\tau}}{n_0\lambda} - \frac{\tau}{\lambda}.$$

Let $\lambda = i\bar{\omega}_k, \tau = \tau_k^{(j)}$, then we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}\Bigg|_{\lambda=i\bar{\omega}_k,\tau=\tau_k^{(j)}} = \frac{\delta_1 + i\delta_2}{(n_0\bar{\omega}_k)^2} - \frac{\tau_k^{(j)}}{i\bar{\omega}_k}.$$

where

$$\delta_1 = \left[(m_1 - 3\bar{\omega}_k^2)\sin\bar{\omega}_k\tau_k^{(j)} + 2m_2\bar{\omega}_k\cos\bar{\omega}_k\tau_k^{(j)}\right] n_0\bar{\omega}_k,$$

$$\delta_2 = \left[2m_2\bar{\omega}_k\sin\bar{\omega}_k\tau_k^{(j)} - (m_1 - 3\bar{\omega}_k^2)\cos\bar{\omega}_k\tau_k^{(j)}\right] n_0\bar{\omega}_k.$$

Then

$$\mathrm{Re}\left\{\left[\frac{d\lambda}{d\tau}\right]^{-1}\Bigg|_{\lambda=i\omega_k,\tau=\tau_k^{(j)}}\right\} = \mathrm{Re}\left[\frac{\delta_1 + i\delta_2}{(n_0\bar{\omega}_k)^2}\right] = \frac{\delta_1}{(n_0\bar{\omega}_k)^2},$$

We assume that the following condition holds.

$$\text{(H6) } \delta_1 \neq 0.$$

Based on above analysis and the results of Kuang [20] and Hale [21], we have

Theorem 2.8. If (H5) holds, then the positive equilibrium $E(x_0, y_0, z_0)$ of system (24) is asymptotically stable for $\tau \in [0, \tau_0)$. Under the condition (H5), if the condition (H6) holds, then system (24) undergoes a Hopf bifurcation at the positive equilibrium $E(x_0, y_0, z_0)$ when $\tau = \tau_0^{(j)}, j = 0, 1, 2, \cdots$.

Case 4. Delayed feedback on the three equations

In this case, we will investigate the system (3) which the variables x, y and z are influenced by the delayed feedback $k(x(t)-x(t-\tau)), k(y(t)-y(t-\tau))$ and $k(z(t)-z(t-\tau))$, respectively, then system (3) takes the form

$$\begin{cases} \dot{x}(t) = rx(1-x) - \dfrac{xy}{x+y} + k(x(t) - x(t-\tau)), \\ \dot{y}(t) = \dfrac{10xy}{x+y} - y - \dfrac{11xy}{y+z} + k(y(t) - y(t-\tau)), \\ \dot{z}(t) = \dfrac{2xy}{y+z} - z + k(z(t) - z(t-\tau)). \end{cases} \tag{34}$$

The linearized system of Eq.(34) around $E(x_0, y_0, z_0)$ is given by

$$\begin{cases} \dot{x} = k_1 x - kx(t-\tau) + y, \\ \dot{y} = k_1 x + z - kx(t-\tau), \\ \dot{z} = -2x^* x - y - x^* z. \end{cases} \tag{35}$$

The characteristic equation of (35) takes the form

$$\begin{aligned} &\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + (n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau} \\ &+(s_1\lambda + s_0)e^{-2\lambda\tau} + t_0 e^{-3\lambda\tau} = 0, \end{aligned} \tag{36}$$

where

$$\begin{aligned} m_0 &= (\alpha_1 + k)^2(\alpha_4 + k) + \alpha_2\alpha_3(\alpha_1 + k) \\ &\quad -\alpha_5\alpha_6(\alpha_1 + k), \\ m_1 &= (\alpha_1 + k)(\alpha_4 + k) - \alpha_2\alpha_3 + \alpha_5\alpha_5 \\ &\quad +(\alpha_1 + k)(\alpha_1 + \alpha_4 + 2k), \\ m_2 &= -(2\alpha_1 + \alpha_4 + 3k), n_2 = 3k^2, t_0 = k^3, \\ n_0 &= k(\alpha_1 + k)(\alpha_1 + \alpha_4 + 2k) + k(\alpha_1 + k)(\alpha_4 + k) \\ &\quad -\alpha_2\alpha_3 k + \alpha_5\alpha_5 k, \\ n_1 &= 2k(\alpha_1 + k) - 2k(\alpha_1 + \alpha_4 + 2k). \end{aligned}$$

Multiplying $e^{\lambda\tau}$ on both sides of (36), it is obvious to obtain

$$\begin{aligned} &(\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0)e^{\lambda\tau} + (n_2\lambda^2 + n_1\lambda + n_0) \\ &+(s_1\lambda + s_0)e^{-\lambda\tau} + t_0 e^{-2\lambda\tau} = 0. \end{aligned} \tag{37}$$

Now we will analyze the distribution of roots of the transcendental equation (37).

When $\tau = 0$, (37) has the form

$$\lambda^3 + (m_2 + n_2)\lambda^2 + (m_1 + n_1 + s_1)\lambda + m_0 + n_0 + s_0 + t_0 = 0. \tag{38}$$

It is easy to see that all roots of (38) have a negative real part if the following condition

(H7) $m_2+n_2>0, m_0+n_0+s_0+t_0>0,$

$$(m_2+n_2)(m_1+n_1+s_1)>m_0+n_0+s_0+t_0$$

is satisfied. Then the equilibrium point $E(x_0,y_0,z_0)$ is locally asymptotically stable when the condition (H7) holds.

For $\hat{\omega}>0, i\hat{\omega}$ is a root of (37) if and only if

$$(-\hat{\omega}^3 i - m_2\hat{\omega}^2 + m_1\hat{\omega}i + m_0)e^{\hat{\omega}\tau i} + t_0 e^{-2\hat{\omega}\tau i}$$
$$+(-n_2\hat{\omega}^2 + n_1\hat{\omega}i + n_0) + (s_1\hat{\omega}i + s_0)e^{-\hat{\omega}\tau i} = 0.$$

Separating the real and imaginary parts gives

$$\begin{cases} \rho_1\cos\hat{\omega}\tau + \varrho_1\sin\hat{\omega}\tau + \sigma_1 = -t_0\cos 2\hat{\omega}\tau, \\ \rho_2\cos\hat{\omega}\tau + \varrho_2\sin\hat{\omega}\tau + \sigma_2 = -t_0\sin 2\hat{\omega}\tau, \end{cases} \tag{39}$$

where

$$\begin{aligned} \rho_1 &= m_0 - m_2\hat{\omega}^2 + s_0, \varrho_1 = s_1\hat{\omega} - m_1\hat{\omega} + \hat{\omega}^3, \\ \sigma_1 &= n_0 - n_2\bar{\omega}^2, \rho_2 = s_1\hat{\omega} + m_1\bar{\omega} - \bar{\omega}^3, \\ \varrho_2 &= m_0 - m_2\hat{\omega}^2 - s_0, \sigma_2 = n_1\hat{\omega}. \end{aligned}$$

According to $\sin\hat{\omega}\tau = \pm\sqrt{1-\cos^2\hat{\omega}\tau}$, then (39) takes the following form:

$$\begin{aligned} &[\rho_1\cos\hat{\omega}\tau + \varrho_1\sin\hat{\omega}\tau + \sigma_1]^2 \\ &+[\rho_2\cos\hat{\omega}\tau + \varrho_2\sin\hat{\omega}\tau + \sigma_2]^2 = t_0^2. \end{aligned} \tag{40}$$

It is easy to see that (40) is equivalent to

$$q_1\cos^4\hat{\omega}\tau + q_2\cos^3\hat{\omega}\tau + q_3\cos^2\hat{\omega}\tau + q_4\cos\hat{\omega}\tau + q_5 = 0, \tag{41}$$

where

$$\begin{aligned} q_1 &= (\rho_1^2+\rho_2^2-\varrho_1^2-\varrho_2^2+4(\rho_1\varrho_1+\rho_2\varrho_2)^2, \\ q_2 &= 4(\rho_1\sigma_1+\rho_2\sigma_2)(\rho_1^2+\rho_2^2-\varrho_1^2-\varrho_2^2) \\ &\quad +4(\rho_1\varrho_1+\rho_2\varrho_2), \\ q_3 &= [4(\rho_1\sigma_1+\rho_2\sigma_2)^2+2(\sigma_1^2+\sigma_2^2-t_0^2) \\ &\quad \times(\rho_1^2+\rho_2^2-\varrho_1^2-\varrho_2^2)-4(\rho_1\varrho_1+\rho_2\varrho_2)^2 \\ &\quad +(\varrho_1\sigma_1+\varrho_2\sigma_2)^2]^2, \\ q_4 &= 4(\sigma_1^2+\sigma_2^2-t_0^2)(\rho_1\sigma_1+\rho_2\sigma_2) \\ &\quad +8(\varrho_1\sigma_1+\varrho_2\sigma_2)(\rho_1\varrho_1+\rho_2\varrho_2), \\ q_5 &= (\sigma_1^2+\sigma_2^2-t_0^2)^2-4(\varrho_1\sigma_1+\varrho_2\sigma_2)^2. \end{aligned}$$

Let $\cos\omega\tau = r$ and denote

$$l(r) = r^4 + \frac{q_2}{q_1}r^3 + \frac{q_3}{q_1}r^2 + \frac{q_4}{q_1}r + \frac{q_5}{q_1}.$$

It is easy to obtain that

$$\frac{l(r)}{dr} = 4r^3 + \frac{3q_2}{q_1}r^2 + \frac{2q_3}{q_1}r + \frac{q_4}{q_1}.$$

Set

$$4r^3 + \frac{3q_2}{q_1}r^2 + \frac{2q_3}{q_1}r + \frac{q_4}{q_1} = 0. \tag{42}$$

Let $y = r + \frac{q_2}{4q_1}$. Then Eq.(42) becomes

$$y^3 + \gamma_1 y + \gamma_2 = 0, \tag{43}$$

where $\gamma_1 = \frac{q_3}{2q_1} - \frac{3q_2^2}{16q_1^2}, \gamma_2 = \frac{q_2^3}{32q_1^3} - \frac{q_2q_3}{8q_1^2} + \frac{q_4}{4q_1}$.

Define $\beta_1 = \left(\frac{\gamma_2}{2}\right)^2 + \left(\frac{\gamma_1}{3}\right)^3, \beta_2 = \frac{-1+i\sqrt{3}}{2}$. By (2.40), Then we obtain

$$\begin{aligned} y_1 &= \sqrt[3]{-\frac{\gamma_2}{2}+\sqrt{\beta_1}} + \sqrt[3]{-\frac{\gamma_2}{2}-\sqrt{\beta_1}}, \\ y_2 &= \sqrt[3]{-\frac{\gamma_2}{2}+\sqrt{\beta_1}\beta_2} + \sqrt[3]{-\frac{\gamma_2}{2}-\sqrt{\beta_1}\beta_2^2}, \\ y_3 &= \sqrt[3]{-\frac{\gamma_2}{2}+\sqrt{\beta_1}\beta_2^2} + \sqrt[3]{-\frac{\gamma_2}{2}-\sqrt{\beta_1}\beta_2}. \end{aligned}$$

By the discussion above, we can obtain the expression of $\cos\omega\tau$, say

$$\cos\hat{\omega}\tau = f_1(\hat{\omega}), \tag{44}$$

where $f_1(\hat{\omega})$ is a function with respect to $\hat{\omega}$. Substitute (44) into (40), then we can get the expression of $\sin\omega\tau$, say

$$\sin\hat{\omega}\tau = f_2(\hat{\omega}), \tag{45}$$

where $f_2(\hat{\omega})$ is a function with respect to $\hat{\omega}$. Thus we obtain

$$f_1{}^2(\hat{\omega}) + f_2{}^2(\hat{\omega}) = 1. \tag{46}$$

If all the coefficients of the system (3) are given, it is easy to use computer to calculate the roots of (46) (say $\hat{\omega}$). Then from (44), we derive

$$\tau^{(k)} = \frac{1}{\hat{\omega}}[\arccos f_1(\hat{\omega}) + 2k\pi] \ (k=0,1,2,\cdots). \tag{47}$$

Define $\tau_0 = \min\{\tau^{(k)}\}$. Let $\lambda(\tau) = \hat{\alpha}(\tau) + i\hat{\omega}(\tau)$ be a root of (37) around $\tau = \tau_0^{(j)}$, and $\hat{\alpha}(\tau_0^{(j)}) = 0$ and $\hat{\omega}(\tau_0^{(j)}) = \hat{\omega}_k$. Differentiating both sides of (37) with respect to τ yields

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{\Theta_1}{\Theta_2} - \frac{\tau}{\lambda},$$

where

$$\begin{aligned} \Theta_1 &= (3\lambda^2 + 2m_2\lambda + m_1)e^{\lambda\tau} \\ &\quad +2n_2\lambda + n_1 + s_1e^{-\lambda\tau}, \\ \Theta_2 &= -\lambda(\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0)e^{\lambda\tau} \\ &\quad +\lambda(s_1\lambda + s_0)e^{-\lambda\tau} + 2t_0e^{-2\lambda\tau}\lambda. \end{aligned}$$

Let $\lambda = i\hat{\omega}_k, \tau = \tau_k^{(j)}$, then we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}\Bigg|_{\lambda=i\hat{\omega}_k,\tau=\tau_k^{(j)}} = \frac{\chi_1+i\chi_2}{\chi_3+i\chi_4} - \frac{\tau_k^{(j)}}{i\hat{\omega}_k}.$$

where

$$\begin{aligned} \chi_1 &= (m_1 - 3\hat{\omega}_k^2 + s_1)\cos\hat{\omega}_k\tau_k^{(j)} - 2m_2\hat{\omega}_k\cos\hat{\omega}_k\tau_k^{(j)}, \\ \chi_2 &= 2m_2\hat{\omega}_k^2\cos\hat{\omega}_k\tau_k^{(j)} + (m_1 - 3\hat{\omega}_k^2 - s_1)\sin\hat{\omega}_k\tau_k^{(j)} \\ &\quad +2n_2\hat{\omega}_k, \\ \chi_3 &= \hat{\omega}_k[(n_1\hat{\omega}_k - \hat{\omega}_k^3 - s_1\hat{\omega}_k)\cos\hat{\omega}_k\tau_k^{(j)} \\ &\quad +(m_0 - m_1\hat{\omega}_k^2 + s_0)\sin\hat{\omega}_k\tau_k^{(j)} + 2t_0\cos 2\hat{\omega}_k\tau_k^{(j)}], \\ \chi_2 &= \hat{\omega}_k[(s_0 - m_0 + m_2\hat{\omega}_k^2)\cos\hat{\omega}_k\tau_k^{(j)}, \\ &\quad +(s_1\hat{\omega}_k + n_1\hat{\omega}_k - \hat{\omega}_k^3)\sin\hat{\omega}_k\tau_k^{(j)} - 2t_0\sin 2\hat{\omega}_k\tau_k^{(j)}]. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{Re}\left\{\left[\frac{d\lambda}{d\tau}\right]^{-1}\Bigg|_{\lambda=i\omega_k,\tau=\tau_k^{(j)}}\right\} &= \mathrm{Re}\left\{\frac{\chi_1+i\chi_2}{\chi_3+i\chi_4}\right\} \\ &= \frac{\chi_1\chi_3+\chi_2\chi_4}{\chi_1^2+\chi_2^2}, \end{aligned}$$

We assume that the following condition holds.

$$\text{(H8)}\ \chi_1\chi_3 + \chi_2\chi_4 \neq 0.$$

Theorem 2.8. If (H7) holds, then the positive equilibrium $E(x_0, y_0, z_0)$ of system (34) is asymptotically stable for $\tau \in [0, \tau_0)$. Under the condition (H7), if the condition (H8) holds, then system (34) undergoes a Hopf bifurcation at the positive equilibrium $E(x_0, y_0, z_0)$ when $\tau = \tau_0^{(j)}, j = 0, 1, 2, \cdots$.

III. Computer simulations

In this section, we present some numerical results of systems corresponding to Case 1, Case 2, Case 3 and Case 4, respectively, to verify the analytical predictions obtained in the previous section. First, we consider the following system which corresponds to Case 1.

$$\begin{cases} \dot{x}(t) = rx(1-x) - \dfrac{xy}{x+y} \\ \qquad\quad + 1.8(x(t) - x(t-\tau)), \\ \dot{y}(t) = \dfrac{10xy}{x+y} - y - \dfrac{11xy}{y+z}, \\ \dot{z}(t) = \dfrac{2xy}{y+z} - z. \end{cases} \tag{48}$$

We can see that (H1)-(H2) are fulfilled. Let $j = 0$ and by means of Matlab 7.0 software, we derive $\omega_0 \approx 0.3042, \tau_0 \approx 0.28$. Thus the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$ of system (48) is asymptotically stable for $\tau < \tau_0 \approx 0.28$ which is illustrated in Fig.2. When $\tau = \tau_0 \approx 0.28$, Eq.(48) undergoes a Hopf bifurcation at the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$, i.e., a small amplitude periodic solution occurs near $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$. When τ is close to $\tau_0 \approx 0.28$ which can be shown in Fig.3.

Second, we consider the following system which corresponds to Case 2:

$$\begin{cases} \dot{x}(t) = rx(1-x) - \dfrac{xy}{x+y}, \\ \dot{y}(t) = \dfrac{10xy}{x+y} - y - \dfrac{11xy}{y+z} \\ \qquad\quad + 0.2(y(t) - y(t-\tau)), \\ \dot{z}(t) = \dfrac{2xy}{y+z} - z. \end{cases} \tag{49}$$

We can see that (H3)-(H4) are fulfilled. Let $j = 0$ and by means of Matlab 7.0 software, we derive $\omega_0 \approx 0.5108, \tau_0 \approx 0.07$. Thus the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$ of system (49) is asymptotically stable for $\tau < \tau_0 \approx 0.07$ which is illustrated in Fig.4. When $\tau = \tau_0 \approx 0.07$, Eq.(49) undergoes a Hopf bifurcation at the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$. When τ is close to $\tau_0 \approx 0.07$ which can be shown in Fig.5.

Third, we consider the following system which corresponds to Case 3:

$$\begin{cases} \dot{x}(t) = rx(1-x) - \dfrac{xy}{x+y}, \\ \dot{y}(t) = \dfrac{10xy}{x+y} - y - \dfrac{11xy}{y+z}, \\ \dot{z}(t) = \dfrac{2xy}{y+z} - z + k(z(t) - z(t-\tau)) \\ \qquad\quad + 2(z(t) - z(t-\tau)). \end{cases} \tag{50}$$

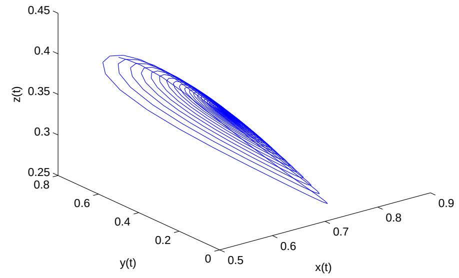

Fig. 2. Chaos vanishes when $\tau = 0.2 < \tau_0 \approx 0.28$. The positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$ is asymptotically stable. The initial value is (0.5,0.5,0.5).

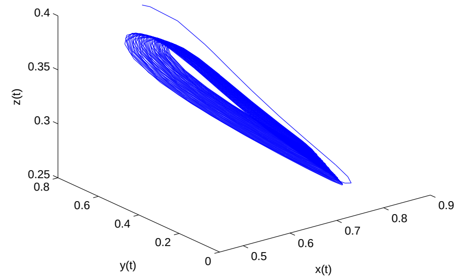

Fig. 3. Chaos vanishes when $\tau = 0.33 > \tau_0 \approx 0.28$. Hopf bifurcation occurs from the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$. The initial value is (0.5,0.5,0.5).

It can be seen that (H5)-(H6) are fulfilled. Let $j = 0$ and by means of Matlab 7.0 software, we derive $\omega_0 \approx 0.3566, \tau_0 \approx 0.27$. Thus the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$ of system (50) is asymptotically stable for $\tau < \tau_0 \approx 0.27$ which is illustrated in Fig.6. When $\tau = \tau_0 \approx 0.27$, Eq.(50) undergoes a Hopf bifurcation at the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$. When τ is close to $\tau_0 \approx 0.27$ which can be shown in Fig.7.

Finally, we consider the following system which corresponds to Case 4:

$$\begin{cases} \dot{x}(t) = rx(1-x) - \dfrac{xy}{x+y} \\ \qquad\quad - 0.2(x(t) - x(t-\tau)), \\ \dot{y}(t) = \dfrac{10xy}{x+y} - y - \dfrac{11xy}{y+z} \\ \qquad\quad - 0.2(y(t) - y(t-\tau)), \\ \dot{z}(t) = \dfrac{2xy}{y+z} - z + k(z(t) - z(t-\tau)) \\ \qquad\quad - 0.2(z(t) - z(t-\tau)). \end{cases} \tag{51}$$

It is easy to verify that (H7)-(H8) are fulfilled. Let $j = 0$ and by means of Matlab 7.0 software, we derive $\omega_0 \approx 0.7042, \tau_0 \approx 0.3$. Thus the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$ of system (50) is asymptotically stable for $\tau < \tau_0 \approx 0.3$ which is illustrated in Fig.8. When $\tau = \tau_0 \approx 0.3$, Eq.(50) undergoes a Hopf bifurcation at the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$. When τ is close to $\tau_0 \approx 0.3$ which can be shown in Fig.9.

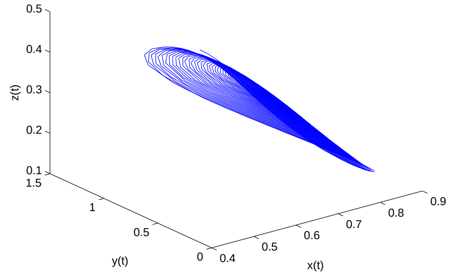

Fig. 4. Chaos vanishes when $\tau = 0.01 < \tau_{2_0} \approx 0.07$. The positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$ is asymptotically stable. The initial value is (0.5,0.5,0.5).

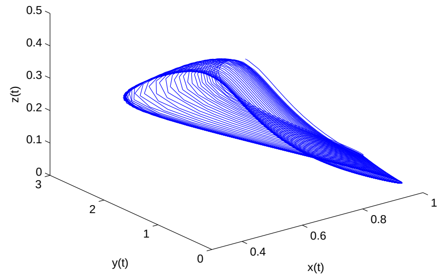

Fig. 5. Chaos vanishes when $\tau = 0.1 > \tau_0 \approx 0.07$. Hopf bifurcation occurs from the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$. The initial value is (0.5,0.5,0.5).

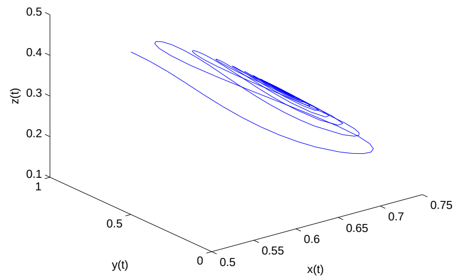

Fig. 6. Chaos vanishes when $\tau = 0.15 < \tau_0 \approx 0.27$. The positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$ is asymptotically stable. The initial value is (0.5,0.5,0.5).

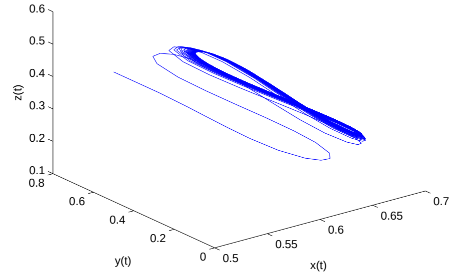

Fig. 7. Chaos vanishes when $\tau = 0.3 > \tau_0 \approx 0.27$.Hopf bifurcation occurs from the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$. The initial value is (0.5,0.5,0.5).

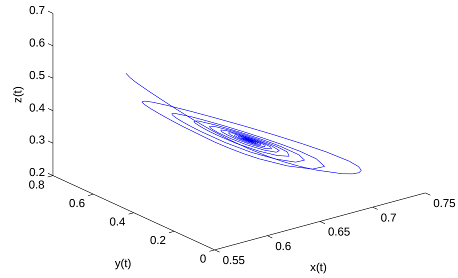

Fig. 8. Chaos vanishes when $\tau = 0.15 < \tau_0 \approx 0.3$. The positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$ is asymptotically stable. The initial value is (0.5,0.5,0.5).

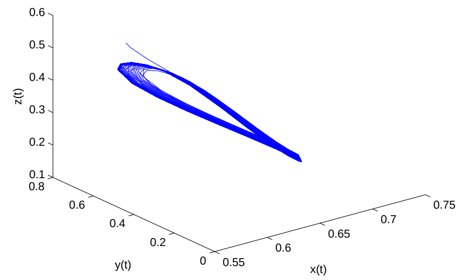

Fig. 9. Chaos vanishes when $\tau = 0.32 > \tau_0 \approx 0.3$. Hopf bifurcation occurs from the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$. The initial value is (0.5,0.5,0.5).

IV. CONCLUSIONS

In this paper, a feedback control method is applied to suppress chaotic behavior of a 3D ratio-dependent food chain system within the chaotic attractor. By adding a time-delayed force to the first equation of the 3D ratio-dependent food chain system, we have focused on the local stability of the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$ and local Hopf bifurcation of the delayed 3D ratio-dependent food chain system. It is showed that if the condition (H1) is satisfied, then the 3D ratio-dependent food chain system is asymptotically stable for $\tau \in [0, \tau_0)$. If (H1) and (H2) hold true, a sequence of Hopf bifurcations occur around the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$, that is, a family of periodic orbits bifurcate from the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$. By adding a time-delayed force to the second equation of the 3D ratio-dependent food chain system, we have analyzed the local stability of the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$ and local Hopf bifurcation of the delayed 3D ratio-dependent food chain system. It is showed that if the condition (H3) is satisfied, then the 3D ratio-dependent food chain system is asymptotically stable for $\tau \in [0, \tau_0)$. If (H3) and (H4) hold true, a sequence of Hopf bifurcations occur around the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$. By adding a time-delayed force to the third equation of 3D ratio-dependent food chain system, we have discussed the local stability of the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$ and local Hopf bifurcation of the delayed 3D ratio-dependent food chain system. We showed that if the condition (H5) is fulfilled, then the 3D ratio-dependent food chain system is asymptotically stable for $\tau \in [0, \tau_0)$. If (H5) and (H6) hold true, a sequence of Hopf bifurcations occur around the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$. By adding a time-delayed forces to the three equations of the 3D ratio-dependent food chain system, we have discussed the local stability of the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$ and local Hopf bifurcation of the delayed 3D ratio-dependent food chain system. We showed that if the condition (H7) is fulfilled, then the 3D ratio-dependent food chain system is asymptotically stable for $\tau \in [0, \tau_0)$. If (H7) and (H8) hold true, a sequence of Hopf bifurcations occur around the positive equilibrium $E\left(\frac{7}{10}, \frac{49}{130}, \frac{49}{130}\right)$. All the cases show that chaos vanishes and can be suppressed. Some computer simulations are carried out to visualize the theoretical results.

It is well known that the topic of chaos and chaotic control are growing rapidly in many fields such as biological models and ecological and chemical models [34-35] The desirability of chaos dependents on the particular application. Thus it is important that the chaotic response of system can be controlled [36]. Among the methods of chaos control, delayed feedback controller (DFC) is an effective method for chaos control. The basic idea of DFC is to realize a continuous control for a dynamical system by applying a feedback signal which is proportional to the difference between the dynamical variable and its delayed value [37]. The delayed feedback control method has its merit: it does not require any computer analysis and can be simply implemented in various experiments. Still there are a lot of excellent prospects in bifurcation and control area, In the future, we will further investigate the bifurcation nature and control by taking the time delay or other parameter of the system.

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