Weak Solutions via Lagrange Multipliers for a Slip-dependent Frictional Contact Model

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Abstract— **We consider a 3D elastostatic slip-dependent frictional contact problem which consists of a system of partial differential equations associated with a homogeneous displacement boundary condition, a traction boundary condition and a frictional contact boundary condition involving a slip-dependent friction bound. After describing the mechanical model, we deliver a variational formulation as a mixed variational problem whose Lagrange multipliers set is solution-dependent. Then, the existence and the boundedness of the solutions is investigated. The proof is based on a recent result for an abstract mixed variational problem with solution-dependent set of Lagrange multipliers.**

Keywords: slip-dependent frictional contact problem, mixed variational problems, solution-dependent set of Lagrange multipliers, weak solutions.

1 Introduction

The contact phenomenon is as frequent as complex. A lot of work was devoted to the modeling in contact mechanics, see e.g. [5, 6, 17, 18, 28, 29, 30, 31, 32, 35]. It is worth to underline that the solvability of contact models relies on calculus of variations. The weak formulations of contact problems are related to the theory of variational inequalities, see e.g. [8, 32], or to the theory of saddle point problems, see e.g. [7, 9]. In the last years several papers were devoted to the weak solvability via Lagrange multipliers of contact models, see e.g. [21, 22, 23, 24, 25, 26, 27] for qualitative analysis and [2, 10, 11, 12, 13, 14, 15] for modern numerical approaches.

In the present work we focus on a 3D contact model with slip-dependent friction bound, for linearly elastic materials. The first mathematical results on contact problems with slip displacements dependent friction in elastostatics were obtained in [16]. The model we discuss in the present paper is mathematically described by a system of partial differential equations associated with a displacement boundary condition, a traction boundary condition and a frictional contact boundary condition with slip-dependent friction bound. This model was already analyzed into the framework of quasi-variational inequalities, see [4]. The novelty in the present paper consists in the variational approach we use; herein, a mixed variational formulation is proposed in a form of a generalized saddle point problem, the set of the Lagrange multipliers being solution-dependent. In contrast to [4], where the solution of the quasi-variational inequality is the displacement field $\boldsymbol{u}$, in this approach, the unknown is the pair of the displacement field $\boldsymbol{u}$ and a Lagrange multiplier $\boldsymbol{\lambda}$ related to the frictional force ($\boldsymbol{\lambda} = -\boldsymbol{\sigma}_\tau|_{\Gamma_3}$ where Γ_3 is the contact zone and $\boldsymbol{\sigma}_\tau$ is the tangential part of the Cauchy vector). If we replace the slip-dependent frictional contact law by Tresca's law with given friction bound $g = g(\boldsymbol{x})$, the set of the Lagrange multipliers becomes a fix set Λ (a priori known) and the weak formulation is related to the classical saddle point formulation; see for instance [9] for classical saddle point problems related to contact models. The present work draws the attention on the mathematical difficulties which appear giving a formulation in which the set of the Lagrange multipliers is a solution-dependent set $\Lambda(\boldsymbol{u})$ (a priori unknown). We investigate the existence and the uniqueness of the solutions based on an abstract result recently obtained in [23]. The abstract result was obtained by using a saddle point technique, see e.g. [9] combined with a fixed point technique of type Schauder-Tychonoff, see [3]. For the convenience of the reader we shall recall this abstract result in Section 2 of the present paper; see [23] for details.

The mixed variational formulations are related to modern numerical techniques in order to approximate the weak solutions of contact models and this motivates the present study. Referring to numerical techniques for approximating weak solutions of contact problems via saddle point technique, we send the reader to, e.g., [10, 33, 34].

The present paper is the extended and revised version of the conference paper [20].

The rest of the paper is structured as follows. In Section 2 we present the main theoretical tool we use in order to weakly solve the mechanical model. In Section 3 we describe the mechanical model for a 3D slip-dependent frictional contact process. In Section 4 we make the assumptions. In Section 5 we deliver a weak formulation

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via Lagrange multiplier and we define the weak solution for our mechanical model. Finally, in Section 6 we investigate the existence and the boundedness of the weak solutions.

2 An abstract auxiliary result

Let us consider the following abstract mixed variational problem.

Problem 1 *Given $f \in X$, $f \neq 0_X$, find $(u, \lambda) \in X \times Y$ such that $\lambda \in \Lambda(u) \subset Y$ and*

$$\begin{aligned} a(u,v) + b(v,\lambda) &= (f,v)_X \ \textit{for all } v \in X, && (1)\\ b(u,\mu-\lambda) &\leq 0 \ \textit{for all } \mu \in \Lambda(u). && (2) \end{aligned}$$

We made the following assumptions.

Assumption 1 $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ *and* $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ *are two Hilbert spaces.*

Assumption 2 $a(\cdot,\cdot) : X \times X \to R$ *is a symmetric bilinear form such that*

(i_1) *there exists* $M_a > 0$:

$|a(u,v)| \leq M_a \|u\|_X \|v\|_X$ *for all* $u, v \in X$,

(i_2) *there exists* $m_a > 0$:

$a(v,v) \geq m_a \|v\|_X^2$ *for all* $v \in X$.

Assumption 3 $b(\cdot,\cdot) : X \times Y \to R$ *is a bilinear form such that*

(j_1) *there exists* $M_b > 0$:

$|b(v,\mu)| \leq M_b \|v\|_X \|\mu\|_Y$ *for all* $v \in X$, $\mu \in Y$,

(j_2) *there exists* $\alpha > 0$:

$$\inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha.$$

Assumption 4 *For each* $\varphi \in X$, $\Lambda(\varphi)$ *is a closed convex subset of* Y *such that* $0_Y \in \Lambda(\varphi)$.

Assumption 5 *Let* $(\eta_n)_n \subset X$ *and* $(u_n)_n \subset X$ *be two weakly convergent sequences,* $\eta_n \rightharpoonup \eta$ *in* X *and* $u_n \rightharpoonup u$ *in* X, *as* $n \to \infty$.

(k_1) *For each* $\mu \in \Lambda(\eta)$, *there exists a sequence* $(\mu_n)_n \subset Y$ *such that* $\mu_n \in \Lambda(\eta_n)$ *and*

$\liminf_{n\to\infty} b(u_n, \mu_n - \mu) \geq 0$.

(k_2) *For each subsequence* $(\Lambda(\eta_{n'}))_{n'}$ *of the sequence* $(\Lambda(\eta_n))_n$, *if* $(\mu_{n'})_{n'} \subset Y$ *such that*

$\mu_{n'} \in \Lambda(\eta_{n'})$ *and* $\mu_{n'} \rightharpoonup \mu$ *in* Y *as* $n' \to \infty$, *then* $\mu \in \Lambda(\eta)$.

Theorem 1 *If Assumptions* 1-5 *hold true, then Problem 1 has a solution. In addition, if* $(u,\lambda) \in X \times \Lambda(u)$ *is a solution of Problem 1, then*

$$(u,\lambda) \in K_1 \times \big(\Lambda(u) \cap K_2\big),$$

where

$$\begin{aligned} K_1 &= \{v \in X \,|\, \|v\|_X \leq \frac{1}{m_a}\|f\|_X\};\\ K_2 &= \{\mu \in Y \,|\, \|\mu\|_Y \leq \frac{m_a + M_a}{\alpha\, m_a}\|f\|_X\}, \end{aligned}$$

m_a, α *and* M_a *being the constants in Assumptions 2-3.*

For the proof of this theorem we refer to [23].

3 The model

We consider a deformable body that occupies the bounded domain $\Omega \subset R^3$ with smooth (say Lipschitz continuous) boundary Γ partitioned into three measurable parts, Γ_1, Γ_2 and Γ_3, such that $meas(\Gamma_1) > 0$. The unit outward normal vector to Γ is denoted by $\boldsymbol{\nu}$ and is defined almost everywhere. The body is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω and surface traction of density $\boldsymbol{f}_2$ acts on Γ_2. On Γ_3 the body is in slip-dependent frictional contact with a rigid foundation. We denote by $\boldsymbol{u} = (u_i)$ the displacement field and by $\boldsymbol{\sigma} = (\sigma_{ij})$ the Cauchy stress tensor. Everywhere below $\|\cdot\|$ denotes the Euclidean norm in R^3.

The 3D slip-dependent frictional contact model is mathematically described as follows.

Problem 2 *Find* $\boldsymbol{u} : \overline{\Omega} \to R^3$ *and* $\boldsymbol{\sigma} : \overline{\Omega} \to S^3$ *such that*

$$\begin{aligned} & Div\,\boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{0} && \textit{in } \Omega, && (3)\\ & \boldsymbol{\sigma}(\boldsymbol{x}) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) && \textit{in } \Omega, && (4)\\ & \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} && \textit{on } \Gamma_1, && (5)\\ & \boldsymbol{\sigma\nu}(\boldsymbol{x}) = \boldsymbol{f}_2(\boldsymbol{x}) && \textit{on } \Gamma_2, && (6)\\ & u_\nu(\boldsymbol{x}) = 0 && \textit{on } \Gamma_3, && (7)\\ & \|\boldsymbol{\sigma}_\tau(\boldsymbol{x})\| \leq g(\boldsymbol{x}, \|\boldsymbol{u}_\tau(\boldsymbol{x})\|), && &&\\ & \boldsymbol{\sigma}_\tau(\boldsymbol{x}) = -g(\boldsymbol{x}, \|\boldsymbol{u}_\tau(\boldsymbol{x})\|)\,\frac{\boldsymbol{u}_\tau(\boldsymbol{x})}{\|\boldsymbol{u}_\tau(\boldsymbol{x})\|} && &&\\ & \textit{if } \boldsymbol{u}_\tau(\boldsymbol{x}) \neq \boldsymbol{0} && \textit{on } \Gamma_3. && (8) \end{aligned}$$

Herein $\overline{\Omega} = \Omega \cup \partial\Omega$. As usual, Div denotes the divergence operator; for every $i \in \{1,2,3\}$,

$$(Div\,\boldsymbol{\sigma})_i = \frac{\partial\sigma_{i1}}{\partial x_1} + \frac{\partial\sigma_{i2}}{\partial x_2} + \frac{\partial\sigma_{i3}}{\partial x_3}.$$

By $\mathcal{E} = (\mathcal{E}_{ijkl})$ we denotes the fourth order elastic tensor and $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ is the infinitezimal strain tensor with components

$$\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \text{ for all } i, j \in \{1, 2, 3\}.$$

Problem 2 has the following structure: (3) represents the equilibrium equation, (4) represents the constitutive law for linearly elastic materials, (5) represents the homogeneous displacements boundary condition, (6) represents the traction boundary condition and (7)-(8) model the bilateral contact with friction, the friction law involving a slip-dependent friction bound g. Notice that $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$, $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$, where " $\cdot$ " denotes the inner product of two vectors in R^3. For more details on this model we refer to [4].

4 Assumptions

In order to weakly solve Problem 2 we make the following assumptions.

Assumption 6 $\mathcal{E} = (\mathcal{E}_{ijls}) : \Omega \times S^3 \to S^3$,

- $\mathcal{E}_{ijls} = \mathcal{E}_{ijsl} = \mathcal{E}_{lsij} \in L^\infty(\Omega)$,
- *There exists* $m_{\mathcal{E}} > 0$ *such that* $\mathcal{E}_{ijls}\varepsilon_{ij}\varepsilon_{ls} \geq m_{\mathcal{E}}\,|\boldsymbol{\varepsilon}|^2$, $\boldsymbol{\varepsilon} \in S^3$, *a.e. in* Ω.

Assumption 7 $\boldsymbol{f}_0 \in L^2(\Omega)^3, \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^3$.

Assumption 8 $g : \Gamma_3 \times R_+ \to R_+$,

- *there exists* $L_g > 0$: $|g(\boldsymbol{x}, r_1) - g(\boldsymbol{x}, r_2)| \leq L_g\,|r_1 - r_2| \quad r_1, r_2 \in R_+$, *a.e.* $\boldsymbol{x} \in \Gamma_3$;
- *the mapping* $\boldsymbol{x} \mapsto g(\boldsymbol{x}, r)$ *is Lebesgue measurable on* Γ_3, *for all* $r \in R_+$;
- *the mapping* $\boldsymbol{x} \mapsto g(\boldsymbol{x}, 0)$ *belongs to* $L^2(\Gamma_3)$.

5 Weak formulation

Let us introduce the following functional space.

$$V = \{\boldsymbol{v} \in H^1(\Omega)^3 \,|\, \boldsymbol{\gamma v} = 0 \text{ on } \Gamma_1,\ v_\nu = 0 \text{ on } \Gamma_3\}. \quad (9)$$

Notice that, everywhere in this paper, for each $\boldsymbol{w} \in V$, $w_\nu = \boldsymbol{\gamma w}\cdot\boldsymbol{\nu}$ and $\boldsymbol{w}_\tau = \boldsymbol{\gamma w} - w_\nu\boldsymbol{\nu}$ a.e. on Γ, where $\boldsymbol{\gamma}$ denotes the Sobolev trace operator for vectors. We use standard notation for the Lebesgue and Sobolev spaces associated to Ω and Γ.

The space V is a real Hilbert space endowed with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx,$$

and the associated norm $\|\cdot\|_V$. The completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $meas\,\Gamma_1 > 0$, which allows us to use of Korn's inequality. Notice that " : " denotes the inner product of two tensors.

Define $\boldsymbol{f} \in V$ using Riesz's representation theorem,

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x})\cdot\boldsymbol{\gamma v}(\boldsymbol{x})\,d\Gamma \quad (10)$$

for all $\boldsymbol{v} \in V$.

Let $\boldsymbol{u}$ be a sufficiently regular solution of Problem 2. By a Green formula we get, for all $\boldsymbol{v} \in V$,

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_V + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x})\cdot\boldsymbol{v}_\tau(\boldsymbol{x})\,d\Gamma \quad (11)$$

where $a(\cdot,\cdot) : V \times V \to R$,

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) : \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))\,dx. \quad (12)$$

Let us introduce the space

$$S = \{\boldsymbol{\gamma w}|_{\Gamma_3} \,|\, \boldsymbol{w} \in V\}, \quad (13)$$

where $\boldsymbol{\gamma w}|_{\Gamma_3}$ denotes the restriction of the trace of the element $\boldsymbol{\gamma w} \in V$ to Γ_3. Thus, $S \subset H^{1/2}(\Gamma_3; R^3)$ where $H^{1/2}(\Gamma_3; R^3)$ is the space of the restrictions on Γ_3 of traces on Γ of functions of $H^1(\Omega)^3$. It is known that S can be organized as a real Hilbert space, see for instance [1, 19]. We use the Sobolev-Slobodeckii norm

$$\|\boldsymbol{\zeta}\|_S = \Big(\int_{\Gamma_3}\int_{\Gamma_3} \frac{\|\boldsymbol{\zeta}(\boldsymbol{x}) - \boldsymbol{\zeta}(\boldsymbol{y})\|^2}{\|\boldsymbol{x} - \boldsymbol{y}\|^3}\,ds_x\,ds_y\Big)^{1/2}.$$

Let us introduce now the following real Hilber space,

$$D = S' \text{ (the dual of the space } S). \quad (14)$$

The duality paring between D and S will be denoted by $\langle\cdot,\cdot\rangle$.

For each $\boldsymbol{\varphi} \in V$ we define

$$\begin{aligned} \boldsymbol{\Lambda}(\boldsymbol{\varphi}) = \{\boldsymbol{\mu} \in D \,|\, \langle\boldsymbol{\mu}, \boldsymbol{\gamma v}|_{\Gamma_3}\rangle \leq \quad (15) \\ \int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{\varphi}_\tau(\boldsymbol{x})\|)\|\boldsymbol{v}_\tau(\boldsymbol{x})\|\,d\Gamma \quad \boldsymbol{v} \in V\}. \end{aligned}$$

Let us define a Lagrange multiplier $\boldsymbol{\lambda} \in D$,

$$\langle\boldsymbol{\lambda}, \boldsymbol{\zeta}\rangle = -\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x})\cdot\boldsymbol{\zeta}(\boldsymbol{x})\,d\Gamma \quad (16)$$

for all $\boldsymbol{\zeta} \in S$.

By (16) and (8) we deduce that $\boldsymbol{\lambda} \in \Lambda(\boldsymbol{u})$.

We also define

$$b : V \times D \to R \quad b(\boldsymbol{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{\gamma v}|_{\Gamma_3} \rangle. \tag{17}$$

Let us rewrite (11) as

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_V - \langle \boldsymbol{\lambda}, \boldsymbol{\gamma v}|_{\Gamma_3} \rangle \quad \text{for all } \boldsymbol{v} \in V.$$

By the definition of the form $b(\cdot, \cdot)$, we obtain

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{\lambda}) = (\boldsymbol{f}, \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V. \tag{18}$$

The friction law (8) leads us to the identity

$$\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{u}_\tau(\boldsymbol{x})\, d\Gamma = -\int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{u}_\tau(\boldsymbol{x})\|)\|\boldsymbol{u}_\tau(\boldsymbol{x})\|\, d\Gamma.$$

Thus,

$$b(\boldsymbol{u}, \boldsymbol{\lambda}) = \int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{u}_\tau(\boldsymbol{x})\|)\|\boldsymbol{u}_\tau(\boldsymbol{x})\|\, d\Gamma. \tag{19}$$

By (15) with $\boldsymbol{\varphi} = \boldsymbol{u}$ we are led to

$$b(\boldsymbol{u}, \boldsymbol{\zeta}) \le \int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{u}_\tau(\boldsymbol{x})\|)\|\boldsymbol{u}_\tau(\boldsymbol{x})\|\, d\Gamma \tag{20}$$

for all $\boldsymbol{\zeta} \in \Lambda(\boldsymbol{u})$. Subtract now (19) from (20) to obtain the inequality

$$b(\boldsymbol{u}, \boldsymbol{\zeta} - \boldsymbol{\lambda}) \le 0 \quad \text{for all } \boldsymbol{\zeta} \in \Lambda(\boldsymbol{u}). \tag{21}$$

Therefore, Problem 2 has the following weak formulation.

Problem 3 *Find* $\boldsymbol{u} \in V$ *and* $\boldsymbol{\lambda} \in \Lambda(\boldsymbol{u}) \subset D$ *such that* (18) *and* (21) *hold true.*

Each solution of Problem 3 is called *weak solution* of Problem 2.

6 Existence and boundedness results

Theorem 2 (An existence result) *If Assumptions 6 -8 hold true, then Problem 2 has a solution.*

Proof. Since Problem 3 and the abstract problem, Problem 1, are the same type, the idea of the proof is to use the abstract result, Theorem 1.

As the spaces V and D are real Hilbert spaces then Assumption 1 is fulfilled with $X = V$ and $Y = D$.

The form $a(\cdot, \cdot)$ defined in (12) verifies Assumption 2 with

$$M_a = \|\mathcal{E}\|_\infty \text{ and } m_a = m_{\mathcal{E}}, \tag{22}$$

where

$$\|\mathcal{E}\|_\infty = \max_{0 \le i,j,k,l \le d} \|E_{ijkl}\|_{L^\infty(\Omega)}.$$

Let us prove (j_1) in Assumption 3. Since for each $\boldsymbol{v} \in V$ $\|\boldsymbol{\gamma v}|_{\Gamma_3}\|_S \le \|\boldsymbol{\gamma v}\|_{H_\Gamma}$, taking into account the definition of the form $b(\cdot, \cdot)$, we can write

$$|b(\boldsymbol{v}, \boldsymbol{\mu})| \le \|\boldsymbol{\mu}\|_D \|\boldsymbol{\gamma v}\|_{H_\Gamma} \quad \text{for all } \boldsymbol{v} \in V, \boldsymbol{\mu} \in D.$$

We recall that $H_\Gamma = \boldsymbol{\gamma}(H^1(\Omega)^3)$ and the Sobolev trace operator $\boldsymbol{\gamma} : H^1(\Omega)^3 \to H_\Gamma$ is a linear and continuous operator. Also, we recall the Sobolev-Slobodeckii norm on H_Γ:

$$\|\boldsymbol{\zeta}\|_{H_\Gamma} = \Big(\int_\Gamma \int_\Gamma \frac{\|\boldsymbol{\zeta}(\boldsymbol{x}) - \boldsymbol{\zeta}(\boldsymbol{y})\|^2}{\|\boldsymbol{x} - \boldsymbol{y}\|^3}\, ds_x\, ds_y \Big)^{1/2}.$$

Due to the fact that $\|\cdot\|_V$ and $\|\cdot\|_{H^1(\Omega)^3}$ are equivalent norms, we deduce that there exists $M_b > 0$ such that (j_1) holds true.

We also recall that there exists a linear and continuous operator $\mathcal{Z}$ such that

$$\mathcal{Z} : H_\Gamma \to H^1(\Omega)^3 \quad \boldsymbol{\gamma}(\mathcal{Z}(\boldsymbol{\zeta})) = \boldsymbol{\zeta} \quad \text{for all } \boldsymbol{\zeta} \in H_\Gamma.$$

The operator $\mathcal{Z}$ is called *the right inverse* of the operator $\boldsymbol{\gamma}$. Notice that,

$$\boldsymbol{\gamma}(\mathcal{Z}(\boldsymbol{\gamma w})) = \boldsymbol{\gamma w} \qquad \text{for all } \boldsymbol{w} \in V.$$

Since, for each $\boldsymbol{w} \in V$, $\mathcal{Z}(\boldsymbol{\gamma w})$ has the same trace as $\boldsymbol{w}$, we deduce that for each $\boldsymbol{w} \in V$, $\mathcal{Z}(\boldsymbol{\gamma w}) \in V$. On the other hand, we note that for each $\boldsymbol{w} \in V$, there exists $\boldsymbol{w}^* \in V$ such that $\boldsymbol{\gamma w} = \boldsymbol{\gamma w}^*$ a.e. on Γ_3 and $\boldsymbol{\gamma w}^* = 0$ a.e. on Γ_2. Notice that $\|\boldsymbol{\gamma w}|_{\Gamma_3}\|_S = \|\boldsymbol{\gamma w}^*\|_{H_\Gamma}$.

Let us prove now (j_2) in Assumption 3.

$$\begin{aligned}
\|\boldsymbol{\mu}\|_D &= \sup_{\boldsymbol{\gamma w}|_{\Gamma_3} \in S,\, \boldsymbol{\gamma w}|_{\Gamma_3} \neq 0_S} \frac{\langle \boldsymbol{\mu}, \boldsymbol{\gamma w}|_{\Gamma_3} \rangle}{\|\boldsymbol{\gamma w}|_{\Gamma_3}\|_S} \\
&= \sup_{\boldsymbol{\gamma w}|_{\Gamma_3} \in S,\, \boldsymbol{\gamma w}|_{\Gamma_3} \neq 0_S} \frac{\langle \boldsymbol{\mu}, \boldsymbol{\gamma w}^*|_{\Gamma_3} \rangle}{\|\boldsymbol{\gamma w}^*\|_{H_\Gamma}} \\
&\le c \sup_{\boldsymbol{\gamma w}|_{\Gamma_3} \in S,\, \boldsymbol{\gamma w}|_{\Gamma_3} \neq 0_S} \frac{b(\mathcal{Z}(\boldsymbol{\gamma w}^*), \boldsymbol{\mu})}{\|\mathcal{Z}(\boldsymbol{\gamma w}^*)\|_V} \\
&\le c \sup_{\boldsymbol{v} \in V,\, \boldsymbol{v} \neq \boldsymbol{0}_V} \frac{b(\boldsymbol{v}, \boldsymbol{\mu})}{\|\boldsymbol{v}\|_V},
\end{aligned}$$

where $c > 0$. We can take

$$\alpha = \frac{1}{c}. \tag{23}$$

Obviously, $0_D \in \Lambda(\boldsymbol{\varphi})$. Also, $\Lambda(\boldsymbol{\varphi})$ is a closed convex subset of the space D. Hence, Assumption 4 is fulfilled.

Let us verify Assumption 5. To start, let $(\boldsymbol{\eta}_n)_n \subset V$ and $(\boldsymbol{u}_n)_n \subset V$ be two weakly convergent sequences, $\boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta}$ in V and $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ in V, as $n \to \infty$. Let us take $\boldsymbol{\mu} \in \Lambda(\boldsymbol{\eta})$.

In order to check Assumption 5 a crucial point is the construction of an appropriate sequence in (k_1). Let us define $(\boldsymbol{\mu}_n)_n$ as follows: for each $n \geq 1$,

$$< \boldsymbol{\mu}_n, \boldsymbol{\zeta} >= \int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{\eta}_{\tau n}(\boldsymbol{x})\|) \boldsymbol{\psi}(\boldsymbol{u}_{\tau n}(\boldsymbol{x})) \cdot \boldsymbol{\zeta}(\boldsymbol{x})\, d\Gamma$$
$$- \int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{\eta}_{\tau}(\boldsymbol{x})\|) \|\boldsymbol{u}_{\tau n}(\boldsymbol{x})\|\, d\Gamma + \langle \boldsymbol{\mu}, \boldsymbol{\gamma} \boldsymbol{u}_n|_{\Gamma_3} \rangle,$$

for all $\boldsymbol{\zeta} \in S$, where

$$\boldsymbol{\psi}(\boldsymbol{r}) = \begin{cases} \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|} & \text{if } \boldsymbol{r} \neq \boldsymbol{0}; \\ \boldsymbol{0} & \text{if } \boldsymbol{r} = \boldsymbol{0}. \end{cases}$$

Taking into account the definition of $\Lambda(\boldsymbol{\varphi})$ in (15), we deduce that, for each positive integer n, we have $\boldsymbol{\mu}_n \in \Lambda(\boldsymbol{\eta}_n)$.

We recall here that $\boldsymbol{\gamma} : H^1(\Omega)^3 \to L^2(\Gamma)^3$ is a compact operator. Thus, since $\boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta}$ in V and $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ in V as $n \to \infty$, using the compactness of the trace operator we can write

$$\boldsymbol{\gamma}\boldsymbol{\eta}_n \to \boldsymbol{\gamma}\boldsymbol{\eta} \text{ in } L^2(\Gamma)^3 \text{ as } n \to \infty;$$
$$\boldsymbol{\gamma}\boldsymbol{u}_n \to \boldsymbol{\gamma}\boldsymbol{u} \text{ in } L^2(\Gamma)^3 \text{ as } n \to \infty.$$

Therefore,

$$\boldsymbol{u}_{\tau\, n}(\boldsymbol{x}) \to \boldsymbol{u}_{\tau}(\boldsymbol{x}) \text{ a.e. on } \Gamma_3 \text{ as } n \to \infty$$

and

$$g(\boldsymbol{x}, \|\boldsymbol{\gamma}\boldsymbol{\eta}_n(\boldsymbol{x})\|) \to g(\boldsymbol{x}, \|\boldsymbol{\gamma}\boldsymbol{\eta}(\boldsymbol{x})\|) \text{ a.e. on } \Gamma_3 \text{ as } n \to \infty.$$

Setting $\boldsymbol{\zeta} = \boldsymbol{\gamma}\boldsymbol{u}_n|_{\Gamma_3}$ in (24) we can write

$$\langle \boldsymbol{\mu}_n - \boldsymbol{\mu}, \boldsymbol{\gamma}\boldsymbol{u}_n|_{\Gamma_3} \rangle =$$
$$\int_{\Gamma_3} \Big(g(\boldsymbol{x}, \|\boldsymbol{\gamma}\boldsymbol{\eta}_n(\boldsymbol{x})\|) - g(\boldsymbol{x}, \|\boldsymbol{\gamma}\boldsymbol{\eta}(\boldsymbol{x})\|) \Big) \|\boldsymbol{u}_{\tau\, n}(\boldsymbol{x})\|\, d\Gamma.$$

Hence, passing to the inferior limit as $n \to \infty$, we get

$$\begin{aligned} &\liminf_{n\to\infty} b(\boldsymbol{u}_n, \boldsymbol{\mu}_n - \boldsymbol{\mu}) \\ &= \liminf_{n\to\infty} \int_{\Gamma_3} \Big(g(\boldsymbol{x}, \|\boldsymbol{\eta}_{\tau\, n}(\boldsymbol{x})\|) \\ &- g(\boldsymbol{x}, \|\boldsymbol{\eta}_{\tau}(\boldsymbol{x})\|) \Big) \|\boldsymbol{u}_{\tau\, n}(\boldsymbol{x})\| d\Gamma \\ &= 0. \end{aligned}$$

Using again the properties of the trace operator and the assumptions on the friction bound we deduce that (k_2) in Assumption 5 is also verified.

We apply now Theorem 1.

Let us introduce

$$\boldsymbol{K}_1 \;=\; \{\boldsymbol{v} \in V \,|\, \|\boldsymbol{v}\|_V \leq \frac{1}{m_a} \|\boldsymbol{f}\|_V\}; \qquad (24)$$

$$\boldsymbol{K}_2 \;=\; \{\boldsymbol{\mu} \in D \,|\, \|\boldsymbol{\mu}\|_D \leq \frac{m_a + M_a}{\alpha\, m_a} \|\boldsymbol{f}\|_V\}. \qquad (25)$$

Theorem 3 (A boundedness result) *If $(\boldsymbol{u}, \boldsymbol{\lambda})$ is a weak solution of Problem 2, then*

$$(\boldsymbol{u}, \boldsymbol{\lambda}) \in \boldsymbol{K}_1 \times \big(\Lambda(\boldsymbol{u}) \cap \boldsymbol{K}_2\big)$$

where $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are given by (24)-(25)*, V given by* (9)*, D given by* (14)*, $\boldsymbol{f}$ given by* (10)*, m_a and M_a being the constants in* (22) *and α being the constant in* (23).

Proof. The proof is a straightforward consequence of Theorem 1.

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