Further Results on 3–equitable Labeling

Gaurang V. Ghodasara, Sunil G. Sonchhatra

Abstract—A mapping \( f \) from the vertex set of a graph \( G \) to the set \( \{0, 1, 2\} \) is called 3–equitable labeling, if the edge labels are produced by the absolute difference of labels of end vertices such that the absolute difference of number of edges of \( G \) labeled with 0, 1 and 2 differ by atmost 1 and similarly the absolute difference of number of edges of \( G \) labeled with 0, 1 and 2 differ by atmost 1. A graph which admits 3–equitable labeling is called 3–equitable graph. In this paper we prove that the graph obtained by joining two copies of fan graph by a path of arbitrary length is 3–equitable. We also prove similar results for wheel, helm, gear and cycle with one pendant edge.

Index Terms—3–equitable graph, Fan, Wheel, Helm, Gear.

I. INTRODUCTION

We consider simple, finite, undirected graph \( G = (V, E) \). In this paper \( F_n \) denotes fan graph with \( n + 1 \) vertices, \( W_n \) denotes the wheel graph with \( n + 1 \) vertices, \( H_n \) denotes helm graph with \( 2n + 1 \) vertices and \( G_n \) denotes gear graph with \( 2n + 1 \) vertices. For all other terminology and notations we follow Gross and Yellen[5].

If the vertices of the graph are assigned values subject to certain conditions is known as graph labeling. A survey on graph labeling is given by Gallian[4].

Definition 1.1 A fan graph, denoted by \( F_n \), is the graph with \( n + 1 \) vertices which is the join of the graphs \( P_n \) and \( K_1 \), i.e. \( F_n = P_n + K_1 \).

Definition 1.2 A wheel graph, denoted by \( W_n \), is the join of the graphs \( C_n \) and \( K_1 \), i.e. \( W_n = C_n + K_1 \). Here vertices correspond to \( C_n \) are called rim vertices and \( C_n \) is called rim of \( W_n \) and the vertex corresponding to \( K_1 \) is called apex vertex.

Definition 1.3 A helm graph, denoted by \( H_n \), is the graph obtained from wheel \( W_n \) by adding a pendant edge at each vertex on rim of \( W_n \).

Definition 1.4 A gear graph, denoted by \( G_n \), is obtained from the wheel \( W_n \) by adding a vertex between every pair of adjacent vertices of the rim of wheel.

Definition 1.5 Let \( G = (V, E) \) be a graph. A mapping \( f : V(G) \to \{0, 1, 2\} \) is called ternary vertex labeling of \( G \) and \( f(v) \) is called label of the vertex \( v \) of \( G \) under \( f \).

Let \( f^* : E(G) \to \{0, 1, 2\} \) be the induced edge labeling function defined by \( f^*(e) = |f(u) - f(v)| \), for an edge \( e = uv \) of \( G \). Let us denote \( v_f(i) \) is the number of vertices of \( G \) with label \( i \) under \( f \) and \( e_f(i) \) is the number of edges of \( G \) with label \( i \) under \( f^* \), \( 0 \leq i \), \( j \leq 2 \).

Definition 1.6 A ternary vertex labeling of a graph \( G \) is called 3–equitable labeling if \( |v_f(i) - v_f(j)| \leq 1 \) and \( |e_f(i) - e_f(j)| \leq 1, 0 \leq i, j \leq 2 \).

A graph \( G \) is 3–equitable if it admits 3–equitable labeling. The concept of 3–equitable labelings was introduced by Cahit[2] in 1990. Seoud and Abdel Maqsooud[6] proved that all fans except \( P_2 + K_1 \) are 3–equitable. Bapat and Limaye[1] proved that Helm \( H_n \) is 3–equitable for \( n \geq 4 \). Youssef[11] proved that Wheels \( W_n = C_n + K_1 \) and \( W_n \) are 3–equitable for \( n \geq 4 \). In this paper we prove that the graph obtained by joining two copies of fan graphs by a path of arbitrary length is 3–equitable. We also prove similar results for wheel, helm, gear and cycle with one pendant edge.

II. MAIN RESULTS

Theorem 1 The graph obtained by joining two copies of fan graph \( F_n \) by a path of arbitrary length is 3–equitable.

Proof: Let \( G \) be the graph obtained by joining two copies of fan graph \( F_n = P_n + K_1 \) by a path \( P_k \) of length \( k - 1 \). Let us denote the successive vertices of first copy of fan graph by \( u_1, u_2, \ldots, u_{n+1} \) (where \( u_1 \) is the vertex corresponding to \( K_1 \)) and the successive vertices of second copy of fan graph by \( v_1, v_2, \ldots, v_{n+1} \) (where \( v_1 \) is the vertex corresponding to \( K_1 \)). Let \( w_1, w_2, \ldots, w_k \) be the vertices of path \( P_k \) with \( w_1 = u_1 \) and \( w_k = v_1 \).

We define labeling function \( f : V(G) \to \{0, 1, 2\} \) as follows.

Case 1: \( n \equiv 0 \mod 6 \).

Subcase I: \( k \equiv 0 \mod 6 \).

\( f(u_i) = 0; \) if \( i \equiv 1, 4 \mod 6 \)

\( = 1; \) if \( i \equiv 2, 3 \mod 6 \)

\( = 2; \) if \( i \equiv 0, 5 \mod 6 \), \( 1 \leq i \leq n + 1 \).

\( f(v_i) = 0; \) if \( i \equiv 0, 3 \mod 6 \)

\( = 1; \) if \( i \equiv 4, 5 \mod 6 \)

\( = 2; \) if \( i \equiv 1, 2 \mod 6 \), \( 1 \leq i \leq n + 1 \).

\( f(w_j) = 0; \) if \( j \equiv 1, 4 \mod 6 \)

\( = 1; \) if \( j \equiv 2, 3 \mod 6 \)

\( = 2; \) if \( j \equiv 0, 5 \mod 6 \), \( 1 \leq j \leq k - 1 \).

Subcase II: \( k \equiv 1 \mod 6 \).

\( f(w_k) = 2. \)

\( f(w_j) = 0; \) if \( j \equiv 1, 4 \mod 6 \)

\( = 1; \) if \( j \equiv 2, 3 \mod 6 \)

\( = 2; \) if \( j \equiv 0, 5 \mod 6 \), \( 1 \leq j \leq k - 1 \).

The remaining vertices are labeled same as in Subcase I.

Subcase III: \( k \equiv 2 \mod 6 \).

\( f(u_i) = 0; \) if \( i \equiv 1, 4 \mod 6 \)

\( = 1; \) if \( i \equiv 2, 3 \mod 6 \)

\( = 2; \) if \( i \equiv 0, 5 \mod 6 \), \( 1 \leq i \leq n + 1 \).

\( f(v_i) = 0; \) if \( i \equiv 2, 5 \mod 6 \)

\( = 1; \) if \( i \equiv 3, 4 \mod 6 \)

\( = 2; \) if \( i \equiv 0, 1 \mod 6 \), \( 1 \leq i \leq n + 1 \).

\( f(w_j) = 0; \) if \( j \equiv 1, 4 \mod 6 \)

\( = 1; \) if \( j \equiv 0, 5 \mod 6 \)

\( = 2; \) if \( j \equiv 2, 3 \mod 6 \), \( 1 \leq j \leq k \).

Subcase IV: \( k \equiv 3 \mod 6 \).

\( f(v_1) = 1, f(v_4) = 0, f(v_{n+1}) = 1 \).

\( f(v_i) = 0; \) if \( i \equiv 2, 5 \mod 6 \)

(Advance online publication: 17 February 2015)
The remaining vertices are labeled same as in Subcase III.

Subcase V: \( k \equiv 4 \pmod{6} \).
\[ f(v_{n+1}) = 1, \quad f(w_k) = 2. \]
The remaining vertices are labeled same as in Subcase III.

Subcase VI: \( k \equiv 5 \pmod{6} \).
\[ f(u_i) = 0; \quad \text{if } i \equiv 0, 3 \pmod{6} \]
\[ = 1; \quad \text{if } i \equiv 4, 5 \pmod{6} \]
\[ = 2; \quad \text{if } i \equiv 1, 2 \pmod{6}, \ 1 \leq i \leq n + 1. \]
\[ f(v_i) = 0; \quad \text{if } i \equiv 1, 4 \pmod{6} \]
\[ = 1; \quad \text{if } i \equiv 2, 3 \pmod{6} \]
\[ = 2; \quad \text{if } i \equiv 0, 5 \pmod{6}, \ 1 \leq i \leq n + 1. \]
\[ f(w_j) = 0; \quad \text{if } j \equiv 1, 4 \pmod{6} \]
\[ = 1; \quad \text{if } j \equiv 2, 3 \pmod{6} \]
\[ = 2; \quad \text{if } j \equiv 0, 1 \pmod{6}, \ 1 \leq j \leq k. \]

Case 3: \( n \equiv 2 \pmod{6} \).

Subcase I: \( k \equiv 0 \pmod{6} \).
\[ f(u_i) = 0; \quad \text{if } i \equiv 1, 4 \pmod{6} \]
\[ = 1; \quad \text{if } i \equiv 0, 5 \pmod{6} \]
\[ = 2; \quad \text{if } i \equiv 2, 3 \pmod{6}, \ 1 \leq i \leq n + 1. \]
\[ f(v_i) = 0; \quad \text{if } i \equiv 1, 4 \pmod{6} \]
\[ = 1; \quad \text{if } i \equiv 2, 3 \pmod{6} \]
\[ = 2; \quad \text{if } i \equiv 0, 1 \pmod{6}, \ 1 \leq i \leq n + 1. \]
\[ f(w_j) = 0; \quad \text{if } j \equiv 1, 4 \pmod{6} \]
\[ = 1; \quad \text{if } j \equiv 2, 3 \pmod{6} \]
\[ = 2; \quad \text{if } j \equiv 0, 5 \pmod{6}, \ 1 \leq j \leq k. \]

Case 4: \( n \equiv 3 \pmod{6} \).

Subcase I: \( k \equiv 0 \pmod{6} \).
\[ f(u_i) = 0; \quad \text{if } i \equiv 1, 4 \pmod{6} \]
\[ = 1; \quad \text{if } i \equiv 2, 3 \pmod{6} \]
\[ = 2; \quad \text{if } i \equiv 0, 5 \pmod{6}, \ 1 \leq i \leq n + 1. \]
\[ f(v_i) = 0; \quad \text{if } i \equiv 1, 4 \pmod{6} \]
\[ = 1; \quad \text{if } i \equiv 2, 3 \pmod{6} \]
\[ = 2; \quad \text{if } i \equiv 0, 5 \pmod{6}, \ 1 \leq i \leq n + 1. \]
\[ f(w_j) = 0; \quad \text{if } j \equiv 1, 4 \pmod{6} \]
\[ = 1; \quad \text{if } j \equiv 2, 3 \pmod{6} \]

(Advance online publication: 17 February 2015)
Subcase II: \( k \equiv 2 \mod 6 \).

\[
\begin{align*}
  f(v_0) &= 0; \quad \text{if } i \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } i \equiv 0, 5 \mod 6 \\
  &= 2; \quad \text{if } i \equiv 2, 3 \mod 6, \ 1 \leq i \leq n + 1.
\end{align*}
\]

The remaining vertices are labeled as in Subcase I.

Subcase III: \( k \equiv 2 \mod 6 \).

\[
\begin{align*}
  f(v_0) &= 0; \quad \text{if } i \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } i \equiv 0, 5 \mod 6 \\
  &= 2; \quad \text{if } i \equiv 2, 3 \mod 6, \ 1 \leq i \leq n + 1.
\end{align*}
\]

\[
\begin{align*}
  f(v_{n-1}) &= 2. \\
  f(v_i) &= 0; \quad \text{if } i \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } i \equiv 0, 5 \mod 6 \\
  &= 2; \quad \text{if } i \equiv 2, 3 \mod 6, \ 1 \leq i \leq n + 1.
\end{align*}
\]

The remaining vertices are labeled as in Subcase I.

Subcase IV: \( k \equiv 3 \mod 6 \).

\[
\begin{align*}
  f(v_0) &= 0; \quad \text{if } i \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } i \equiv 0, 5 \mod 6 \\
  &= 2; \quad \text{if } i \equiv 2, 3 \mod 6, \ 1 \leq i \leq n + 1.
\end{align*}
\]

\[
\begin{align*}
  f(v_{n-1}) &= 2, \quad f(v_{n-3}) = 0. \\
  f(v_i) &= 0; \quad \text{if } i \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } i \equiv 0, 5 \mod 6 \\
  &= 2; \quad \text{if } i \equiv 2, 3 \mod 6, \ 1 \leq i \leq n + 1,
\end{align*}
\]

\[
i \neq n - 3, \ 4 \neq i - n - 1.
\]

\[
\begin{align*}
  f(w_k) &= 0. \\
  f(w_j) &= 0; \quad \text{if } j \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } j \equiv 2 \mod 6 \\
  &= 2; \quad \text{if } j \equiv 0, 5 \mod 6, \ 1 \leq j \leq k - 1.
\end{align*}
\]

Subcase V: \( k \equiv 4 \mod 6 \).

\[
\begin{align*}
  f(w_k) &= 2. \\
  f(w_j) &= 0; \quad \text{if } j \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } j \equiv 2 \mod 6 \\
  &= 2; \quad \text{if } j \equiv 0, 5 \mod 6, \ 1 \leq j \leq k - 1.
\end{align*}
\]

The remaining vertices are labeled as in Subcase I.

Subcase VI: \( k \equiv 5 \mod 6 \).

\[
\begin{align*}
  f(u_i) &= 0; \quad \text{if } i \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } i \equiv 2, 3 \mod 6 \\
  &= 2; \quad \text{if } i \equiv 0, 5 \mod 6, \ 1 \leq i \leq n + 1.
\end{align*}
\]

\[
\begin{align*}
  f(v_i) &= 0; \quad \text{if } i \equiv 0, 3 \mod 6 \\
  &= 1; \quad \text{if } i \equiv 4, 5 \mod 6 \\
  &= 2; \quad \text{if } i \equiv 1, 2 \mod 6, \ 1 \leq i \leq n + 1.
\end{align*}
\]

\[
\begin{align*}
  f(w_k) &= 2. \\
  f(w_j) &= 0; \quad \text{if } j \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } j \equiv 2, 3 \mod 6 \\
  &= 2; \quad \text{if } j \equiv 0, 5 \mod 6, \ 1 \leq j \leq k - 1.
\end{align*}
\]

Case 6: \( n \equiv 5 \mod 6 \).

Subcase I: \( k \equiv 0 \mod 6 \).

\[
\begin{align*}
  f(u_i) &= 0; \quad \text{if } i \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } i \equiv 2, 3 \mod 6 \\
  &= 2; \quad \text{if } i \equiv 0, 5 \mod 6, \ 1 \leq i \leq n + 1.
\end{align*}
\]

\[
\begin{align*}
  f(v_i) &= 0; \quad \text{if } i \equiv 0, 3 \mod 6 \\
  &= 1; \quad \text{if } i \equiv 4, 5 \mod 6 \\
  &= 2; \quad \text{if } i \equiv 1, 2 \mod 6, \ 1 \leq i \leq n + 1.
\end{align*}
\]

\[
\begin{align*}
  f(w_k) &= 2. \\
  f(w_j) &= 0; \quad \text{if } j \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } j \equiv 2, 3 \mod 6 \\
  &= 2; \quad \text{if } j \equiv 0, 5 \mod 6, \ 1 \leq j \leq k - 1.
\end{align*}
\]

Subcase II: \( k \equiv 1 \mod 6 \).

\[
\begin{align*}
  f(u_i) &= 0; \quad \text{if } i \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } i \equiv 2, 3 \mod 6 \\
  &= 2; \quad \text{if } i \equiv 0, 5 \mod 6, \ 1 \leq i \leq n + 1.
\end{align*}
\]

\[
\begin{align*}
  f(v_i) &= 0; \quad \text{if } i \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } i \equiv 2, 3 \mod 6 \\
  &= 2; \quad \text{if } i \equiv 0, 5 \mod 6, \ 1 \leq i \leq n + 1.
\end{align*}
\]

The remaining vertices are labeled as in Subcase I.

Subcase III: \( k \equiv 2 \mod 6 \).

\[
\begin{align*}
  f(u_i) &= 0; \quad \text{if } i \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } i \equiv 0, 5 \mod 6 \\
  &= 2; \quad \text{if } i \equiv 2, 3 \mod 6, \ 1 \leq j \leq k.
\end{align*}
\]

\[
\begin{align*}
  f(v_i) &= 0; \quad \text{if } i \equiv 1, 4 \mod 6 \\
  &= 1; \quad \text{if } i \equiv 0, 5 \mod 6 \\
  &= 2; \quad \text{if } i \equiv 2, 3 \mod 6, \ 1 \leq i \leq n + 1.
\end{align*}
\]

\[
\begin{align*}
  f(w_k) &= 0. \\
  f(w_j) &= 0; \quad \text{if } j \equiv 1, 4 \mod 6
\end{align*}
\]

(Advance online publication: 17 February 2015)
The graph obtained by joining two copies of wheel graph by a path of arbitrary length is 3-equitable.

Proof: Let \( G \) be the graph obtained by joining two copies of wheel graph \( W_n \) by path \( P_k \) of length \( k - 1 \). Let us denote the successive vertices of first copy of wheel graph by \( v_0, u_1, \ldots, u_n \) (where \( u_0 \) is apex vertex) and the successive vertices of second copy of wheel graph by \( v_0, v_1, \ldots, v_n \) (where \( v_0 \) is apex vertex). Let \( w_1, \ldots, w_k \) be the vertices of path \( P_k \) with \( w_1 = u_1 \) and \( w_k = v_1 \).

We define labeling function \( f : V(G) \to \{0, 1, 2\} \) as follows.

**Case 1:** \( n \equiv 0(6) \).

- \( f(u_0) = 0 \).
- \( f(v_0) = 2 \).

**Subcase I:** \( k \equiv 0(6) \).

- \( f(u_i) = 0; \) if \( i \equiv 1, 4 \pmod{6} \)
- \( = 1; \) if \( i \equiv 3, 4 \pmod{6} \)
- \( = 2; \) if \( i \equiv 0, 1 \pmod{6}, 1 \leq i \leq n \).

**Subcase II:** \( k \equiv 1, 2 \pmod{6} \).

- \( f(v_i) = 0; \) if \( i \equiv 0, 3 \pmod{6} \)
- \( = 1; \) if \( i \equiv 4, 5 \pmod{6} \)
- \( = 2; \) if \( i \equiv 1, 2 \pmod{6}, 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase I.

**Subcase III:** \( k \equiv 3 \pmod{6} \).

- \( f(u_i) = 0; \) if \( i \equiv 0, 3 \pmod{6} \)
- \( = 1; \) if \( i \equiv 4, 5 \pmod{6} \)
- \( = 2; \) if \( i \equiv 1, 2 \pmod{6}, 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase I.

**Subcase IV:** \( k \equiv 4 \pmod{6} \).

- \( f(u_i) = 0; \) if \( i \equiv 0, 3 \pmod{6} \)
- \( = 1; \) if \( i \equiv 4, 5 \pmod{6} \)
- \( = 2; \) if \( i \equiv 1, 2 \pmod{6}, 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase I.

**Subcase V:** \( k \equiv 5 \pmod{6} \).

- \( f(u_i) = 0; \) if \( i \equiv 0, 3 \pmod{6} \)
- \( = 1; \) if \( i \equiv 4, 5 \pmod{6} \)
- \( = 2; \) if \( i \equiv 1, 2 \pmod{6}, 1 \leq i \leq k - 1 \).

The remaining vertices are labeled same as in Subcase I.

**Case 2:** \( n \equiv 1(6) \).

**Subcase I:** \( k \equiv 0(6) \).

- \( f(u_i) = 2 \).

**Subcase II:** \( k \equiv 1 \pmod{6} \).

- \( f(u_i) = 0; \) if \( i \equiv 1, 4 \pmod{6} \)
- \( = 1; \) if \( i \equiv 2, 3 \pmod{6} \)
- \( = 2; \) if \( i \equiv 0, 5 \pmod{6}, 1 \leq i \leq n \).

**Subcase III:** \( k \equiv 2 \pmod{6} \).

- \( f(u_i) = 0; \) if \( i \equiv 1, 4 \pmod{6} \)
- \( = 1; \) if \( i \equiv 2, 3 \pmod{6} \)
- \( = 2; \) if \( i \equiv 0, 5 \pmod{6}, 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase II except for \( f(w_k) = 0 \).
The vertices are labeled same as in Subcase II except for $f(u_n) = 0$ and $f(v_{n-1}) = 1$.

Case 3: $n \equiv 2 (mod 6)$.

Subcase I: $k \equiv 0 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$ and $f(v_i) = 1$.

Subcase II: $k \equiv 4 (mod 6)$.

The remaining vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase VI: $k \equiv 5 (mod 6)$.

The remaining vertices are labeled same as in Subcase II except for $f(u_i) = 0$ and $f(v_i) = 1$.

Case 4: $n \equiv 3 (mod 6)$.

Subcase I: $k \equiv 0 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase VI: $k \equiv 5 (mod 6)$.

The remaining vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase V: $k \equiv 4 (mod 6)$.

The remaining vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase IV: $k \equiv 3 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$ and $f(v_i) = 1$.

Subcase II: $k \equiv 1 (mod 6)$.

The remaining vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase III: $k \equiv 2 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase I: $k \equiv 0 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase IV: $k \equiv 3 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$ and $f(v_i) = 1$.

Subcase II: $k \equiv 1 (mod 6)$.

The remaining vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase III: $k \equiv 2 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase I: $k \equiv 0 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase IV: $k \equiv 3 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$ and $f(v_i) = 1$.

Subcase II: $k \equiv 1 (mod 6)$.

The remaining vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase III: $k \equiv 2 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase I: $k \equiv 0 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase IV: $k \equiv 3 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$ and $f(v_i) = 1$.

Subcase II: $k \equiv 1 (mod 6)$.

The remaining vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase III: $k \equiv 2 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase I: $k \equiv 0 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase IV: $k \equiv 3 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$ and $f(v_i) = 1$.

Subcase II: $k \equiv 1 (mod 6)$.

The remaining vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase III: $k \equiv 2 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$.

Subcase I: $k \equiv 0 (mod 6)$.

The vertices are labeled same as in Subcase II except for $f(u_i) = 0$.
The remaining vertices are labeled same as in Subcase I except for \( f(w_4) = 0 \).

**Subcase IV:** \( k \equiv 3, 4 \pmod{6} \).

The vertices are labeled same as in Subcase III except for \( f(w_{k-1}) = 1 \).

**Subcase V:** \( k \equiv 5 \pmod{6} \).

The vertices are labeled same as in Subcase IV except for \( f(w_{k-2}) = 1 \).

**Subcase VI:** \( k \equiv 5 \pmod{6} \).

The vertices are labeled same as in Subcase V except for \( f(w_{k-3}) = 1, f(w_k) = 2 \).

The graph \( G \) under consideration satisfies the conditions \( |v_f(i) - v_f(j)| \leq 1 \) and \( |v_f(i) - v_f(j)| \leq 1 \) in each case. Hence the graph \( G \) under consideration is 3-equitable graph.

**Theorem 3** The graph obtained by joining two copies of helm graph \( H_n \) by a path of arbitrary length is 3-equitable.

**Proof:** Let \( G \) be the graph obtained by joining two copies of helm graph \( H_n \) by a path \( P_k \) of length \( k - 1 \). Let \( u_0 \) be the apex vertex, \( u_1, u_2, \ldots, u_n \) be the rim vertices and \( u'_1, u'_2, \ldots, u'_n \) be the pendant vertices of first copy of helm \( H_n \). Similarly let \( v_0 \) be the apex vertex, \( v_1, v_2, \ldots, v_n \) be the rim vertices and \( v'_1, v'_2, \ldots, v'_n \) be the pendant vertices of second copy of helm \( H_n \). Let \( w_1, w_2, \ldots, w_k \) be the vertices of path \( P_k \) with \( w_1 = v_1 \) and \( w_k = v_1 \). We define labeling function \( f : V(G) \to \{0, 1, 2\} \) as follows.

**Case 1:** \( n \equiv 0 \pmod{6} \).

\[
\begin{align*}
&f(u_0) = 0, f(v_0) = 2. \\
&f(u_i) = 0; \quad f(v_i) = 0; \quad f(u_i') = 0; \quad f(v_i') = 0; \quad f(u_j) = 0; \quad f(v_j) = 0; \quad f(w_j) = 0; \\
&f(u_i) = 0; \quad f(v_i) = 0; \quad f(u_i') = 0; \quad f(v_i') = 0; \quad f(u_j) = 0; \quad f(v_j) = 0; \quad f(w_j) = 0; \\
&f(u_i) = 0; \quad f(v_i) = 0; \quad f(u_i') = 0; \quad f(v_i') = 0; \quad f(u_j) = 0; \quad f(v_j) = 0; \quad f(w_j) = 0.;
\end{align*}
\]

**Subcase I:** \( k \equiv 0 \pmod{6} \).

\[
\begin{align*}
&f(u_0) = 0, f(v_0) = 0, f(w_1) = 2, f(w_k) = 1. \\
&f(u_i) = 1; \quad f(v_i) = 1; \quad f(u_i') = 1; \quad f(v_i') = 1; \quad f(u_j) = 1; \quad f(v_j) = 1; \quad f(w_j) = 1.
\end{align*}
\]

**Subcase II:** \( k \equiv 2 \pmod{6} \).

\[
\begin{align*}
&f(u_0) = 0, f(v_0) = 0, f(w_1) = 2, f(w_k) = 1. \\
&f(u_i) = 1; \quad f(v_i) = 1; \quad f(u_i') = 1; \quad f(v_i') = 1; \quad f(u_j) = 1; \quad f(v_j) = 1; \quad f(w_j) = 1.
\end{align*}
\]

**Subcase III:** \( k \equiv 3 \pmod{6} \).

\[
\begin{align*}
&f(u_0) = 0, f(v_0) = 0, f(w_1) = 2, f(w_k) = 1. \\
&f(u_i) = 1; \quad f(v_i) = 1; \quad f(u_i') = 1; \quad f(v_i') = 1; \quad f(u_j) = 1; \quad f(v_j) = 1; \quad f(w_j) = 1.
\end{align*}
\]

**Subcase IV:** \( k \equiv 4 \pmod{6} \).

\[
\begin{align*}
&f(u_0) = 0, f(v_0) = 0, f(w_1) = 2, f(w_k) = 1. \\
&f(u_i) = 1; \quad f(v_i) = 1; \quad f(u_i') = 1; \quad f(v_i') = 1; \quad f(u_j) = 1; \quad f(v_j) = 1; \quad f(w_j) = 1.
\end{align*}
\]

**Subcase V:** \( k \equiv 5 \pmod{6} \).

\[
\begin{align*}
&f(u_0) = 0, f(v_0) = 0, f(w_1) = 2, f(w_k) = 1. \\
&f(u_i) = 1; \quad f(v_i) = 1; \quad f(u_i') = 1; \quad f(v_i') = 1; \quad f(u_j) = 1; \quad f(v_j) = 1; \quad f(w_j) = 1.
\end{align*}
\]

**Subcase VI:** \( k \equiv 5 \pmod{6} \).

\[
\begin{align*}
&f(u_0) = 0, f(v_0) = 0, f(w_1) = 2, f(w_k) = 1.
\end{align*}
\]
Case 2: $n \equiv 1 \pmod{6}$.

Subcase I: $k \equiv 0 \pmod{6}$.

$f(u_0) = 2$.

$f(u_i) = 0$; if $i \equiv 1, 4 \pmod{6}$

$f(v_i) = 0$; if $i \equiv 0, 3 \pmod{6}$

Subcase IV: $k \equiv 4 \pmod{6}$.

Subcase II: $k \equiv 1 \pmod{6}$.

$f(w_{k-1}) = 1$.

$f(w_j) = 0$; if $j \equiv 3, 4 \pmod{6}$, $1 \leq j \leq k$, $j \neq k - 1$.

Subcase II: $k \equiv 1 \pmod{6}$.

$f(w_{k-1}) = 1$.

$f(w_j) = 0$; if $j \equiv 3, 4 \pmod{6}$, $1 \leq j \leq k$, $j \neq k - 1$.

Subcase III: $k \equiv 2 \pmod{6}$.

The vertices are labeled same as in Subcase II except for $f(v_0) = 2$, $f(w_k) = 0$.

Subcase IV: $k \equiv 3 \pmod{6}$.

$f(w_0) = 0$.

Subcase II: $k \equiv 2 \pmod{6}$.

$f(w_{k-1}) = 1$.

$f(w_j) = 0$; if $j \equiv 3, 4 \pmod{6}$, $1 \leq j \leq k$, $j \neq k - 1$.

Subcase V: $k \equiv 4 \pmod{6}$.

The vertices are labeled same as in Subcase I.

Case 2: $n \equiv 1 \pmod{6}$.

Subcase I: $k \equiv 0 \pmod{6}$.

$f(u_0) = 2$.

$f(u_i) = 0$; if $i \equiv 1, 4 \pmod{6}$

Subcase II: $k \equiv 1 \pmod{6}$.

$f(u_i) = 0$; if $i \equiv 0, 3 \pmod{6}$

$f(v_i) = 0$; if $i \equiv 0, 3 \pmod{6}$

Subcase IV: $k \equiv 4 \pmod{6}$.

$f(v_i) = 0$; if $i \equiv 0, 3 \pmod{6}$

$f(v_i) = 0$; if $i \equiv 2, 3 \pmod{6}$, $1 \leq i \leq n - 1$.

The remaining vertices are labeled same as in Subcase I.

Case 3: $n \equiv 2 \pmod{6}$.

Subcase I: $k \equiv 0 \pmod{6}$.

$f(u_0) = 0$.

Subcase II: $k \equiv 1 \pmod{6}$.

$f(v_i) = 0$; if $i \equiv 1, 4 \pmod{6}$

Subcase IV: $k \equiv 4 \pmod{6}$.

$f(v_i) = 0$; if $i \equiv 2, 3 \pmod{6}$, $1 \leq i \leq n$.

The remaining vertices are labeled same as in Subcase IV except for $f(w_{k-1}) = 1$.

Subcase VI: $k \equiv 5 \pmod{6}$.

$f(u_0) = 0$.

Subcase II: $k \equiv 2 \pmod{6}$.

$f(u_i) = 0$; if $i \equiv 0, 3 \pmod{6}$

$f(v_i) = 0$; if $i \equiv 0, 3 \pmod{6}$

Subcase IV: $k \equiv 4 \pmod{6}$.

$f(v_i) = 0$; if $i \equiv 0, 3 \pmod{6}$

Subcase II: $k \equiv 2 \pmod{6}$.

$f(u_i) = 0$; if $i \equiv 0, 3 \pmod{6}$

$f(v_i) = 0$; if $i \equiv 0, 3 \pmod{6}$

Subcase IV: $k \equiv 4 \pmod{6}$.

$f(v_i) = 0$; if $i \equiv 0, 3 \pmod{6}$

The remaining vertices are labeled same as in Subcase IV except for $f(w_{k-1}) = 1$.

Subcase VI: $k \equiv 5 \pmod{6}$.

$f(u_0) = 0$.

Subcase II: $k \equiv 2 \pmod{6}$.

$f(u_i) = 0$; if $i \equiv 0, 3 \pmod{6}$

Subcase IV: $k \equiv 4 \pmod{6}$.

$f(v_i) = 0$; if $i \equiv 0, 3 \pmod{6}$

Subcase II: $k \equiv 2 \pmod{6}$.
Subcase II: \( k \equiv 1 \pmod{2} \).
\[ f(u_n) = 1. \]
\[ f(u_{n-1}) = 0; \quad f(u_i) = 0; \quad f(v_i) = 1; \quad f(w_j) = 2; \]
\[ 2 \leq i \leq n. \]

Subcase III: \( k \equiv 2 \pmod{2} \).
\[ f(w_k) = 1. \]

The remaining vertices are labeled same as in Subcase II.

Subcase IV: \( k \equiv 3 \pmod{2} \).
\[ f(u_n) = 0; \quad f(u_i) = 0; \quad f(v_i) = 1; \quad f(w_j) = 2; \]
\[ 1 \leq i \leq n. \]

Subcase V: \( k \equiv 4 \pmod{2} \).
\[ f(w_{k-5}) = 1. \]

Subcase VI: \( k \equiv 5 \pmod{2} \).
\[ f(u_n) = 0; \quad f(u_i) = 0; \quad f(v_i) = 1; \quad f(w_j) = 2; \]
\[ 1 \leq i \leq n. \]
Case 5: except for $f$.

Subcase III: $k \equiv 2(\text{mod }6)$.

Subcase IV: $k \equiv 3(\text{mod }6)$.

Subcase V: $k \equiv 4(\text{mod }6)$.

Subcase VI: $k \equiv 5(\text{mod }6)$.

The vertices are labeled same as in Subcase V except for $f(v_k) = 0$.

Case 6: $n \equiv 5(\text{mod }6)$.

Subcase I: $k \equiv 0(\text{mod }6)$.

Subcase II: $k \equiv 1(\text{mod }6)$.

Subcase III: $k \equiv 2(\text{mod }6)$.

Subcase IV: $k \equiv 3(\text{mod }6)$.

Subcase V: $k \equiv 4(\text{mod }6)$.

Subcase VI: $k \equiv 5(\text{mod }6)$.

The vertices are labeled same as in Subcase V except for $f(v_k) = 2$.

The remaining vertices are labeled same as in Subcase II except for $f(v_u) = 0$.

The remaining vertices are labeled same as in Subcase III except for $f(v_i) = 1$.

The remaining vertices are labeled same as in Subcase IV except for $f(v_k) = 0$.

The remaining vertices are labeled same as in Subcase V except for $f(v_k) = 2$.

The remaining vertices are labeled same as in Subcase VI except for $f(v_k) = 0$. 

The remaining vertices are labeled same as in Subcase I except for $f(v_k) = 0$. 

(Advance online publication: 17 February 2015)
Subcase IV: \( k \equiv 3(\text{mod}6) \).

\[ f(u_0) = 0. \]
\[ f(u_i) = 0; \text{if } i \equiv 0,3(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 4,5(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 1,2(\text{mod}6), 1 \leq i \leq n. \]

\[ f(v_0) = 0. \]
\[ f(v_i) = 0; \text{if } i \equiv 2,5(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 3,4(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 0,1(\text{mod}6), 1 \leq i \leq n. \]

Subcase V: \( k \equiv 4(\text{mod}6) \).

\[ f(u_0) = 0. \]
\[ f(u_i) = 0; \text{if } i \equiv 0,3(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 4,5(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 1,2(\text{mod}6), 1 \leq i \leq n. \]

\[ f(v_0) = 2. \]
\[ f(v_i) = 0; \text{if } i \equiv 2,5(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 3,4(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 0,1(\text{mod}6), 1 \leq i \leq n - 1. \]

\[ f(v'_0) = 0. \]
\[ f(v'_i) = 0; \text{if } i \equiv 0,3(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 4,5(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 1,2(\text{mod}6), 1 \leq i \leq n - 1. \]

\[ f(w_k) = 2. \]
\[ f(w_j) = 0; \text{if } j \equiv 1,4(\text{mod}6) \]
\[ = 1; \text{if } j \equiv 5,0(\text{mod}6) \]
\[ = 2; \text{if } j \equiv 2,3(\text{mod}6), 1 \leq j \leq k - 1. \]

Subcase VI: \( k \equiv 5(\text{mod}6) \).

\[ f(u_0) = 2. \]
\[ f(u_i) = 0; \text{if } i \equiv 2,5(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 0,1(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 3,4(\text{mod}6), 1 \leq i \leq n. \]

\[ f(v_0) = 2. \]
\[ f(v_i) = 0; \text{if } i \equiv 2,5(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 0,1(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 3,4(\text{mod}6), 1 \leq i \leq n. \]

\[ f(v'_0) = 0. \]
\[ f(v'_i) = 0; \text{if } i \equiv 0,3(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 4,5(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 1,2(\text{mod}6), 1 \leq i \leq n. \]

Subcase VII: \( k \equiv 0(\text{mod}6) \).

\[ f(u_0) = 0. \]
\[ f(u_i) = 0; \text{if } i \equiv 0,3(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 4,5(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 1,2(\text{mod}6), 1 \leq i \leq n. \]

\[ f(v_0) = 0. \]
\[ f(v_i) = 0; \text{if } i \equiv 1,4(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 2,3(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 0,5(\text{mod}6), 1 \leq i \leq n. \]

\[ f(v'_0) = 0. \]
\[ f(v'_i) = 0; \text{if } i \equiv 0,3(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 4,5(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 2,3(\text{mod}6), 1 \leq i \leq n. \]

\[ f(w_{k-1}) = 1. \]
\[ f(w_j) = 0; \text{if } j \equiv 2,5(\text{mod}6) \]

Theorem 4 The graph obtained by joining two copies of gear graph \( G_n \) by a path of arbitrary length is \( 3 \)-equitable.

Proof: Let \( G \) be the graph obtained by joining two copies of gear graph \( G_n \) by path \( P_k \) of length \( k - 1 \). Let us denote the successive vertices of first copy of gear graph by \( u_0, u_1, \ldots, u_{2n} \), where \( u_0 \) is apex vertex. \( u_1, u_3, \ldots, u_{2n-1} \) are rim vertices of wheel and \( u_2, u_4, \ldots, u_{2n} \) are the vertices inserted between two consecutive rim vertices corresponding to \( u_1, u_3, \ldots, u_{2n-1} \). Similarly let \( v_0, v_1, \ldots, v_{2n} \) be the successive vertices of second copy of gear graph, where \( v_0 \) is apex vertex, \( v_1, v_3, \ldots, v_{2n-1} \) are rim vertices of wheel and \( v_2, v_4, \ldots, v_{2n} \) are the vertices inserted between two consecutive rim vertices corresponding to \( v_1, v_3, \ldots, v_{2n-1} \). We let \( w_1, w_2, \ldots, w_k \) be the vertices of path \( P_k \) with \( w_1 = u_1 \) and \( w_k = v_1 \).

We define labeling function \( f : V(G) \rightarrow \{0, 1, 2\} \) as follows.

Case 1: \( n \equiv 0,3(\text{mod}6) \).

\[ f(u_0) = 0. \]
\[ f(v_0) = 2. \]

Subcase I: \( k \equiv 0(\text{mod}6) \).

\[ f(u_i) = 0; \text{if } i \equiv 2,5(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 3,4(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 0,1(\text{mod}6), 1 \leq i \leq 2n. \]

\[ f(v_i) = 0; \text{if } i \equiv 1,4(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 2,3(\mod6) \]
\[ = 2; \text{if } i \equiv 0,5(\text{mod}6), 1 \leq i \leq 2n. \]

The remaining vertices are labeled same as in Subcase I.

Subcase II: \( k \equiv 1,2(\text{mod}6) \).

\[ f(u_i) = 0; \text{if } i \equiv 0,3(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 4,5(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 1,2(\text{mod}6), 1 \leq i \leq 2n. \]

The remaining vertices are labeled same as in Subcase I.

Subcase III: \( k \equiv 3(\text{mod}6) \).

\[ f(u_i) = 0; \text{if } i \equiv 0,3(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 1,2(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 4,5(\text{mod}6), 1 \leq i \leq 2n. \]

\[ f(v_i) = 0; \text{if } i \equiv 2,5(\text{mod}6) \]
\[ = 1; \text{if } i \equiv 3,4(\text{mod}6) \]
\[ = 2; \text{if } i \equiv 0,1(\text{mod}6), 1 \leq i \leq 2n. \]

\[ f(w_{k-1}) = 1. \]

\[ f(w_j) = 0; \text{if } j \equiv 2,5(\text{mod}6) \]

(Advance online publication: 17 February 2015)
The remaining vertices are labeled same as in Subcase I.

**Subcase IV: \( k \equiv 4 \pmod{6} \).**

- \( f(u_i) = 0 \); if \( i \equiv 0, 1 \pmod{3} \)
- \( f(u_i) = 1 \); if \( i \equiv 2, 4 \pmod{6} \), \( 1 \leq j \leq k \), \( j \neq k - 1 \).

**Subcase V: \( k \equiv 4 \pmod{6} \).**

- \( f(v_i) = 0 \); if \( i \equiv 0, 3 \pmod{6} \)
- \( f(v_i) = 1 \); if \( i \equiv 1, 2 \pmod{6} \)
- \( f(v_i) = 2 \); if \( i \equiv 4, 5 \pmod{6} \), \( 1 \leq i \leq 2n \).

**Subcase VI: \( k \equiv 5 \pmod{6} \).**

- \( f(v_i) = 0 \); if \( i \equiv 2, 5 \pmod{6} \)
- \( f(v_i) = 1 \); if \( i \equiv 3, 4 \pmod{6} \), \( 1 \leq j \leq k \).

The remaining vertices are labeled same as in Subcase I.

**Case 2: \( n \equiv 1, 4 \pmod{6} \).**

**Subcase I: \( k \equiv 0 \pmod{6} \).**

- \( f(u_0) = 0 \)
- \( f(v_i) = 0 \); if \( i \equiv 0, 3 \pmod{6} \)
- \( f(v_i) = 1 \); if \( i \equiv 2, 5 \pmod{6} \)
- \( f(v_i) = 2 \); if \( i \equiv 4, 5 \pmod{6} \), \( 1 \leq i \leq 2n \).

**Subcase II: \( k \equiv 1 \pmod{6} \).**

- \( f(u_0) = 0 \)
- \( f(u_i) = 0 \); if \( i \equiv 2, 5 \pmod{6} \)
- \( f(u_i) = 1 \); if \( i \equiv 3, 4 \pmod{6} \)
- \( f(u_i) = 2 \); if \( i \equiv 0, 1 \pmod{6} \), \( 1 \leq i \leq 2n \).

**Subcase III: \( k \equiv 2 \pmod{6} \).**

- \( f(u_0) = 0 \)
- \( f(v_{2n-1}) = 1 \)
- \( f(u_i) = 0 \); if \( i \equiv 0, 3 \pmod{6} \)
- \( f(u_i) = 1 \); if \( i \equiv 2, 5 \pmod{6} \)
- \( f(u_i) = 2 \); if \( i \equiv 4, 5 \pmod{6} \), \( 1 \leq i \leq 2n - 1 \).

**Subcase IV: \( k \equiv 3 \pmod{6} \).**

- \( f(u_0) = 2, f(v_0) = 0, f(v_{2n}) = 1 \)
- \( f(v_i) = 0 \); if \( i \equiv 2, 5 \pmod{6} \)
- \( f(v_i) = 1 \); if \( i \equiv 0, 1 \pmod{6} \)
- \( f(v_i) = 2 \); if \( i \equiv 3, 4 \pmod{6} \), \( 1 \leq i \leq 2n - 1 \).

**Subcase V: \( k \equiv 4 \pmod{6} \).**

- \( f(u_0) = 0 \)
- \( f(u_i) = 0 \); if \( i \equiv 2, 5 \pmod{6} \)
- \( f(u_i) = 1 \); if \( i \equiv 3, 4 \pmod{6} \)
- \( f(u_i) = 2 \); if \( i \equiv 0, 1 \pmod{6} \), \( 1 \leq i \leq 2n \).

**Subcase VI: \( k \equiv 5 \pmod{6} \).**

- \( f(u_0) = 2 \)
- \( f(u_i) = 0 \); if \( i \equiv 2, 5 \pmod{6} \)
- \( f(u_i) = 1 \); if \( i \equiv 3, 4 \pmod{6} \), \( 1 \leq i \leq 2n \).

**Case 3: \( n \equiv 2, 5 \pmod{6} \).**

- \( f(u_0) = 0 \)
- \( f(v_0) = 2 \)

**Subcase I: \( k \equiv 0 \pmod{6} \).**

- \( f(u_0) = 0 \)
- \( f(v_i) = 0 \); if \( i \equiv 0, 3 \pmod{6} \)
- \( f(v_i) = 1 \); if \( i \equiv 2, 5 \pmod{6} \)
- \( f(v_i) = 2 \); if \( i \equiv 4, 5 \pmod{6} \), \( 1 \leq i \leq 2n \).

**Subcase II: \( k \equiv 1 \pmod{6} \).**

- \( f(u_0) = 0 \)
- \( f(u_i) = 0 \); if \( i \equiv 2, 5 \pmod{6} \)
- \( f(u_i) = 1 \); if \( i \equiv 3, 4 \pmod{6} \)
- \( f(u_i) = 2 \); if \( i \equiv 0, 1 \pmod{6} \), \( 1 \leq i \leq 2n \).

**Subcase III: \( k \equiv 2 \pmod{6} \).**

- \( f(u_0) = 0 \)
- \( f(v_{2n-3}) = 1 \)
- \( f(v_i) = 0 \); if \( i \equiv 0, 3 \pmod{6} \)
- \( f(v_i) = 1 \); if \( i \equiv 2, 5 \pmod{6} \)
- \( f(v_i) = 2 \); if \( i \equiv 4, 5 \pmod{6} \), \( 1 \leq i \leq 2n \).

**Subcase IV: \( k \equiv 3 \pmod{6} \).**

- \( f(u_0) = 2 \)
- \( f(v_i) = 0 \); if \( i \equiv 0, 3 \pmod{6} \)
- \( f(v_i) = 1 \); if \( i \equiv 2, 5 \pmod{6} \)
- \( f(v_i) = 2 \); if \( i \equiv 4, 5 \pmod{6} \), \( 1 \leq i \leq 2n \).

All the vertices are labeled same as in Subcase II except for \( f(w_k) = 0 \).

**Subcase IV: \( k \equiv 3 \pmod{6} \).**

- \( f(u_i) = 0 \); if \( i \equiv 1, 4 \pmod{6} \)
- \( f(u_i) = 1 \); if \( i \equiv 2, 3 \pmod{6} \)
- \( f(u_i) = 2 \); if \( i \equiv 0, 1 \pmod{6} \), \( 1 \leq i \leq 2n \).
The remaining vertices are labeled same as in Subcase IV.

Subcase V: \( k \equiv 4 \pmod{6} \).
\[
f(v_0) = 2.
\]
The remaining vertices are labeled same as in Subcase I.

Case 2: \( n \equiv 1 \pmod{6} \).

Subcase I: \( k \equiv 0 \pmod{6} \).
\[
f(u_0) = 0.
f(v_i) = 0; \text{ if } i \equiv 1, 4 \pmod{6} = 1; \text{ if } i \equiv 0, 5 \pmod{6} = 2; \text{ if } i \equiv 2, 3 \pmod{6}, 1 \leq j \leq k, j \neq k - 1.
\]

Subcase V: \( k \equiv 5 \pmod{6} \).
\[
f(v_0) = 2.
\]
The remaining vertices are labeled same as in Subcase I.

Subcase VI: \( k \equiv 5 \pmod{6} \).
\[
f(v_j) = 0; \text{ if } i \equiv 1, 4 \pmod{6} = 1; \text{ if } i \equiv 0, 5 \pmod{6} = 2; \text{ if } i \equiv 2, 3 \pmod{6}, 1 \leq i \leq 2n - 1.
\]

The remaining vertices are labeled same as in Subcase IV.

Subcase VI: \( k \equiv 5 \pmod{6} \).
\[
f(v_j) = 0; \text{ if } i \equiv 1, 4 \pmod{6} = 1; \text{ if } i \equiv 0, 5 \pmod{6} = 2; \text{ if } i \equiv 2, 3 \pmod{6}, 1 \leq j \leq k.
\]

The remaining vertices are labeled same as in Subcase IV.

Subcase I: \( k \equiv 0 \pmod{6} \).
\[
f(u_0) = 2.
f(v_i) = 0; \text{ if } i \equiv 1, 4 \pmod{6} = 1; \text{ if } i \equiv 0, 5 \pmod{6} = 2; \text{ if } i \equiv 2, 3 \pmod{6}, 1 \leq i \leq n.
\]

Subcase II: \( k \equiv 1 \pmod{6} \).
\[
f(v_0) = 0.
f(v_i) = 0; \text{ if } i \equiv 1, 4 \pmod{6} = 1; \text{ if } i \equiv 0, 5 \pmod{6} = 2; \text{ if } i \equiv 2, 3 \pmod{6}, 1 \leq i \leq n.
\]

Subcase III: \( k \equiv 1 \pmod{6} \).
\[
f(v_0) = 0.
f(v_i) = 0; \text{ if } i \equiv 2, 5 \pmod{6} = 1; \text{ if } i \equiv 3, 4 \pmod{6} = 2; \text{ if } i \equiv 0, 1 \pmod{6}, 1 \leq i \leq n.
\]

Subcase II: \( k \equiv 1 \pmod{6} \).
\[
f(v_0) = 0.
f(v_i) = 0; \text{ if } i \equiv 1, 4 \pmod{6} = 1; \text{ if } i \equiv 0, 5 \pmod{6} = 2; \text{ if } i \equiv 2, 3 \pmod{6}, 1 \leq j \leq k.
\]

Subcase III: \( k \equiv 1 \pmod{6} \).
\[
f(v_0) = 0.
f(v_i) = 0; \text{ if } i \equiv 2, 5 \pmod{6} = 1; \text{ if } i \equiv 3, 4 \pmod{6} = 2; \text{ if } i \equiv 0, 1 \pmod{6}, 1 \leq i \leq n.
\]

Subcase III: \( k \equiv 1 \pmod{6} \).
\[
f(v_0) = 0.
f(v_i) = 0; \text{ if } i \equiv 2, 5 \pmod{6} = 1; \text{ if } i \equiv 3, 4 \pmod{6} = 2; \text{ if } i \equiv 0, 1 \pmod{6}, 1 \leq i \leq n.
\]

Subcase IV: \( k \equiv 3 \pmod{6} \).
\[
f(v_0) = 0.
f(v_i) = 0; \text{ if } i \equiv 1, 4 \pmod{6} = 1; \text{ if } i \equiv 0, 5 \pmod{6} = 2; \text{ if } i \equiv 2, 3 \pmod{6}, 1 \leq i \leq n.
\]

Subcase IV: \( k \equiv 3 \pmod{6} \).
\[
f(v_0) = 0.
f(v_i) = 0; \text{ if } i \equiv 1, 4 \pmod{6} = 1; \text{ if } i \equiv 0, 5 \pmod{6} = 2; \text{ if } i \equiv 2, 3 \pmod{6}, 1 \leq i \leq n.
\]

Subcase V: \( k \equiv 5 \pmod{6} \).
\[
f(v_0) = 2.
\]
The remaining vertices are labeled same as in Subcase I.

Case 3: \( n \equiv 2 \pmod{6} \).

Subcase I: \( k \equiv 0 \pmod{6} \).
\[
f(u_0) = 1.
f(u_i) = 0; \text{ if } i \equiv 1, 4 \pmod{6} = 1; \text{ if } i \equiv 0, 5 \pmod{6} = 2; \text{ if } i \equiv 2, 3 \pmod{6}, 1 \leq i \leq n.
\]

Subcase V: \( k \equiv 5 \pmod{6} \).
\[
f(v_0) = 2.
\]
The remaining vertices are labeled same as in Subcase I.

(Advance online publication: 17 February 2015)
The remaining vertices are labeled same as in Subcase I.

**Subcase III:** \( k \equiv 1 \mod 2 \).

\( f(v_0) = f(v_1) = 0 \).

\( f(v_0) = 0; \) if \( i \equiv 0,1,2 \mod 3 \)
\[ = 1; \] if \( i \equiv 1,2 \mod 3 \)
\[ = 2; \] if \( i \equiv 0,2 \mod 3 \), \( 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase I.

**Subcase IV:** \( k \equiv 3 \mod 2 \).

\( f(v_0) = 2 \).

\( f(v_0) = 0; \) if \( i \equiv 2,5 \mod 6 \)
\[ = 1; \] if \( i \equiv 3,4 \mod 6 \)
\[ = 2; \] if \( i \equiv 0,1 \mod 6 \), \( 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase IV.

**Subcase VI:** \( k \equiv 5 \mod 2 \).

\( f(v_0) = 1 \).

\( f(v_1) = 0; \) if \( i \equiv 2,5 \mod 6 \)
\[ = 1; \] if \( i \equiv 3,4 \mod 6 \)
\[ = 2; \] if \( i \equiv 0,1 \mod 6 \), \( 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase IV.

**Case 4:** \( n \equiv 3 \mod 2 \).

**Subcase I:** \( k \equiv 0 \mod 2 \).

\( f(v_0) = 0 \).

\( f(v_0) = 0; \) if \( i \equiv 1,4 \mod 6 \)
\[ = 1; \] if \( i \equiv 0,5 \mod 6 \)
\[ = 2; \] if \( i \equiv 2,3 \mod 6 \), \( 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase I.

**Subcase II:** \( k \equiv 1 \mod 2 \).

\( f(v_0) = 0; \) if \( i \equiv 1,4 \mod 6 \)
\[ = 1; \] if \( i \equiv 0,5 \mod 6 \)
\[ = 2; \] if \( i \equiv 2,3 \mod 6 \), \( 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase I.

**Subcase III:** \( k \equiv 2 \mod 2 \).

\( f(v_0) = 0; \) if \( i \equiv 2,5 \mod 6 \)
\[ = 1; \] if \( i \equiv 0,1 \mod 6 \)
\[ = 2; \] if \( i \equiv 3,4 \mod 6 \), \( 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase I.

**Subcase IV:** \( k \equiv 4 \mod 2 \).

\( f(v_0) = 0; \) if \( i \equiv 1,4 \mod 6 \)
\[ = 1; \] if \( i \equiv 0,5 \mod 6 \)
\[ = 2; \] if \( i \equiv 2,3 \mod 6 \), \( 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase I.

**Subcase V:** \( k \equiv 5 \mod 2 \).

\( f(v_0) = 0; \) if \( i \equiv 1,4 \mod 6 \)
\[ = 1; \] if \( i \equiv 0,5 \mod 6 \)
\[ = 2; \] if \( i \equiv 2,3 \mod 6 \), \( 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase I.

**Case 5:** \( n \equiv 4 \mod 2 \).

**Subcase I:** \( k \equiv 0 \mod 2 \).

\( f(v_0) = 0; \) if \( i \equiv 1,4 \mod 6 \)
\[ = 1; \] if \( i \equiv 0,5 \mod 6 \)
\[ = 2; \] if \( i \equiv 2,3 \mod 6 \), \( 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase I.

**Subcase II:** \( k \equiv 1 \mod 2 \).

\( f(v_0) = 0; \) if \( i \equiv 1,4 \mod 6 \)
\[ = 1; \] if \( i \equiv 0,5 \mod 6 \)
\[ = 2; \] if \( i \equiv 2,3 \mod 6 \), \( 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase I.

**Case 6:** \( n \equiv 5 \mod 2 \).

**Subcase I:** \( k \equiv 0 \mod 2 \).

\( f(v_0) = 0; \) if \( i \equiv 1,4 \mod 6 \)
\[ = 1; \] if \( i \equiv 0,5 \mod 6 \)
\[ = 2; \] if \( i \equiv 2,3 \mod 6 \), \( 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase I.

**Subcase II:** \( k \equiv 1 \mod 2 \).

\( f(v_0) = 0; \) if \( i \equiv 1,4 \mod 6 \)
\[ = 1; \] if \( i \equiv 0,5 \mod 6 \)
\[ = 2; \] if \( i \equiv 2,3 \mod 6 \), \( 1 \leq i \leq n \).

The remaining vertices are labeled same as in Subcase I.
\begin{align*}
\text{Subcase III: } k &\equiv 2(\text{mod}6), \\
\text{Subcase IV: } k &\equiv 3, 4(\text{mod}6), \\
\text{Subcase V: } k &\equiv 5(\text{mod}6).
\end{align*}

The remaining vertices are labeled same as in Subcase II.

\begin{align*}
\text{The graph } G \text{ under consideration satisfies the conditions } |v_i(i) - v_j(j)| &\leq 1 \text{ and } |e_f(i) - e_f(j)| \leq 1, 0 \leq j \leq 2
\end{align*}
in each case. Hence the graph \( G \) under consideration is 3-equitable graph.

\section{Illustrations}

\textbf{Illustration 1} As an illustration of Theorem 1, 3-equitable labeling of the graph \( G \) obtained by joining two copies of fan graph \( F_7 \) by path \( P_8 \) is shown in Fig. 1. It is the case related to \( n \equiv 1(\text{mod}6) \) and \( k \equiv 3(\text{mod}6) \).

\textbf{Illustration 2} As an illustration of labeling pattern defined in Theorem 2, 3-equitable labeling of the graph \( G \) obtained by joining two copies of wheel graph \( W_8 \) by path \( P_8 \) is shown in Fig. 2. It is the case related to \( n \equiv 2(\text{mod}6) \) and \( k \equiv 0(\text{mod}6) \).

\textbf{Illustration 3} As an illustration of labeling pattern defined in Theorem 3, 3-equitable labeling of the graph \( G \) obtained by joining two copies of helm graph \( H_6 \) by path \( P_8 \) is shown in Fig. 3. It is the case related to \( n \equiv 0(\text{mod}6) \) and \( k \equiv 0(\text{mod}6) \).

\textbf{Illustration 4} As an illustration of labeling pattern defined in Theorem 4, 3-equitable labeling of the graph \( G \) obtained by joining two copies of gear graph \( G_6 \) by path \( P_8 \) is shown in Fig. 4. It is the case related to \( n \equiv 0(\text{mod}6) \) and \( k \equiv 0(\text{mod}6) \).

\textbf{Illustration 5} As an illustration of labeling pattern defined in Theorem 5, 3-equitable labeling of the graph \( G \) obtained by joining two copies of cycle \( C_6 \) with one pendant edge by path \( P_8 \) is shown in Fig. 5. It is the case related to \( n \equiv 0(\text{mod}6) \) and \( k \equiv 0(\text{mod}6) \).

\section{Conclusion}

The research work presented here provide five new results in the theory of 3-equitable labeling of graphs. The entire work is focused on joining two copies of some graph by a path of arbitrary length. In this work two copies of fans, wheels helms, gears and cycle with one pendant edge are considered.

(Advance online publication: 17 February 2015)
ACKNOWLEDGEMENT

The authors are grateful to the anonymous referee for valuable suggestions and comments.

REFERENCES