Kron's Method and Cell Complexes for Magnetomotive and Electromotive Forces

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Abstract—**Starting from topological principles we first recall the elementary ones giving Kirchhoff's laws for current conservation. Using in a second step the properties of spaning tree, we show that currents are under one hypothesis intrinsically boundaries of surfaces flux. Naturally flux appears as the object from which the edge comes from. The current becomes the magnetomotive force (mmf) that creates the flux in the magnetostatic representation. Using a metric and an Hodge's operator, this flux creates an electromotive force (emf). This emf is finally linked with the current to give the fundamental tensor - or "metric" - of the Kron's tensorial analysis of networks. As it results in a link between currents of cycles (surface boundaries) and energy sources in the network, we propose to symbolize this cross talk using chords between cycles in the graph structure on which the topology is based. Starting then from energies relations we show that this metric is the Lagrange's operator of the circuit. But introducing moment space, the previous results can be extended to non local interactions as far field one. And to conclude, we use the same principle to create general relation of information exchange between networks as functors between categories.**

Index Terms—**EMC, Kron's formalism, MKME, tensorial analysis of networks.**

I. Introduction

GABRIEL KRON (1900 - 1968) has transfered the tensorial analysis developed in the framework of general relativity to the world of applied electromagnetism [1], [2]. He had felt the straight relations between Kirchhoff's laws and topology. To replace his reflexions at this time, we try to give a brief history of these works, from Kron to nowdays. As many words exist to call the various elements available on graphs, we propose some of them that we use in our paper. Our first purpose is to find again these straight relations as simplest as possible. Making this exercise and under the hypothesis that no spaning tree current sources are involved, only electrostatic and magnetostatic phenomenons are considered, it appears that the current can be seen as a fundamental element of the faces boundary space. Through this vision, the electromotive force belongs to the faces space. The current seen as a boundary of faces, becomes equivalent to magnetomotive forces. By a physical understanding, a relation between magnetomotive forces and electromotive forces must exists. A problem appears because both doesn't belong to the same differential form dimension. Thanks to the Hodge operator, we make this link, following previous existing works. But if the electromotive force, which give the network its energy, belongs to the faces space, self inductance reaction must belongs to the same space and mutual inductance interaction translates the relation of Hodge. These components, as sensed by Kron, represent the metric that we take to compute these interactions. This leads to the Lagrangian expression of the whole graph and to the "chords" elements that symbolize these interactions. This Lagrangian must then be increased taking into account the spaning tree sources, added to the faces ones. This new space give us the complete base to take into account far field interactions. Lamellar fields create current sources, rotationnal fields create electromotive forces. This continuous fields are connected to our topology using moment space. This space is the frontier between our first bounded manifolds which generate the graph and continuous not bounded manifolds which are radiated and propagated fields. The first topological discussion gives all the base to include this new interaction through a generalized definition of the chords. At each step, we first give a topological approach before to "translate" it in expressions more usually given by physicists.

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II. Tensorial Analysis of Networks (TAN) history

Gabriel Kron has written is famous "tensorial analysis of networks" in 1939. Before this work, he has written in 1931 a first remarkable paper "Non-Riemannian Dynamics of Rotating Electrical Machinery" [3]. For this work he had the Montefiore price of the university of Liege in 1933 and the M.I.T. journal of mathematics and physics publishes the entire paper in the May 1934 [4]. This paper instantly produces wide-spread discussion and contreversy [5]. Kron uses its own notation without regarding established ones. This leads to some mathematicians contempt. But some of them were clear enought to understand and study Kron's work. Hoffmann [6], Roth [7] make links between Kron's concept and topology ones. Physicists like Branin, Happ, and in France Denis-Papin and Kaufmann [8] promote Kron's work for electrical engineers. Many studies were done after around the concept of *Diakoptic* initiated by Kron [9]. But to focus on topology, less references are available. Major lecture was made by Balasubramanian, Lynn and Sen Gupta [2]. Recently, there is the work done by Gross and Kotiuga [10], following first one of Bossavit [11]. These last two works was made more particularly for finite element method. But they give fundamental bases through algebraic topology, following previous works of Roth and others, clarified using benefit of years passing. In this paper we try to take benefit

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of all this story and to present as clear as we can our understanding on these concepts.

III. Notations

We note $\mathbb{N}$ the set of integers, $\mathbb{R}$ the set of real numbers, $\mathbb{C}$ for complex, $\mathcal{T}$ for cells, etc. We work in a complex cellular $\mathcal{T}^\infty$ made of vertexes $s \in \mathcal{T}^0$, edges $a \in \mathcal{T}^1$ and faces $f \in \mathcal{T}^2$, etc. $\mathcal{T}$ is the whole set of these geometrical or chain objects. Low indices refer to chains, high ones to the geometric objects. Geometric objects are classical forms. Chains are abstract objects embedding properties added to the geometric objects in order to represent symbolically a real thing. A set of currents runing in a system can be linked with a set of chains associated with edges. This set of currents are components of a unique current vector. The current vector constitute a chain, image of some real currents on an electronic system. More generally, vectors can be associated with each of these geometric objects and their bases: $|s\rangle$ for vertexes, $|a\rangle$ for edges, $|f\rangle$ for faces, etc. We note $\mathcal{T}_j$ the vectorial space created by the geometric vectors $|x\rangle$ of $\mathcal{T}^{D(x)}$ ($D(x) = 0$ if x similar to s, etc.). $\mathcal{T}_j$ can be developped as:

$$\mathcal{T}_j = \left\{ \sum_{\sigma \in \mathcal{T}^j} \alpha_\sigma |\sigma\rangle\,,\ \alpha_\sigma \in \mathbb{R},\ \mathbb{C} \right\},\ j \in N \tag{1}$$

In this definition, we see here a generalized formulation of the classical writing of a vector, using the mute index notation (each time an index is repeated, the summation symbol on the index can be omitted) in [12]: $\vec{f} = f^\sigma \vec{u}_\sigma$. Here $\vec{f}$ is a vector developed on the base $\vec{u}_\sigma$ of components f^σ.

IV. Boundary operator

We now introduce the boundary operator. It translates the intuitive understanding of object boundary. The boundary of a segment is a pair of two points, the one of a surface is a closed line, and so on. The boundary operator is the base of all Whitney's concepts [13]. To define integration through bounded objects, anyone needs boundaries. Once more, this boundary concept is natural. It is an application from $\mathcal{T}$ to $\mathcal{T}$, $\partial : \mathcal{T} \to \mathcal{T}$, and more precisely, ∂ is an operator from $\mathcal{T}_j$ to $\mathcal{T}_{j-1}$. Its self composition leads to zero: $\forall \theta \in \mathcal{T}^{D(\theta)}$, $\partial \circ \partial |\theta\rangle = 0$. For example, $\partial \circ \partial |a\rangle = 0$: the boudary of an edge is a vertex and the boundary of a vertex is null, or $\partial \circ \partial |f\rangle = 0$: the boundary of a face is a closed line, and the boundary of this closed line is null (remember $\partial \circ \partial = 0$). As we will see in next paragraph, this operator is linked to various connectivities in tensorial algebra. A face boundary is a cycle (a closed line). When we have an edge $|a\rangle$, we can consider its boundary $\partial |a\rangle$ which is a couple of vertices. The boundaries can be developed on the zero chain of vertices :

$$\partial |a\rangle = \sum_{s \in \mathcal{T}^0} B_a^s |a\rangle\ . \tag{2}$$

Take a look to the graph figure 1. We can easily find its incidence B making relations between the vertices s and the edges a:

$$B = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}. \tag{3}$$

Each row is linked to an vertex and each column to an edge.

V. Seeing electrical current as a 1-chain and the potential as a 0-cochain

In the following we consider the electrical current i as an element $|i\rangle$ of the space $\mathcal{T}_1$: on each edge k, the current has a component i^k which is a real number:

$$|i\rangle = \sum_{k \in \mathcal{T}^1} i^k |k\rangle\ . \tag{4}$$

In the tensorial analysis of networks [1] (as previously in classical nodal techniques [12]), the boundary operator applied to edges is called the "incidence". Using the sign rule saying that a current entering a vertex is affected of a plus sign and a current leaving a vertex is affected of a minus sign, it is a matrix that gives the relations between vertices and edges. It is possible to create for each vertex s a linear form $\langle s|$ acting on all the vertices: $\langle s|\sigma\rangle = 0$ if s and σ differ, $\langle s|\sigma\rangle = 1$ if $s = \sigma$. This form belongs to $\mathcal{T}_0^*$: the dual space of $\mathcal{T}_0$ composed by the 0-cochains. Moreover to each vertex $s \in \mathcal{T}^0$ is associated a potential value V_s. With this set of numbers we construct the potential V such that:

$$\langle V| = \sum_{s \in \mathcal{T}^0} V_s \langle s|\ . \tag{5}$$

VI. A topological form of Kirchhoff's laws

We propose here to write the Kirchhoff's laws in a single abstract form as:

$$\langle V|\partial i\rangle = 0,\ \forall V \in \mathcal{T}_0^*,\ \forall i \in \mathcal{T}^1\,. \tag{6}$$

This relation applied to electricity is usual for physicists. If we retain one particular node (for example node 1 on figure 1), the algebraic sum of the currents i^k for k an edge that contains the vertex number 1 is equal to zero; in this case $i^1 - i^2 + i^3 = 0$. To interpret relation (6) as the second Kirchhoff's law relative to the mesh law, we need the mathematical notion of co-boundary.

From the chains $\mathcal{T}_j$, we introduce the space $\mathcal{T}_j^*$ of co-chains of degree j: we have defined the duality product $\langle s|\sigma\rangle$ for two vertices s and σ of the cellular complex. We do the same for each geometrical object of dimension j. For $\alpha \in \mathcal{T}^j$, the dual form $\langle\alpha|$ belongs to $\mathcal{T}_j^*$ and is defined for each $\theta \in \mathcal{T}^j$ by $\langle\alpha|\theta\rangle = 0$ if α and θ differ, $\langle\alpha|\theta\rangle = 1$ if $\alpha = \theta$. The co-boundary operator ∂^o is the polar operator of the boundary operator ∂. By definition for $\varphi \in \mathcal{T}_j^*$ and $\theta \in \mathcal{T}_{j+1}$ we have:

$$\langle\partial^o\varphi|\theta\rangle \equiv \langle\varphi|\partial\theta\rangle\,, \quad \forall\varphi \in \mathcal{T}_j^*,\, \theta \in \mathcal{T}_{j+1}\,. \tag{7}$$

The co-boundary operator ∂^o is defined from each $\mathcal{T}_j^*$ and takes its values in the space $\mathcal{T}_{j+1}^*$. The boundary operator makes decreasing the dimension of the chains while the co-boundary operator makes it increasing.

The co-boundary operator is a good tool to express the second Kirchhoff's law. We re-express the fundamental property (6) in terms of the co-boundary operator:

$$\langle\partial^o V|i\rangle = 0,\ \forall V \in \mathcal{T}_0^*,\ \forall i \in \mathcal{T}^1\,. \tag{8}$$

For each edge a we introduce the potential differences in term of the potential values V_s for each vertex s and the incidence matrix B as introduced in (2):

$$U_a = \sum_{s \in \mathcal{T}^0} B_a^s V_s\,. \tag{9}$$

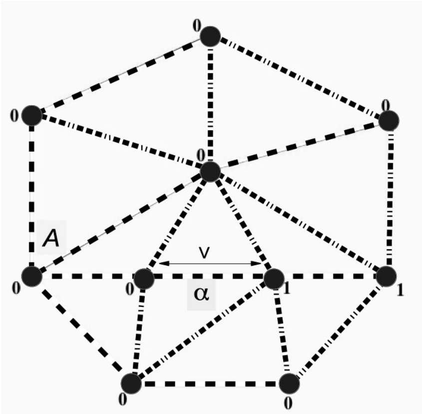

Fig. 1. spanning tree

So we have: $\partial^o V = \sum_{a \in \mathcal{T}^1} U_a \langle a|$. We introduce a closed circuit γ. We test the relation (8) for $i = i_0 \sum_{a \in \gamma} |a\rangle$. Then $\langle \partial^o V | i \rangle >= i_0 \sum_{a \in \mathcal{T}^1} \sum_{b \in \gamma} U_a \langle a|b\rangle = i_0 \sum_{a \in \mathcal{T}^1} U_a = 0$. The mesh Kirchhoff's law express that the sum of the potential differences along a closed circuit is identically equal to zero. For example if we take a look on circuit 2-3-4 figure 1, we have: $U_2 - U_4 - U_3 = 0$.

VII. Spanning tree for pair of nodes currents

We assume now that the network $\mathcal{T}$ is connected. To fix the ideas we suppose more precisely that this network is simply connected *i.e.* does not contain any hole. When this hypothesis is not satisfied (a torus to fix the ideas) we refer to the contribution of Rapetti *et al.* [14]. A spanning tree $\mathcal{A}$ is a subgraph of the set of edges, doesn't contain any cycle, and is composed with a number of edges equal to the number of vertices minus one; if we add to this spanning tree an edge a which doesn't belongs to $\mathcal{A}$ we obtain a cycle γ composed by a and edges of the spanning tree. We refer for a precise definition to the book of Berge [15].

Once a spanning tree $\mathcal{A}$ is fixed, any 1-chain can be decomposed in terms of boundary of faces plus a term associated to the spanning tree. In particular, each current can be decomposed in the previous form:

$$|i\rangle = \sum_{f \in \mathcal{T}^2} \beta_f \partial |f\rangle + \sum_{\alpha \in \mathcal{A}} \theta_\alpha |\alpha\rangle \ . \qquad (10)$$

The first term $\sum_{f \in \mathcal{T}^2} \beta_f \partial |f\rangle$ corresponds to the meshes currents in Kron's terminology, and the second one $\sum_{\alpha \in \mathcal{A}} \theta_\alpha |\alpha\rangle$ to the nodes-pairs currents. The formula (10) describes the direct sum of these two spaces. It is denoted as the "complete space" in Kron's approach.

We have the following theorem: if the current i satisfies the Kirchhoff's law (6) then the nodes pair currents is reduced to zero. We have in relation (10): $\theta_\alpha = 0$ for each edge α of the spanning tree $\mathcal{A}$. The proof can be conducted as follows. Consider an arbitrary edge $\alpha \in \mathcal{A}$. We construct a potential V as the one explicited on the figure 1. We have $\partial^o V_\alpha = \langle \alpha|$ plus a sum related to edges that does not belong to the spanning tree $\mathcal{A}$. Then we have the following calculus: $0 = \langle V_\alpha | \partial i \rangle = \Big\langle V_\alpha | \partial \big(\sum_{f \in \mathcal{T}^2} \beta_f \partial |f\rangle \big) \Big\rangle$ $+ \Big\langle V_\alpha | \partial \big(\sum_{\beta \in \mathcal{A}} \theta_\beta |\beta\rangle \big) \Big\rangle = \Big\langle V_\alpha | \partial \big(\sum_{\beta \in \mathcal{A}} \theta_\beta |\beta\rangle \big) \Big\rangle = \Big\langle \partial^o V_\alpha | \sum_{\beta \in \mathcal{A}} \theta_\beta |\beta \Big\rangle = \Big\langle \alpha | \sum_{\beta \in \mathcal{A}} \theta_\beta |\beta \Big\rangle = \theta_\alpha$. The property is established.

When the Kirchhoff's law are satisfied, the electrical branches currents can be represented by the meshes currents $|i\rangle = \sum_{f \in \mathcal{T}^2} \beta_f \partial |f\rangle$. In the general case, when charges are injected to nodes, the Kirchhoff's laws (6) are no more satisfied. The node pair currents $\sum_{\alpha \in \mathcal{A}} \theta_\alpha |\alpha\rangle$ is not equal to zero and represents these charges variations. In Maxwell's equations, the charge conservation has two terms: "$divJ$" which is represented by the mesh currents and "$\partial_t \rho$" represented by the node pair currents.

VIII. The fundamental space of faces

In electrodynamic we want to make a relation between the meshes currents i and some quantity coming from the flux Φ. It translates the general relation of electrodynamic between currents and electromotive forces. This flux Φ is in relation with a magnetomotive force through i. The induced electrical current described in the previous section can be linked with the meshes currents through $|i\rangle = \sum_{f \in \mathcal{T}^2} \beta_f \partial |f\rangle$. The boundary operator being linear, we can write: $|i\rangle = \partial \left(\sum_{f \in \mathcal{T}^2} \beta_f |f\rangle \right)$. This makes appear clearly the magnetic flux Φ given by:

$$\Phi = \sum_{f \in \mathcal{T}^2} \beta_f |f\rangle \ . \qquad (11)$$

The magnetic flux $\Phi \in \mathcal{T}^2$ is associated with the faces in the complex cellular $\mathcal{T}$. The meshes current i being under this view a boundary current.

We define a dissipation operator W:

$$\mathcal{T}_2 \times \mathcal{T}_2 \ni \ (\Phi, \ \Phi') \longmapsto W(\Phi, \ \Phi') \in \mathbb{R}.$$

It creates a positive defined quadratic form: $W(\Phi, \ \Phi) \geq 0$ and $W(\Phi, \ \Phi) = 0 \Rightarrow \Phi = 0$. This quadratic form W generates a linear application: $\mathcal{T}_2 \ni \Phi \longmapsto \zeta \, \Phi \in \mathcal{T}_2^*$ so that:

$$\langle \zeta \, \Phi | \Psi \rangle = W(\Phi, \ \Psi) \quad \forall \Phi, \ \Psi \in \mathcal{T}_2 \, . \qquad (12)$$

This linear application is nothing else than the impedance operator. On the faces base of $\mathcal{T}_2$ it creates the impedance matrix Z:

$$W\Big(\sum_{f \in \mathcal{T}^2} \alpha_f |f\rangle , \ \sum_{g \in \mathcal{T}^2} \beta_g |g\rangle \Big) \equiv \sum_{f,g \in \mathcal{T}_2} Z_{fg} \alpha_f \beta_g \, . \qquad (13)$$

The energy source for the mesh space is given by the electromotive force e: for $\Phi \in \mathcal{T}_2, \ e \in \mathcal{T}_2^*$ the dual product $\langle e | \Phi \rangle$ is well defined and points out e as the dual source for faces as well as θ for the current source for nodes. The natural space for the electromotive force is cochain of degree 2.

Equilibrating sources and dissipation (in the general sense, *i.e.* used energy, losses or stored ones) we have:

$$W(\Phi, \ \Psi) = \langle e | \Psi \rangle \, , \quad \forall \Psi \in \mathcal{T}_2$$

which means

$$\zeta \, \Phi = e \, . \qquad (14)$$

Then the current can be obtained through:

$$i = \partial \, \zeta^{-1} \, e$$

which is the topological expression for the Kron's tensorial equation

$$i^\mu = y^{\mu\nu} e_\nu \, .$$

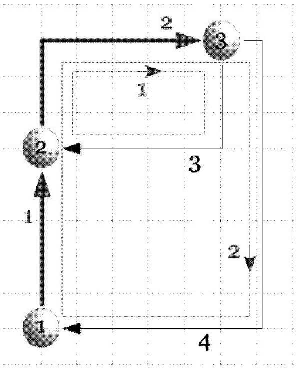

Fig. 2. Graph of a filter

IX. Resolution of networks in complete spaces using the Kron's method

The method starts with a graph. This graph is a engineer view of a real system. Through homotopy, homology, surgery, the problem is projected on a graph [16]. In this operation, we start by finding a ST passing through the various remarkable points of the structure. This tree can be drawn on a sheet. Each point has its own connection with the original 3D space attached to the structure. Figure 2 shows such simple graph obtained from a filter.

On this graph, the ST is repaired by bold lines while closing edges are repaired by thin lines. Meshes are in doted and blue lines. Each edge current can be described depending on ST edges and meshes. For example, current of the edge 1 depends on ST edge 1 and mesh 2. The sign of this dependance is positive, as all currents flows in the same direction. When the meshes are constructed through closing edges from the ST, all the sign of the dependances are positive. This is a remarkable property linked with this construction method. The edge currents projection on ST edges and meshes is synthesized in a connectivity matrix C. If we choose to number the edges firstly as functions of closed edges belonging to the ST, then to meshes, the C matrix has a particular organization:

$$C = \begin{bmatrix} Q & L \\ 0 & I \end{bmatrix} \tag{15}$$

As edges belonging to the ST are firstly numbered, they depend on both closing edges and meshes - that's why the submatrixes Q and L make links between these edges and both current sources and meshes. On the other hand, edges obtained by closing paths depend only on meshes. So a unity matrix links these edges with the meshes, and a zero matrix shows that there's no links between them and the closing paths.

Each edge (a) has its own intrinsic property represented by an operator z_{aa}. This operator can depends on the current value on the edge (non linear one). This operator is a metric component which we talk about in the next paragraph. To present the Kron's method, we accept that this metric has the form:

$$z = \begin{bmatrix} A & B \\ E & D \end{bmatrix} \tag{16}$$

We note i^t the ST edges, i^c the closing path edges, J the current sources belonging to the ST, k the mesh currents. The connectivity is:

$$\begin{bmatrix} i^t \\ i^c \end{bmatrix} = \begin{bmatrix} Q & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} J \\ k \end{bmatrix} \tag{17}$$

The Kirchhoff's law for any edges can be written [12] (it can be used for any physics):

$$\begin{bmatrix} 0 \\ S \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix} + z \begin{bmatrix} i^t \\ i^c \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix} + z \begin{bmatrix} Q & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} J \\ k \end{bmatrix} \tag{18}$$

S are the mesh sources on which we come back later, V the ST potential differences. By multiplying on the left by C^T (index T here is for transpose operation) we make appearing a bilinear transformation $C^T zC$. Noting A', B', E', D' the components of this triple products (they are the component of the metric in the complete space: ST plus meshes), we obtain finally:

$$\begin{bmatrix} 0 \\ S \end{bmatrix} = \begin{bmatrix} QV \\ LV \end{bmatrix} + \begin{bmatrix} A' & B' \\ E' & D' \end{bmatrix} \begin{bmatrix} J \\ k \end{bmatrix} \tag{19}$$

As previously demonstrated $LV = 0$. The first equation resolved is: $S = E'J + D'k \Rightarrow k = (D')^{-1} [S - E'J]$. Then, knowing k, the edges voltages of the ST current sources can be obtained: $QV = -(A'J + B'k)$.

X. About the metric

Usually, a metric is a matrix that can be half-positive, singular [17]. In Kron's theory, as we want to manipulate complex operator through this tensor, we accept non strictly positive matrixes as metric and non symetric. Many mathematicians can object that we are finally far from a metric? But the notion stills very relevant as it describes a distance under preferential paths for currents in the topology, and it leads to the generalized power of the network studied. As said in [8] page 309 (this is a translation from French): *A general theory for linear networks, symetrics or not, should be based on a metric space definition with fundamental tensor...* The connectivity C can be seen as a group transformation that belongs to SO ([17] page 271).

We accept from now to call metric the fundamental tensor obtained from μ. We mean by this that to create the Hodge's relation between the magnetic flux and the electromotive force (emf) we need to give ourselves a metric. This metric is like a rule to make a correspondance between the vectorial surface flux and the scalar emf. The metric makes the link between the first space of mesh currents and the dual quantity of emf obtained from an Hodge process.

From years, electronicians use the *mutual inductance* to translate cross talk between both isolated circuits of a transformer. As we said before, emf and so self-inductances are for us deeply linked with the mesh space. Figure 3 shows the graph for a transformer cross talking two simple circuits. On the classical schematic presented fig. 3, we have two simple circuits having each of them a self-inductance and cross talked through a mutual inductance u. For each faces of each circuit, we can, following Maxwell's laws, compute the electric field circulation [18]:

$$\oint_{\Gamma \equiv \partial f} \vec{E} \cdot d\vec{\Gamma} = \int_{A \in \Gamma} d\Gamma \frac{J}{\sigma} + \int_{B \in \Gamma} d\Gamma \left[\int_t dt \frac{J}{\epsilon} \right] - hE_0 \tag{20}$$

A and B begin part of the boundary ∂f of the face f, hE_0 begin the source electrical work giving energy to the

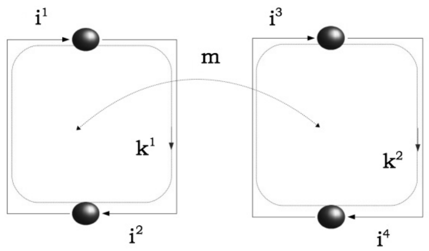

Fig. 3. Graph of a transformer

circuit. Each kind of energy (dissipative through σ, potential through ϵ) can belong to a single edge in the edge space. At the beginning of the circuit description, we have separate information for edges. Always for one circuit, one could be dedicate for dissipative energy (they are the classical resistors in electrical circuits) and another for potential energy (capacitances). Each elementary circuit can be for example a RC one, one edge A being a resistor, and another edge B being the capacitor. When we construct both circuits, we first begin to associate couple of edges in a single mesh. Using the mesh connectivity: $L^T = \left[\begin{array}{cc} 1 & 1 \end{array} \right]$, we obtain a first expression of the metric in the mesh space given for one circuit g_n by:

$$g_n = L^T z L = \left[\begin{array}{cc} 1 & 1 \end{array} \right] \left[\begin{array}{cc} R_n & 0 \\ 0 & \frac{1}{C_n p} \end{array} \right] \left[\begin{array}{c} 1 \\ 1 \end{array} \right] \qquad (21)$$

(p is the Laplace's operator, and R_n and C_n the resistance and capacitance of the circuit n). The metric for the two circuits comes from the union of all the metrics of separate circuits involved: $g = \bigcup_n g_n$. For us it gives :

$$g = \left[\begin{array}{cc} R_1 + \frac{1}{C_1 p} & 0 \\ 0 & R_2 + \frac{1}{C_2 p} \end{array} \right] \qquad (22)$$

As inductances belong only to the mesh space, we must add them to the previous matrix through a new one μ, one including the inductance parts L_n:

$$g = g + \mu, \ \mu = \left[\begin{array}{cc} L_1 p & 0 \\ 0 & L_2 p \end{array} \right] \qquad (23)$$

Now if the first circuit has its own energy source hE_0 - we don't care from where it comes, the second circuit has no self energy generator. But a cross talk creates in its mesh an emf e. This emf comes from the mmf F of the first circuit. Using the previous relations we can write finally a function between the mesh current of the first circuit k^1 and the emf of the second onde e_2: $e_2 = -upk^1$. The cross talk is symetric, it means that the current in the second circuit k^2 creates an emf in the first one: $e_1 = -upk^2$. Finally, the complete magnetic energy tensor added to the one obtain in the edges space becomes :

$$\mu \left[\begin{array}{cc} L_1 p & -up \\ -up & L_2 p \end{array} \right] \qquad (24)$$

We compute the emf added to the edge first circuit in a second step, after transformation of the edge metric in the mesh space and based on a function depending on the mesh current of another circuit. The mutual inductance appears here to be directly an application of the reluctance physic

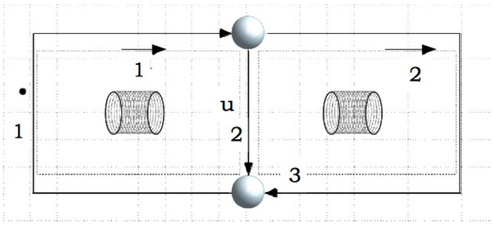

Fig. 4. Isometric graph

[19]. A reluctance network and the associated graph can be construct where edges are tubes and vertex are meshes. This graph is the "chord" that is associated with the function u of mutual coupling. In next paragraph we will generalized this approach for Maxwell's fields. Before we obtain the extradiagonal component of our metric by another way. Another graph gives the same metric that the one of figure 3. Two spaces can be construct with these topologies. An isometric bijection exists between them that doesn't preserve the graph structure, but keep their common metric [20]. We consider the graph presented figure 4.

Always without ST sources, the current can be expressed as: $I = \beta_1 \partial \left| f_1 \right\rangle + \beta_2 \partial \left| f_2 \right\rangle$. The boundaries of the faces can be developed on the edges, depending on the directions chosen: $\partial \left| f_1 \right\rangle = \left| a_1 \right\rangle + \left| a_{12} \right\rangle, \ \partial \left| f_2 \right\rangle = \left| a_2 \right\rangle - \left| a_{12} \right\rangle$ where a_1 is edge 1 on the graph, a_{12} edge 2 and a_3 edge 3. f_1 and f_2 are meshes 1 and 2 (doted blue lines fig.5). If z_1, z_2 and z_3 are the three components of the metric for the three edges 1, 2 and 3, the metric tensor in the mesh space ($L_m : z \rightarrow g$) becomes:

$$g = \left[\begin{array}{cc} z_1 + z_2 & -z_2 \\ -z_2 & z_2 + z_3 \end{array} \right] \qquad (25)$$

Comparing with previous one, we see that: $z_1 = R_1 + (C_1 p)^{-1} + L_1 p - up$, $z_2 = R_2 + (C_2 p)^{-1} + L_2 p - up$ and $z_2 = up$. The natural energy distribution leads to the lagrangian operator obtained in the mesh space using the Kron's description of the networks [21]. Graphs from figure 3 and 4 have not the same topology, but they do have the same metric, that's why we call *isometric* the transformation from one to the other.

XI. Extension of chords to radiative fields

All our previous discussion is based on graphs to describe a topology. When linked with real objects, these objects are reduced to edges through some algebraic topology operations and concepts like homotopy, homology, etc. Representation of reality is made through agencement of edges as pieces. The whole object modeled is a group of R connex networks parts of a global graph. Each of these networks is a set of B edges, joined by N vertexes. The major characteristic of the object is its number of meshes M by respect of the equivalent Poincaré's law for complex cellular: $M = B - N + R$. In each connex network transmission of energy is perfectly controled because it goes from one vertex to another. By this way, the mathematical concepts can be applied to Maxwell's fields quite easily, under the hypothesis that the field behave like a bounded volume able to be represented by an edge. This is true for near and evanescent fields - for which macroscopic modelling are resistors, capacitances, inductances, reluctances, but this is not true for the free radiative field that cannot radiate in a bounded volume independently of the distance. We recall the fundamental difference between these two kinds of fields, and then we apply our previous method to these fields.

A. *Radiated versus evanescent fields*

Basic demonstration for photon starts from the potential vector in the Coulomb's gauge [22]. Under this gauge, there is a formal separation of the transverse part of the field with the other components. Transverse part of the field leads to the photon concept of quantum mechanics. All the evanescent parts are the longitudinal components of the potential and scalar vectors. They can be modeled using inductances, capacitances, etc., that are properties (and components of the metric) of edges or meshes. Strictly speaking, evanescent modes of the field are virtual photon [23]. The big difference for us is that the free field cannot be enclosed in a bounded edge (or a cycle of bounded edge, i.e. a mesh). It radiates in the infinity space. Another remarkable property of the far field is the radiation resistance. On the edge radiating, the property is increased of this radiation resistance. It can be shown that the radiation resistance is intrinsic to the radiated edge. When enclosed in a shielded room, the same edge has its metric modified by the metallic walls of the room. Reflecting the radiated field, the radiation resistance disapears due to the back induction coming from the walls. This process allows to desmontrate that the radiation resistance is natively existing and is modified by the environment [24] [25].

B. *emf and chords process for free radiated fields*

From years, engineers use antennas very simply: an input impedance gives the antenna equivalent circuit seen from the electronics. A radiation diagram describes the radiation of the antenna in all the free space. A gain, gives the relation between the free field and the power received. This very efficient modelling can be translated in our topology description. Basic principles are well explosed in [26]. We present these principles before to translate them in topology.

C. *Antennas principles*

Depending on the antennas gain, the receiving antenna can integrate the energy coming from the emitting antenna. The total radiated power S_r in free space (whatever the distance r from the emitter) is:

$$S_r = \frac{P_t G_t}{4\pi r^2} e^{-\tau p} \tag{26}$$

P_t is the power delivered to the emitter, G_t is its own antenna gain. The last term makes sure of the causality. The available power in reception is given by: $P_r = S_r A_r$. A_r is the effective surface of reception of the antenna. The relation between gain and effective surface is:

$$G_x(\theta,\phi) = \frac{4\pi}{\lambda^2} A_x(\theta,\phi) \tag{27}$$

Both gain or effective aperture (the other name for the effective surface of reception or emission) are functions of the 3D space. Using all the previous relations we find what is the Friss' equation:

$$\frac{P_r}{P_t} = A_r(\theta,\phi)\left[\frac{e^{-\tau p}}{\lambda^2 r^2}\right] A_t(\theta',\phi') \tag{28}$$

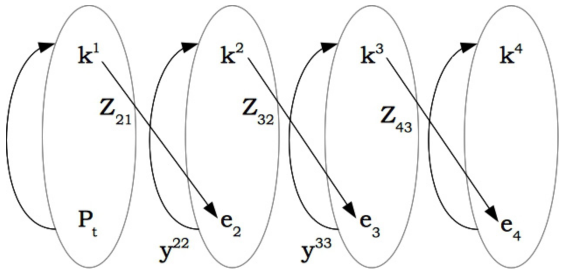

Fig. 5. General free field process

Now we just have to make the links between the power and the topology. For that, fundamental concepts previously stated should be used.

D. *From power to topology*

The total power of the radiated field is obtain through the radiative resistance on an edge. This component represents all the losses due to the radiation. As inductance and capacitance are linked with evanescent and lamellar fields, the radiated and transverse one which leaves the circuit (we consider no losses due to Joule effect in the wires) is linked with a resistance. For example we have: $P_t = R_{11}k^1$ and $P_r = \frac{(e_2)^2}{R_{22}}$. These relations give us the fundamental coupling impedance for far field:

$$z_{21} = \frac{e_2}{k^1} = \sqrt{R_{11}R_{22}A_r(\theta,\phi)\left[G_{\theta,\phi,\theta',\phi'}\right]A_t(\theta',\phi')} \tag{29}$$

where $G_{\theta,\phi,\theta',\phi'}$ is a Green's kernel for the Friis formula. Once more we see that e_2 derive from an integration on a face that belongs to $\mathcal{T}^2$. This kind of interaction can be generalized, whatever the modes of the field propagated. Take a look to figure 5.

We see four simple meshes in interactions. For each of them we can identify the emf and mesh currents, and both emitted and received power. Going from the source of dynamic field k^1 to the induced emf e_4 we can describe the path: $e_4 = z_{43}y^{33}z_{32}y^{22}z_{21}k^1$. The z_{ij} are the coupling metric components and y^{ij} the intrinsic inverse metric of meshes 2 and 3. A_t and A_r are properties attached to the graphs 1 and 4, i.e. to faces that belong to each graph of these circuits. In fact e_2 is the emf linked with A_t: it's a face resulting from a co-boundary applied to the cycle k^1. Each point on this face can be linked with a wave vector. The set of these wave vectors generate the flux S_t (corresponding to k^2): the cotangent manifold of $\mathbb{R}^4$ generated by k^1 using an application ϕ. The component of this set are transformed by the propagation operator yzy to have its image through S_r (k^3 and ϕ^{-1}) where this time, z is not a metric attached to the cellular complex $\mathcal{T}^\infty$ but to the 4D space-time fields propagation $\mathbb{R}^4$. The interface with the receiver is covered by the scalar product with $A_r \to k^3$. Then the emf e_4 is created by k^3. Figure 6 shows the general process involved in the connection between the cell complex $\mathcal{T}^\infty$ and the 4D space $\mathbb{R}^4$

The process can be described by the symbolic equation:

$$\frac{e}{k} = |A_t\rangle\langle S_t|\, G|\, S_r\rangle\langle A_r| \tag{30}$$

This process is very general for all electromagnetical or mechanical phenomenons that involve radiative process. In

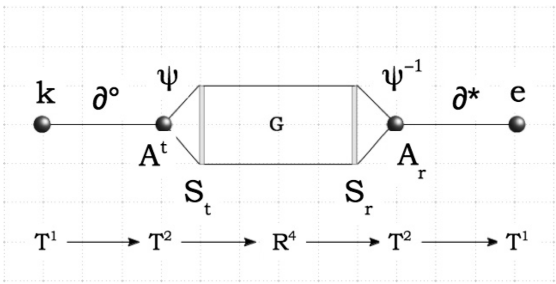

Fig. 6. Sequence of operation for free field

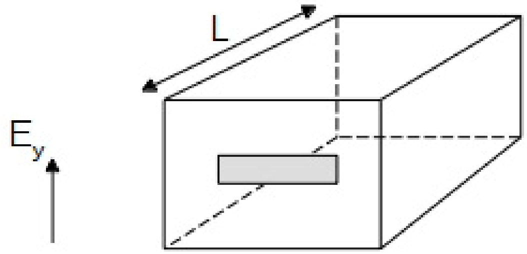

Fig. 7. Electric field on cavity with an aperture

some cases, it can be simplified in three products zyz [27]. But what is remarkable, is that the process should be extendable to any information transmission, starting from the hypothesis that a function ϕ can be created. That's the purpose of the next paragraph.
The $\mathcal{T}^2$ space including the A_x was called the *moment space* [18] because it makes reference to the moment modelling of the radiated field (the magnetic moment is defined by the product of the current i and the emission surface S: $m = iS$ [26]).

XII. Extension to functors

Previous function ϕ is a correspondance between both topological spaces $\mathcal{T}^2$ and $\mathbb{R}^4$: $\phi : \mathcal{T}^2 \rightarrow \mathbb{R}^4$. We define as morphism of graphs, functions that preserve the graph structure, i.e. keep the numbering of vertices, edges, etc. Cell complexes can be used to project any physical phenomenon on graphs. It was already usual to employ electroanalogy for various physics like thermal [28], mechanics [29], quantum mechanics [31] and even biological information [30], [32]. In this last case, the author introduces the two categories of continuous and discrete spaces. This theory seems to be relevant for us, as it gives an algebraic approach for the method of chords [33]. A possible extension of his approach could be to generalize our previous work to functors between these two categories. By the way the method of chords replaces a large spatial domain where free electromagnetic energy propagates by a discrete link between topological objects. This kind of model could be extended to other physical configurations.

XIII. A cavity problem as example

Considering a cavity with an aperture and an impeging parasitic wave, the problem is to compute the internal field induced by the external wave (figure 7). In the following, only the vertical polarization of the incident field will be considered.

This problem can be solved using the formalism developed above. Thanks to the analogy between respectively the electrical field and the voltage and then the magnetic field and the current, it is possible to define electrical equivalent scheme of the problem. In order to make such a representation, we will begin by the definition of different topological domain, corresponding to different regions that will be modeled separatly and then connected together to reconstruct the entire cavity. Three object have to be modeled:

- the incident field that can be assimilated to a voltage source in series with a 377 ohms impedance. This model represents the electrical field propagating in the free space that is equivalent to the mathematical representation in a plane wave shape of the incident field arriving with a normal incidence.

- the aperture, that will be seen from its middle point and that can be represented as two short circuited half transmission lines. The impedance seen from the center has already been extensively studied by different authors. The famous formulation given by Gupta ([2]) has proved to efficiently represent the aperture impedance by a simple formula:

$$Z_a = 120\pi^2[ln(2\frac{1+\sqrt[4]{1-(w_e/b)^2}}{1-\sqrt[4]{1-(w_e/b)^2}})]^{-1} \quad (31)$$

- the cavity that can be seen as a transmission line along the main axis with some particular terminal conditions. On the side containing the aperture, the line will be directly connected in parallel with the aperture impedance model.The opposite side will be short circuited. In the present case, only the fundamental TE propagating mode will be considered. If we want to have results in a more large frequency bandwidth, we will have to consider each mode as an individual transmission line and to connect them in parallel. In order to complete our model, we will add the possibility to realize a measurement in one point in the internal cavity. As in a real experiment, we will introduce a sensor that is modeled here as a resistor having a high value in order to avoid field perturbations inside the cavity. This can be made by cutting the line into two half transmission lines located on each side of the transmission line

After having introduced the topology of our system, it is now important to give the Kron's transmission line model. In fact the more easier way to represent a transmission line is to give a quadrupole model. This quadrupole can be represented by two branches coupled by driven voltage sources. In fact, we can notice that such a representation is nothing else a circuit model of the impedance matrix of the line seen from its two extremities:

$$\begin{cases} V_1 = Z_{11}i^1 + Z_{12}i^2 \\ V_2 = Z_{21}i^1 + Z_{22}i^2 \end{cases} \quad (32)$$

For example, from the branch 1, the first equation shows two terms: the first one represents the influence of the current from this branch and the second one the effect of the current on the first branch. This coupling can be represented by a voltage source, the equation becomes: $V_1 = Z_{11}I_1 + V_c$. As a conclusion, the Kron's model of the cavity is given figure 8.

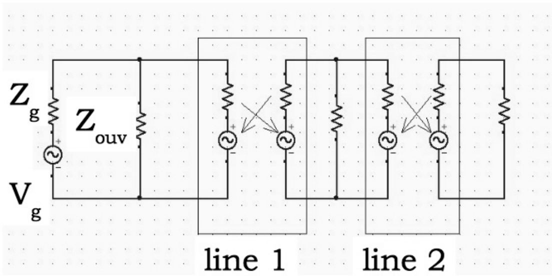

Fig. 8. Kron's model for a cavity

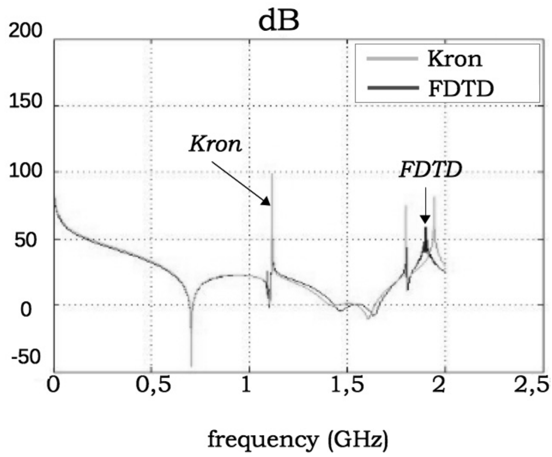

Fig. 9. Comparison with FDTD code

In such an application, the important parameter that is usually modeled is the shielding effectiveness. This quantity was compared with the one obtained with a FDTD code. A very good agreement can be observed figure 9. Other examples between many others are given in [34], [35], [36].

XIV. Antenna modelling

This second example shows the use of the formalism to compute the interaction between an antenna and a metallic wall. The objective of this experience was to understand the effect of a reflexion of energy on the radiation impedance of an antenna. One horn antenna is powered through an amplifier that delivers a amplitude modulated waveform at 10 GHz. 50 cm in front of the antenna, there is a metalic wall or absorbers. The set-up of the experiments is shown figure 10.

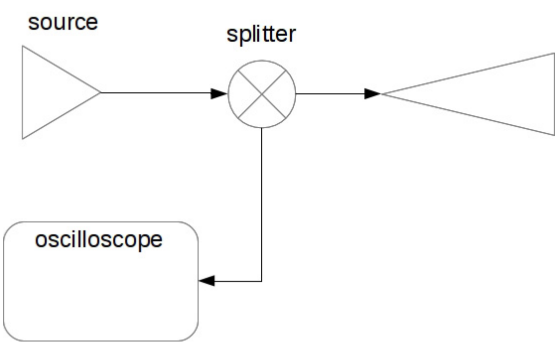

Fig. 10. Experiment set-up

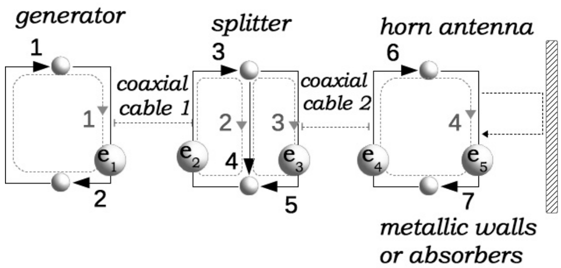

Fig. 11. Experiment equivalent graph

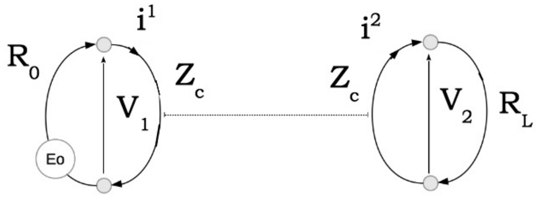

Fig. 12. Simple line

All the cable lengths were measured and the splitter resistors values characterized. Figure 11 shows the graph of the experience.

The splitter is made of three resistors of 17 ohms. Cables are simulated using Branin's model. For a cable of electrical length τ, characteristic impedance Z_c, the Branin's model consists in two equations defining the electromotive force at each extremity of the line:

$$\left\{ \begin{array}{l} e_1 = \left(V_2 - Z_c i^2\right) e^{-\tau p} \\ \\ e_2 = \left(V_1 + Z_c i^1\right) e^{-\tau p} \end{array} \right. \tag{33}$$

V_1 and V_2 are voltages respectively at the left and right of the line. i^1 and i^2 are the currents at the same extremities. Replacing V_1 and V_2 by their expressions depending on the loads and currents, Looking at the circuit figure 12, we obtain:

$$\left\{ \begin{array}{l} V_1 = E_0 - R_0 i^1 \\ \\ V_2 = RLi^2 \end{array} \right. \tag{34}$$

So, by replacement in (33) we understand that:

$$\left\{ \begin{array}{l} e_1 = \left(RL - Z_c\right) e^{-\tau p} i^2 \\ \\ e_2 - E_0 e^{-\tau p} = \left(Z_c - R_0\right) e^{-\tau p} i^1 \end{array} \right. \tag{35}$$

Any expression involving forms like e_i/I^j can be replaced by an impedance interaction z_{ij}. Any line or guided wave structure can be replaced by an impedance tensor as:

$$\left[\begin{array}{cc} R_O + Z_c & \left(RL - Z_c\right) e^{-\tau p} \\ \\ \left(Z_c - R_0\right) e^{-\tau p} & RL + Z_c \end{array} \right] \tag{36}$$

In the graph figure 11, two of these structures were used. The last edge of the graph represents the emitting horn antenna. It is a radiation resistance of fifty ohms. After a delayed time, when the field is reflected by the metallic wall in front of the antenna, a reflected field wave comes back in the horn and creates an electromotive force given by:

$e_5 = G\lambda\left(4\pi 2R\right)^{-1}\sigma\sqrt{R_r}i^4$ (R is the distance to the wall, σ the reflection coefficient on the wall, G the antenna gain and R_r the radiation resistance). This creates an interaction given by e_5/i^4. The complete experience is detailed in [37]. Various measurements were made. Some with the wall equipped of a metallic plate. Some others with absorbers. The variation of radio frequency signal envelop at 10 GHz shows both cases of short circuit on the metallic wall or free space radiation on the absorbers. Difference between the signal compute under the Kron's formalism and measurements is of 1,2%. This performance like others [38] was obtain using the method which allows to couple various accurate equations taken from different previous works.

XV. Conclusion

From fundamental definitions of discrete topology like the boundary operator and the notion of duality, we propose a mathematical model to formalize major results expressed by Gabriel Kron in his "Tensorial Analysis of Networks". One remarkable fact is that the chords introduced in a previous work appear as links between electromotive forces and meshes currents. We present an application of this result to an electromagnetic cavity without the help of three dimensional Maxwell's solver. Another application shows free radiated interaction between an antenna and a wall. This algebraic concept is flexible and the extension is under work for the modeling of multidisciplinary systems with networks.

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