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A Compact Finite Difference Method for Solving the General Rosenau–RLW Equation

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***Abstract*—In this paper, a compact finite difference method to solve the Rosenau–RLW equation is proposed. A numerical tool is applied to the model by using a three–level average implicit finite difference technique. The fundamental conservative property of the equation is preserved by the presented numerical scheme, and the existence and uniqueness of the numerical solution are proved. Moreover, the convergence and stability of the numerical solution are also shown. The new method gives second– and fourth–order accuracy in time and space, respectively. The algorithm uses five–point stencil to approximate the derivatives for the space discretization. The numerical experiments show that the proposed method improves the accuracy of the solution significantly.**

***Index Terms*—finite difference method, Rosenau–RLW equation.**

I. Introduction

A nonlinear wave phenomenon is the important area of scientific research, which many scientists in the past have studied about mathematical models explaining the wave behavior. There are mathematical models which describe the dynamic of wave behaviors–for example, the KdV equation, the RLW equation, the Rosenau equation, and many others [1]–[10]. The KdV equation has been used in very wide applications, such as magnetic fluid waves, ion sound waves, and longitudinal astigmatic waves [4]–[6]. The RLW equation, which was first proposed by Peregrine [7], [8] provides an explanation on a different situation of a nonlinear dispersive wave from the more classical KdV equation. The RLW equation is one of models which are encountered in many areas, e.g. ion–acoustic plasma waves, magnetohydrodynamic plasma waves, and shallow water waves. Since the case of wave–wave and wave–wall interactions cannot be described by the KdV equation, Rosenau [9], [10] proposed an equation for describing the dynamic of dense discrete systems; it is known as the Rosenue equation. The existence and uniqueness of the solution for the Rosenau equation were proved by Park [11], [12]. For the further consideration of the nonlinear wave, a viscous term u_{xxt} needs to be included:

$$u_t - u_{xxt} + u_{xxxxt} + u_x + (u^p)_x = 0, \tag{1}$$

where $p \geq 2$ is an integer and $u_0(x)$ is a known smooth function. This equation is usually called the Rosenua–RLW equation. If $p = 2$, then Eq. (1) is called the usual Rosenau–RLW equation. Moreover, if $p = 3$, then Eq. (1) is called the modified Rosenau–RLW equation. The behavior of the solution to the Rosenau–RLW equation with the Cauchy problem has been well studied for the past years [13]–[18]. It is known that the solitary wave solution for Eq. (1) is

$$u(x,t) = e^{ln\{(p+3)(3p+1)(p+1)/[2(p^2+3)(p^2+4p+7)]\}/(p+1)} \times \operatorname{sech}^{4/(p+1)}\left[\frac{p-1}{\sqrt{4p^2+8p+20}}(x-ct)\right],$$

where $p \geq 2$ is an integer and $c = (p^4 + 4p^3 + 14p^2 + 20p + 25)/(p^4 + 4p^3 + 10p^2 + 12p + 21)$.

The Rosenau–RLW equation has been solved numerically by various methods (for example, see [13]–[18]). Zuo et al. [13] have proposed the Crank–Nocolson finite difference scheme for the equation. The convergence and stability of the proposed method were also discussed. Obviously, the scheme in [13] requires heavy iterative computations because the scheme is nonlinear implicit. Pan and Zhang [14], [15] developed linearized difference schemes which are three–level and conservative implicit for both the usual Rosenau–RLW ($p = 2$) and the general Rosenau–RLW ($p \geq 2$) equations. The second–order accuracy and unconditional stability were also proved.

In this paper, we consider the following initial–boundary value problem of the general Rosenau–RLW equation with an initial condition:

$$u(x,0) = u_0(x), \quad (x_l \leq x \leq x_r), \tag{2}$$

and boundary conditions

$$u(x_l,t) = u(x_r,t) = 0,$$
$$u_{xx}(x_l,t) = u_{xx}(x_r,t) = 0, \quad (0 \leq t \leq T). \tag{3}$$

The initial–boundary value problem possesses the following conservative properties:

$$Q(t) = \int_{x_l}^{x_r} u(x,t)dx = \int_{x_l}^{x_r} u_0(x,0)dx = Q(0),$$

and

$$E(t) = \|u\|_{L_2}^2 + \|u_x\|_{L_2}^2 + \|u_{xx}\|_{L_2}^2 = E(0).$$

When $-x_l \gg 0$ and $x_r \gg 0$, the initial–boundary value problem (1)–(3) is consistent, so the boundary condition (3) is reasonable.

By observation, the total accuracy of a specific method is affected by not only the order of accuracy of the numerical method but also other factors. That is, the conservative approximation property of the method is another factor that has the same or possibly even more impact on results. Better solutions can be expected from numerical schemes which have effective conservative approximation properties rather than the ones which have nonconservative properties [19], [20]. To create the discretization equation, a finite difference

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method is applied in the present research since conservative approximation analysis by the mathematical tools has been developing until now.

The content of this paper is organized as follows. In the next section, we describe a conservative implicit finite difference scheme for the general Rosenau–RLW equation (1) with the initial and boundary conditions (2)–(3). Some preliminary lemmas and discrete norms are given and the invariant property Q^n is proved. We discuss about the solvability of the finite difference scheme, and the existence and uniqueness of the solution are also proved in the Section 3. Section 4 presents complete proofs on the convergence and stability of the proposed method with convergence rate $O(\tau^2+h^4)$. The results of validation for the finite difference scheme are presented in Section 5, where we make a detailed comparison with available data, to confirm and illustrate our theoretical analysis. Finally, we finish our paper by concluding remarks in Section 6.

II. Finite Difference Scheme

In this section, we introduce a finite difference scheme for the formulation of Eqs. (1)–(3). The solution domain $\Omega = \{(x,t)|\ x_l \le x \le x_r,\ 0 \le t \le T\}$ is covered by a uniform grid:

$$\Omega_h = \{(x_i,t_n)|\ x_i = x_l + ih,\ t_n = n\tau, 0 \le i \le M,\ 0 \le n \le N\},$$

with spacings $h = (x_r - x_l)/M$ and $\tau = T/N$. Denote $u_i^n \approx u(x_i,t_n)$,

$$\bar{\Omega}_h = \{(x_i,t_n)|\ x_i = x_l+ih,\ t_n = n\tau,\ -1 \le i \le M+1,\ 0 \le n \le N\},$$

and $Z_h^0 = \{u^n = (u_i^n)|\ u_0 = u_M = 0,\ -1 \le i \le M+1\}$. We use the following notations for simplicity:

$$u_i^{n+\frac{1}{2}} = \frac{u_i^{n+1}+u_i^n}{2}, \qquad \bar{u}_i^n = \frac{u_i^{n+1}+u_i^{n-1}}{2},$$
$$(u_i^n)_t = \frac{u_i^{n+1}-u_i^n}{\tau}, \qquad (u_i^n)_{\hat{t}} = \frac{u_i^{n+1}-u_i^{n-1}}{2\tau},$$
$$(u_i^n)_x = \frac{u_{i+1}^n-u_i^n}{h}, \qquad (u_i^n)_{\bar{x}} = \frac{u_i^n-u_{i-1}^n}{h},$$
$$(u_i^n)_{\hat{x}} = \frac{u_{i+1}^n-u_{i-1}^n}{2h}, \qquad (u^n,v^n) = h\sum_{i=1}^{M-1} u_i^n v_i^n,$$
$$\|u^n\|^2 = (u^n,u^n), \qquad \|u^n\|_\infty = \max_{1\le i\le M-1} |u_i^n|.$$

By setting $w = u_{xxt} - u_x - u_{xxxxt} - (u^p)_x$, Eq. (1) can be written as $w = u_t$. By the Taylor expansion, we obtain

$$w_i^n = (\partial_t u)_i^n = (u_i^n)_{\hat{t}} + O\left(\tau^2\right), \tag{4}$$

and

$$\begin{aligned} w_i^n =& \left[(u_i^n)_{x\bar{x}\hat{t}} - \frac{h^2}{12}\left(\partial_x^4\partial_t u\right)_i^n\right] - \left[(u_i^n)_{\hat{x}} - \frac{h^2}{6}\left(\partial_x^3 u\right)_i^n\right] \\ &- \left[(u_i^n)_{xx\bar{x}\bar{x}\hat{t}} - \frac{h^2}{6}\left(\partial_x^6\partial_t u\right)_i^n\right] \\ &- \left[[(u_i^n)^p]_{\hat{x}} - \frac{h^2}{6}\left(\partial_x^3 u^p\right)_i^n\right] + O\left(h^4\right). \end{aligned} \tag{5}$$

From Eq. (4), we have

$$\left(\partial_x^6\partial_t u\right)_i^n = \left(\partial_x^4\partial_t u\right)_i^n - \left(\partial_x^3 u\right)_i^n - \left(\partial_x^3 u^p\right)_i^n - \left(\partial_x^2 w\right)_i^n. \tag{6}$$

Then,

$$\begin{aligned} w_i^n =& \left[(u_i^n)_{x\bar{x}\hat{t}} - \frac{h^2}{12}\left(\partial_x^4\partial_t u\right)_i^n\right] - \left[(u_i^n)_{\hat{x}} - \frac{h^2}{6}\left(\partial_x^3 u\right)_i^n\right] \\ &- \left[[(u_i^n)^p]_{\hat{x}} - \frac{h^2}{6}\left(\partial_x^3 u^p\right)_i^n\right] - \left[(u_i^n)_{xx\bar{x}\bar{x}\hat{t}}\right. \\ &\left.- \frac{h^2}{6}\left[\left(\partial_x^4\partial_t u\right)_i^n - \left(\partial_x^3 u\right)_i^n - \left(\partial_x^3 u^p\right)_i^n - \left(\partial_x^2 w\right)_i^n\right]\right] \\ &+ O\left(h^4\right). \end{aligned} \tag{7}$$

This implies that

$$\begin{aligned} w_i^n =& (u_i^n)_{x\bar{x}\hat{t}} + \frac{h^2}{12}\left(\partial_x^4\partial_t u\right)_i^n - (u_i^n)_{\hat{x}} - [(u_i^n)^p]_{\hat{x}} \\ &- (u_i^n)_{xx\bar{x}\bar{x}\hat{t}} - \frac{h^2}{6}\left(\partial_x^2 w\right)_i^n + O\left(h^4\right). \end{aligned} \tag{8}$$

Using second–order accuracy for approximation, we obtain

$$\left(\partial_x^4 u\right)_i^n = (u_i^n)_{xx\bar{x}\bar{x}} + O\left(h^2\right),$$
$$\left(\partial_x^2 w\right)_i^n = (w_i^n)_{x\bar{x}} + O\left(h^2\right).$$

The following method is a proposed finite difference scheme to solve the problem (1)–(3):

$$\begin{aligned} (u_i^n)_{\hat{t}} - \left(1-\frac{h^2}{6}\right)(u_i^n)_{x\bar{x}\hat{t}} + \left(1-\frac{h^2}{12}\right)(u_i^n)_{xx\bar{x}\bar{x}\hat{t}} \\ + (u_i^n)_{\hat{x}} + [(u_i^n)^p]_{\hat{x}} = 0; \\ 1 \le i \le M-1,\ 1 \le n \le N-1, \end{aligned} \tag{9}$$

where

$$u_i^0 = u_0(x_i), \quad 0 \le i \le M, \tag{10}$$

$$u_0^n = u_M^n = 0, \ (u_0^n)_{x\bar{x}} = (u_M^n)_{x\bar{x}} = 0, \quad 1 \le n \le N. \tag{11}$$

A three–step method is used for the time discretization of the above described scheme. After the new time discretization of Eq. (9) is performed, three– and five–point stencils approximating the derivatives for the space discretization are used to obtain an algebraic system. The matrix system of Eq. (9) is banded with penta–diagonals and we use a standard routine of the MATLAB to solve the system (9)–(11). The nonlinear term of Eq. (1) is handled by using the linear implicit scheme. Therefore, the equations are solved easily by using the presented method since it does not require extra effort to deal with the nonlinear term.

Lemma 1: (Pan and Zhang [15]) For any two mesh functions $u, v \in Z_h^0$, we have

$$(u_{\hat{x}}, v) = -(u, v_{\hat{x}}),$$
$$(u_x, v) = -(u, v_{\bar{x}}),$$
$$(v, u_{x\bar{x}}) = -(v_x, u_x),$$
$$(u, u_{x\bar{x}}) = -(u_x, u_x) = -\|u_x\|^2.$$

Furthermore, if $(u_0^n)_{x\bar{x}} = (u_M^n)_{x\bar{x}} = 0$, then it implies

$$(u, u_{xx\bar{x}\bar{x}}) = \|u_{x\bar{x}}\|^2.$$

Theorem 2: Suppose that $u_0 \in H_0^2$, then the scheme (9)–(11) is conservative in sense:

$$Q^n = \frac{h}{2}\sum_{i=1}^{M-1}\left(u_i^{n+1} + u_i^n\right) = Q^{n-1} = \ldots = Q^0, \quad (12)$$

under assumptions $u_{-1} = u_1 = 0$ and $u_{M-1} = u_{M+1} = 0$.

Proof: By multiplying Eq. (9) by h, summing up for i from 0 to $M-1$, considering the boundary conditions, and assuming $u_{-1} = u_1 = 0$ and $u_{M-1} = u_{M+1} = 0$, we get

$$\frac{h}{2}\sum_{i=1}^{M-1}\left(u_i^{n+1} - u_i^{n-1}\right) = 0.$$

Then, this gives Eq. (12). ■

Lemma 3: (Discrete Sobolev's inequality [21]) There exist two constants C_1 and C_2 such that

$$\|u^n\|_\infty \le C_1\|u^n\| + C_2\|u_x^n\|.$$

Theorem 4: Suppose $u_0 \in H_0^2[x_l, x_r]$, then the solution u^n satisfies $\|u^n\| \le C$ and $\|u_{xx}^n\| \le C$, which yields $\|u^n\|_\infty \le C$.

Proof: It follows from the initial condition (10) that $u^0 \le C$. The first level u^1 is computed by the fourth–order method. Hence, the following estimates are gotten about $\left\|u^1\right\| \le C$ and $\left\|u^1\right\|_\infty \le C$. Now, we use the induction argument to prove the estimate. We assume that

$$\left\|u^k\right\|_\infty \le C \ \text{ for } \ k = 0, 1, 2, \ldots, n. \quad (13)$$

Taking the inner product of Eq. (9) with $2\bar{u}^n$ and using Lemma 1, we obtain

$$\begin{aligned}\left\|u^{n+1}\right\|^2 - \left\|u^{n-1}\right\|^2 + \left(1 - \frac{h^2}{6}\right)\left(\left\|u_x^{n+1}\right\|^2 - \left\|u_x^{n-1}\right\|^2\right)\\ + \left(1 - \frac{h^2}{12}\right)\left(\left\|u_{x\bar{x}}^{n+1}\right\|^2 - \left\|u_{x\bar{x}}^{n-1}\right\|^2\right)\\ = -2\tau\left((u^n)_{\hat{x}}, 2\bar{u}^n\right) - 2\tau\left([(u^n)^p]_{\hat{x}}, 2\bar{u}^n\right).\end{aligned}$$

According to the Cauchy–Schwarz inequality and direct calculation, it gives

$$\|u_{\hat{x}}^n\| \le \|u_x^n\|,$$

and

$$\left((u^n)_{\hat{x}}, 2\bar{u}^n\right) \le \left(\|u_x^n\|^2 + \frac{1}{2}\left\|u^{n+1}\right\|^2 + \frac{1}{2}\left\|u^{n-1}\right\|^2\right).$$

From Eq. (13), the Cauchy–Schwarz inequality, and Lemma 1, we get

$$\begin{aligned}\left([(u^n)^p]_{\hat{x}}, 2\bar{u}^n\right) &= -h\sum_{i=1}^{M-1}(u_i^n)^p\left(u_i^{n+1} + u_i^{n-1}\right)_{\hat{x}}\\ &\le C\left(\|u^n\|^2 + \frac{1}{2}\left\|u_x^{n+1}\right\|^2 + \frac{1}{2}\left\|u_x^{n-1}\right\|^2\right).\end{aligned}$$

Setting

$$\begin{aligned}B^n = \|u^n\|^2 + \left\|u^{n-1}\right\|^2 + \left(1 - \frac{h^2}{6}\right)\left(\|u_x^n\|^2 + \left\|u_x^{n-1}\right\|^2\right)\\ + \left(1 - \frac{h^2}{12}\right)\left(\|u_{x\bar{x}}^n\|^2 + \left\|u_{x\bar{x}}^{n-1}\right\|^2\right),\end{aligned}$$

then

$$B^{n+1} - B^n \le \tau C\left(B^{n+1} + B^n\right).$$

If τ is sufficiently small, which satisfies $\tau \le \dfrac{k-2}{kC}$ and $k > 2$, then

$$B^{n+1} \le \frac{(1 + \tau C)}{(1 - \tau C)}B^n \le (1 + \tau kC)\,B^n \le \exp(kCT)\,B^0.$$

Hence $\left\|u^{n+1}\right\| \le C$, $\left\|u_x^{n+1}\right\| \le C$, and $\left\|u_{x\bar{x}}^{n+1}\right\| \le C$, which yield $\left\|u^{n+1}\right\|_\infty \le C$ by Lemma 3. ■

III. Solvability

In this section, we prove the existence and uniqueness of our proposed scheme that implies the unique solvability.

Theorem 5: The finite difference scheme (9)–(11) is uniquely solvable.

Proof: By using the mathematical induction, we can determine u^0 uniquely by an initial condition and then choose a fourth–order method to compute u^1. Now, suppose $u^0, u^1, u^2, ..., u^n$ to be solved uniquely. By considering Eq. (9) for u^{n+1}, we have

$$\begin{aligned}\frac{1}{2\tau}u_i^{n+1} - \frac{1}{2\tau}\left(1 - \frac{h^2}{6}\right)\left(u_i^{n+1}\right)_{x\bar{x}} +\\ \frac{1}{2\tau}\left(1 - \frac{h^2}{12}\right)\left(u_i^{n+1}\right)_{xx\bar{x}\bar{x}} = 0. \quad (14)\end{aligned}$$

By taking an inner product of Eq. (14) with u^{n+1}, we obtain

$$\begin{aligned}\frac{1}{2\tau}\left\|u^{n+1}\right\|^2 - \frac{1}{2\tau}\left(1 - \frac{h^2}{6}\right)\left\|u_x^{n+1}\right\|^2\\ + \frac{1}{2\tau}\left(1 - \frac{h^2}{12}\right)\left\|u_{x\bar{x}}^{n+1}\right\|^2 = 0.\end{aligned}$$

By the Cauchy–Schwarz inequality and Lemma 1, we have

$$\left\|u_x^{n+1}\right\|^2 = \left(u^{n+1}, u_{x\bar{x}}^{n+1}\right) \le \frac{1}{2}\left\|u^{n+1}\right\|^2 + \frac{1}{2}\left\|u_{x\bar{x}}^{n+1}\right\|^2.$$

Then,

$$\frac{1}{2}\left\|u^{n+1}\right\|^2 + \left(\frac{1}{2} - \frac{h^2}{12}\right)\left\|u_{x\bar{x}}^{n+1}\right\|^2 = 0.$$

Therefore, Eq. (14) has the only one solution and Eq. (9) u^{n+1} is uniquely solvable. This completes the proof of Theorem 5. ■

IV. Convergence and stability

In this section, we prove the convergence and stability of the scheme (9)–(11). Let $e_i^n = v_i^n - u_i^n$, where v_i^n and u_i^n are the solutions of (1)–(3) and (9)–(11), respectively. Then, we obtain the following error equations:

$$\begin{aligned}r_i^n = (e_i^n)_{\hat{t}} - \left(1 - \frac{h^2}{6}\right)(e_i^n)_{x\bar{x}\hat{t}} + \left(1 - \frac{h^2}{12}\right)(e_i^n)_{xx\bar{x}\bar{x}\hat{t}}\\ + (e_i^n)_{\hat{x}} + [(v_i^n)^p]_{\hat{x}} - [(u_i^n)^p]_{\hat{x}}, \quad (15)\end{aligned}$$

where r_i^n denotes the truncation error. By using the Taylor expansion, it is easy to see that $r_i^n = O(\tau^2 + h^4)$ holds as $\tau, h \to 0$. The following lemmas are essential for the proof of convergence and stability of our scheme.

Lemma 6: (Discrete Gronwall's inequality [21]) Suppose that $\omega(k)$ and $\rho(k)$ are nonnegative functions and $\rho(k)$ is nondecreasing. If $C > 0$ and

$$\omega(k) \le \rho(k) + C\tau\sum_{l=0}^{k-1}\omega(l), \quad \forall k,$$

then

$$\omega(k) \le \rho(k) e^{C\tau k}, \quad \forall k.$$

Lemma 7: (Pan and Zhang [15]) Suppose that $u_0 \in H_0^2[x_l, x_r]$, then the solution u^n of Eqs. (1)–(3) satisfies

$$\|u\|_{L_2} \le C, \qquad \|u_x\|_{L_2} \le C,$$
$$\|u_{xx}\|_{L_2} \le C, \qquad \|u\|_{L_\infty} \le C.$$

The following theorem shows that our scheme converges to the solution with convergence rate $O(\tau^2 + h^4)$.

Theorem 8: Suppose $u_0 \in H_0^2[x_l, x_r]$, then the solution u^n converges to the solution for the problem in the sense of $\|\cdot\|_\infty$ and the rate of convergence is $O(\tau^2 + h^4)$.

Proof: By taking an inner product on both sides of Eq. (15) with $2\bar{e}^n \equiv (e^{n+1} + e^{n-1})$, we get

$$\begin{aligned}
&\left(\|e^{n+1}\|^2 - \|e^{n-1}\|^2\right) + \left(1 - \frac{h^2}{6}\right)\left(\|e_x^{n+1}\|^2 - \|e_x^{n-1}\|^2\right) \\
&+ \left(1 - \frac{h^2}{12}\right)\left(\|e_{x\bar{x}}^{n+1}\|^2 - \|e_{x\bar{x}}^{n-1}\|^2\right) = 2\tau\left(r^n, 2\bar{e}^n\right) \\
&- 2\tau\left(e_{\hat{x}}^n, 2\bar{e}^n\right) - 2\tau\left([(v^n)^p]_{\hat{x}} - [(u^n)^p]_{\hat{x}}, 2\bar{e}^n\right). \quad (16)
\end{aligned}$$

According to the Schwarz inequality, Lemma 1, Theorem 2, and Lemma 7, we obtain

$$\begin{aligned}
&\left([(v^n)^p]_{\hat{x}} - [(u^n)^p]_{\hat{x}}, 2\bar{e}^n\right) \\
&= 2h \sum_{i=1}^{M-1}\left[[(v_i^n)^p]_{\hat{x}} - [(u_i^n)^p]_{\hat{x}}\right]\bar{e}_i^n \\
&= -2h \sum_{i=1}^{M-1}\left[(v_i^n)^p - (u_i^n)^p\right](\bar{e}_i^n)_{\hat{x}} \\
&= 2h \sum_{i=1}^{M-1}\left[e_i^n \sum_{k=1}^{p-2}(v_i^n)^{p-k-2}(u_i^n)^k\right](\bar{e}_i^n)_{\hat{x}} \\
&\le C\left(\|e^n\|^2 + \|\bar{e}_{\hat{x}}^n\|^2\right) \\
&\le C\left(\|e^n\|^2 + \|e_{\hat{x}}^{n-1}\|^2 + \|e_{\hat{x}}^{n+1}\|^2\right). \quad (17)
\end{aligned}$$

By the Cauchy–Schwarz inequality, Lemma 1, and a direct calculation, we obtain

$$\|e_{\hat{x}}^n\| \le \|e_x^n\|, \quad (18)$$

$$\|e_x^n\| = -\left(e^n, e_{x\bar{x}}^n\right) \le \frac{1}{2}\left(\|e^n\|^2 + \|e_{x\bar{x}}^n\|^2\right), \quad (19)$$

$$\left(e_{\hat{x}}^n, 2\bar{e}^n\right) \le \|e_{\hat{x}}^n\|^2 + \frac{1}{2}\left(\|e^{n+1}\|^2 + \|e^{n-1}\|^2\right), \quad (20)$$

$$\left(r^n, 2\bar{e}^n\right) \le \|r^n\|^2 + \frac{1}{2}\left(\|e^{n+1}\|^2 + \|e^{n-1}\|^2\right). \quad (21)$$

From Eqs.(16)–(21), they yield

$$\begin{aligned}
&\left(\|e^{n+1}\|^2 - \|e^{n-1}\|^2\right) \\
&+ \left(1 - \frac{h^2}{6}\right)\left(\|e_x^{n+1}\|^2 - \|e_x^{n-1}\|^2\right) \\
&+ \left(1 - \frac{h^2}{12}\right)\left(\|e_{x\bar{x}}^{n+1}\|^2 - \|e_{x\bar{x}}^{n-1}\|^2\right) \\
&\le 2\tau\|r^n\|^2 + \tau C\Big(\|e^{n-1}\|^2 + \|e^n\|^2 + \|e^{n+1}\|^2 \\
&+ \|e_x^{n-1}\|^2 + \|e_x^n\|^2 + \|e_x^{n+1}\|^2\Big). \quad (22)
\end{aligned}$$

Setting

$$\begin{aligned}
E^n = &\|e^n\|^2 + \|e^{n-1}\|^2 + \left(1 - \frac{h^2}{6}\right)\left(\|e_x^n\|^2 + \|e_x^{n-1}\|^2\right) \\
&+ \left(1 - \frac{h^2}{12}\right)\left(\|e_{x\bar{x}}^n\|^2 + \|e_{x\bar{x}}^{n-1}\|^2\right),
\end{aligned}$$

then Eq. (22) can be rewritten as

$$E^{n+1} - E^n \le 2\tau\|r^n\|^2 + \tau C\left(E^{n+1} + E^n\right),$$

and

$$(1 - 2\tau C)\left(E^{n+1} - E^n\right) \le \tau\|r^n\|^2 + 2\tau C E^n.$$

If τ is sufficiently small, which satisfies $1 - 2C\tau > 0$, then

$$E^{n+1} - E^n \le \tau C\|r^n\|^2 + \tau C E^n. \quad (23)$$

Summing up Eq. (23) from 1 to n, we have

$$E^{n+1} \le E^1 + C\tau\sum_{k=1}^{n}\|r^k\|^2 + C\tau\sum_{k=1}^{n}E^k. \quad (24)$$

Thus, we can use a fourth–order method to compute u^1 such that

$$E^1 \le O(\tau^2 + h^4)^2,$$

and

$$\tau\sum_{k=1}^{n}\|r^k\|^2 \le n\tau \max_{0 \le l \le n-1}\|r^l\|^2 \le T \cdot O(\tau^2 + h^4)^2.$$

By Lemma 6, we obtain $E^n \le O(\tau^2 + h^4)^2$, that is

$$\|e^n\| \le O(\tau^2 + h^4), \quad \|e_{x\bar{x}}^n\| \le O(\tau^2 + h^4).$$

From Eq. (20), we obtain

$$\|e^n\| \le O(\tau^2 + h^4), \ \|e_x^n\| \le O(\tau^2 + h^4),$$

and

$$\|e_{x\bar{x}}^n\| \le O(\tau^2 + h^4).$$

By Lemma 3,

$$\|e^n\|_\infty \le O(\tau^2 + h^4).$$

This completes the proof. ■

Theorem 9: Under the conditions of Theorem 8, the solution u^n of Eqs. (9)–(11) is stable in norm $\|\cdot\|_\infty$.

V. Numerical Experiments

In this section, we present numerical experiments on a test problem to confirm and illustrate the accuracy of our proposed method. The accuracy of the method is measured by the comparison of numerical solutions with exact solutions as well as other numerical solutions from the method in the literature [15] by using $\|\cdot\|$ and $\|\cdot\|_\infty$ norm. The initial condition associated for the Rosenau–RLW equation takes the form:

$$u_0(x) = e^{ln\{(p+3)(3p+1)(p+1)/[2(p^2+3)(p^2+4p+7)]\}/(p+1)} \times \operatorname{sech}^{4/(p+1)}\left[\frac{p-1}{\sqrt{4p^2+8p+20}}(x)\right].$$

TABLE I
COMPARISON OF ERRORS WITH $\tau = 0.1$, $h = 0.25$, $x_l = -60$, AND $x_r = 120$ AT $t = 40$.

	$\|e\| \times 10^{-2}$		$\|e\|_\infty \times 10^{-3}$	
p	Present	Pan&Zhang	Present	Pan&Zhang
2	0.23608	0.78777	0.88670	2.88972
4	0.47254	1.73066	1.81252	6.47969
8	0.46713	1.80583	1.75739	6.66740
16	0.38438	1.37857	1.30630	5.05919

TABLE II
COMPARISON OF ERRORS WITH $\tau = 0.1$, $h = 0.5$, $x_l = -60$, AND $x_r = 120$ AT $t = 40$.

	$\|e\| \times 10^{-2}$		$\|e\|_\infty \times 10^{-2}$	
p	Present	Pan&Zhang	Present	Pan&Zhang
2	0.230294	3.25288	0.086284	1.19460
4	0.447881	7.45173	0.171122	2.78712
8	0.431841	8.03730	0.161891	2.95337
16	0.357253	6.13044	0.118759	2.25471

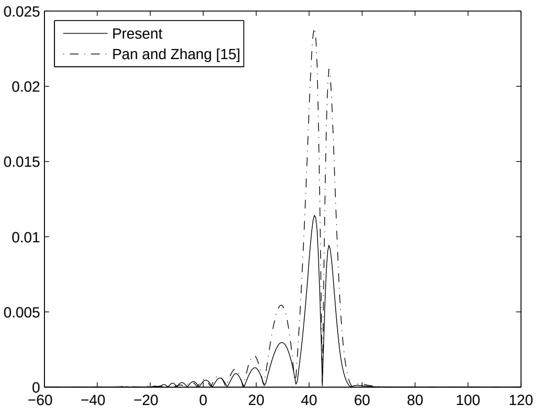

Fig. 1. Absolute error distribution at $p = 4, h = 0.5, \tau = h^2$, and $t = 40$.

For u^1, we employ a two–level method to estimate the solution by

$$(u_i^n)_t - \left(1 - \frac{h^2}{6}\right)(u_i^n)_{x\bar{x}t} + \left(1 - \frac{h^2}{12}\right)(u_i^n)_{xx\bar{x}\bar{x}t} + \left(u_i^{n+\frac{1}{2}}\right)_{\hat{x}} + [(u_i^n)^p]_{\hat{x}} = 0;$$
$$1 \le i \le M-1, \;\; 1 \le n \le N-1. \quad (25)$$

We make a comparison between the scheme (9)–(11) and the scheme proposed in [15]. The rate of convergence is computed using two grids, according to the formula:

$$\text{Rate} = \log_2 \frac{\|e_h\|}{\|e_{h/2}\|}.$$

The results in term of errors at $t = 40$, $\tau = 0.1$, and different p, by using $x_l = -60$ and $x_r = 120$, with $h = 0.25$ and $h = 0.5$ are reported in Tables I and II. It is clear that the results obtained by the scheme (9)–(11) are more accurate than the ones obtained by the scheme in [15].

Absolute error distributions for the two methods with $\tau = 0.25, h = 0.5$, and $t = 40$ are drawn at $p = 4$ and 8 in Figs. 1 and 2, respectively. The results obtained by the scheme (9)–(11) are greatly improved when compared to those by the scheme in [15]. It can be easily observed that the maximum error is taken place around the peak amplitude of the solitary wave and then the scheme (9)–(11) is applied in this area.

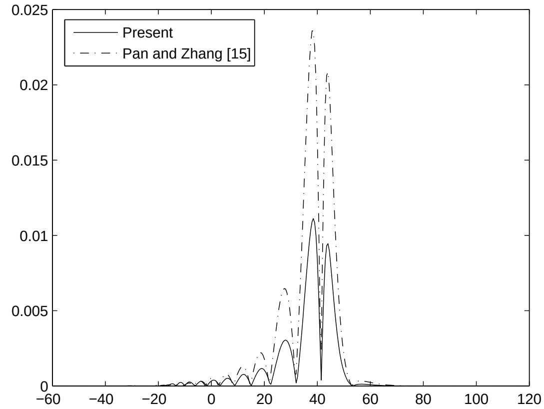

Fig. 2. Absolute error distribution at $p = 8, h = 0.5, \tau = h^2$, and $t = 40$.

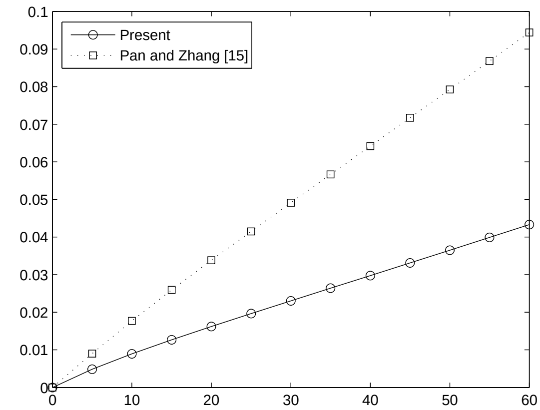

Fig. 3. Error $\|e\|$ versus t at $p = 4, h = 0.5$, and $\tau = h^2$.

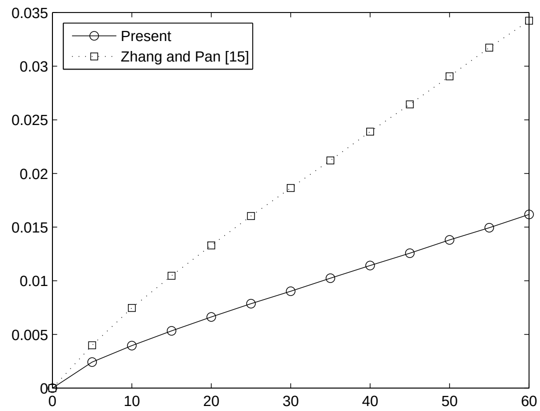

Fig. 4. Error $\|e\|_\infty$ versus t at $p = 4, h = 0.5$, and $\tau = h^2$.

Figs. 3–6 show errors at $t \in [0, 60]$ with $\tau = 0.25$, $h = 0.5$, and $p = 4, 8$ by comparing with the Pan&Zhang method [15]. It is observed that both errors increase with time quite linearly but the error of the present method is less than that of the Pan&Zhang method [15].

As shown in Tables III and IV, on one particular choice of the parameters, the estimated rate is close to the theoretically

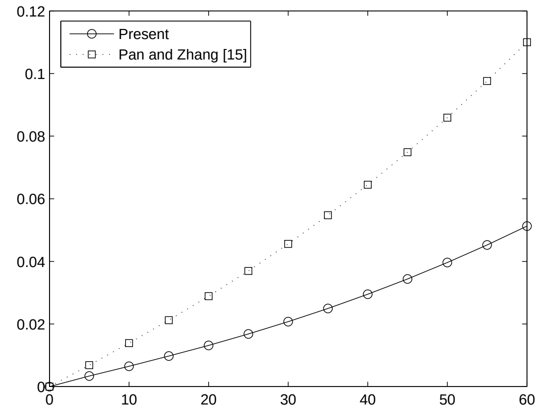

Fig. 5. Error $\|e\|$ versus t at $p = 8, h = 0.5$, and $\tau = h^2$.

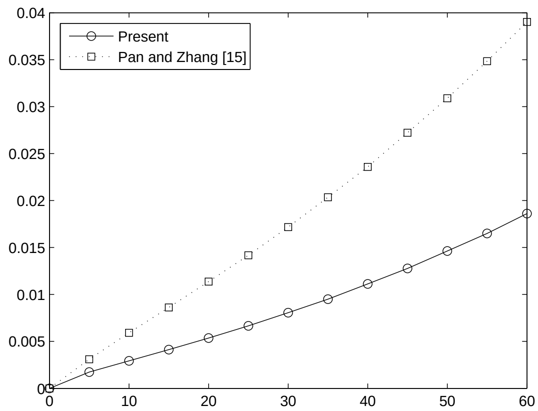

Fig. 6. Error $\|e\|_\infty$ versus t at $p = 8, h = 0.5$, and $\tau = h^2$.

predicted fourth–order rate of convergence. We can also say that when we use smaller time and space steps, numerical values are almost the same as exact values. The CPU time for two methods are listed in Tables III and IV. It can be seen that the computational efficiency of the present method are slightly better than that of Pan&Zhang method [15], in term of CPU time. However, the construction of the novel scheme requires only a regular five–point stencil at a higher time level, which is similar to the standard second–order Crank–Nicolson scheme and Pan&Zhang scheme [15].

As in Tables V and VI, the values of Q^n and E^n at any time $t \in [0, 40]$, which results from the present method, coincide with the theory. The quantities Q^n and E^n seem to be conserved on the average, i.e. they are contained in a small interval but there are fluctuations.

Figs. 7 and 8 show numerical solutions at $t = 200$ with $p = 4$ and 8. The results from the Pan&Zhang method [15] are slightly oscillate at the left side of the solitary wave in case of $p = 8$. However, the results from the present method are almost perfectly sharp in both cases $p = 4$ and 8. From the point of view for the long time behavior of the resolution, the present method can be seen to be much better than the method in [15].

The solitary waves obtained by the present scheme are plotted in Figs. 9 and 10 using $\tau = 0.25, h = 0.5, x_l = -60, x_r = 200$, and $p = 4, 8$. The solitons at $t = 60$ and

TABLE III
RATE OF CONVERGENCE AND CPU TIME WITH $p = 4$ AND $t = 40$.

	$\tau = 0.25,\ h = 0.5$		
	τ, h	$\frac{\tau}{4}, \frac{h}{2}$	$\frac{\tau}{16}, \frac{h}{4}$
Present			
$\|e\| \times 10^{-2}$	3.20548	0.197080	0.0123084
Rate		4.02369	4.00106
$\|e\|_\infty \times 10^{-2}$	1.22483	0.0752290	0.00469781
Rate		4.02515	4.00123
CPU time (s)	1.153389	12.866165	155.967273
Pan&Zhang			
$\|e\| \times 10^{-2}$	6.41825	1.85385	0.479643
Rate		1.79165	1.95050
$\|e\|_\infty \times 10^{-2}$	2.38960	0.696030	0.180409
Rate		1.77955	1.94788
CPU time (s)	1.251865	13.534488	157.561488

TABLE IV
RATE OF CONVERGENCE AND CPU TIME WITH $p = 8$ AND $t = 40$.

	$\tau = 0.25,\ h = 0.5$		
	τ, h	$\frac{\tau}{4}, \frac{h}{2}$	$\frac{\tau}{16}, \frac{h}{4}$
Present			
$\|e\| \times 10^{-2}$	3.18080	0.194284	0.0121337
Rate		4.03315	4.00108
$\|e\|_\infty \times 10^{-2}$	1.19513	0.0727869	0.00454621
Rate		4.03734	4.00094
CPU time	1.21464	13.868260	174.397644
Pan&Zhang			
$\|e\| \times 10^{-2}$	6.44908	1.99919	0.525426
Rate		1.68968	1.92785
$\|e\|_\infty \times 10^{-2}$	2.35870	0.739615	0.194938
Rate		1.67314	1.92376
CPU time	1.371416	14.862871	175.068007

TABLE V
DISCRETE MASS Q^n.

	$\tau = 0.25,\ h = 0.5$	
t	$p = 4$	$p = 8$
$t = 10$	6.26580620079700	9.74208591413665
$t = 20$	6.26580620078861	9.74208595412127
$t = 30$	6.26580619948382	9.74208578472995
$t = 40$	6.26580617252808	9.74208558745239
$Q(0)$	6.26580620079328	9.74208618205024

TABLE VI
DISCRETE ENERGY E^n.

	$\tau = 0.25,\ h = 0.5$	
t	$p = 4$	$p = 8$
$t = 10$	2.86723006370139	4.73479863443071
$t = 20$	2.86725271321602	4.73481771538282
$t = 30$	2.86726739317968	4.73483391314363
$t = 40$	2.86727839480750	4.73485101919594
$E(0)$	2.86718872840474	4.73477831492679

120 agree with the soliton at $t = 0$ quite well, which also shows the accuracy of the scheme.

VI. CONCLUSIONS

The new conservative finite difference scheme for the Rosenau–RLW equation is introduced and analyzed. The present method gives an implicit linear system, which can be easily implemented. This method shows the second– and fourth–order accuracy in time and space, respectively. In

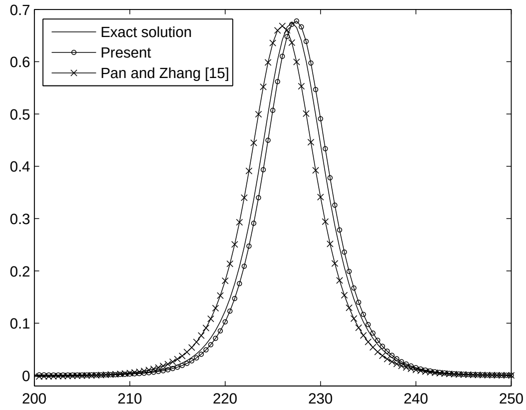

Fig. 7. Numerical solutions at $p = 4, x_l = -60, x_r = 300, h = 0.5$, $\tau = h^2$, and $t = 200$.

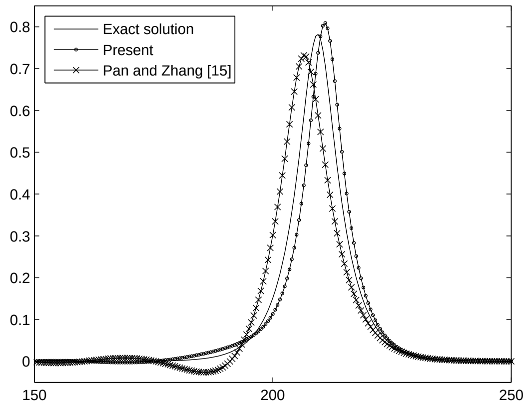

Fig. 8. Numerical solutions at $p = 8, x_l = -60, x_r = 300, h = 0.5$, $\tau = h^2$, and $t = 200$.

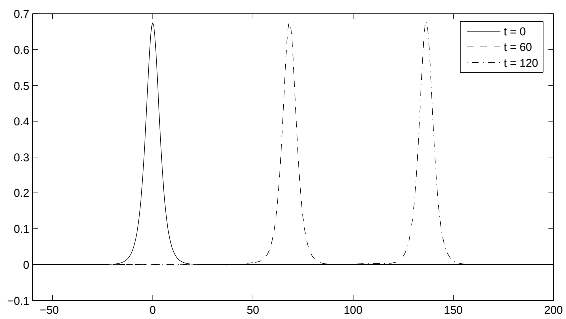

Fig. 9. Numerical solutions at $p = 4$.

addition, the numerical experiments show that the present method supports the analysis of convergence rate.

It is obvious from numerical experiments that the present method, the scheme (9)–(11), gives the well resolution for the Rosenau–RLW equation. It is possible that the solitary wave obtained by this novel method can be smoothed out, at long time, by type of the high–order accuracy.

ACKNOWLEDGMENT

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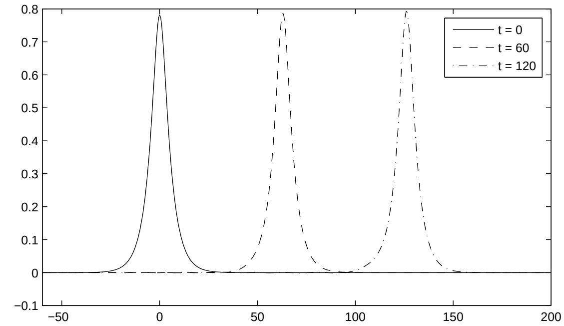

Fig. 10. Numerical solutions at $p = 8$.

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