Solving STO And KD Equations with Modified Riemann-Liouville Derivative Using Improved (G'/G)-expansion Function Method

Youwei Zhang*

***Abstract*—This present article applies fractional complex transformation to convert nonlinear fractional evolution equations to nonlinear ordinary differential equations, and obtain the exact solitary wave solutions of space-time fractional Sharma-Tasso-Olever (STO) and Konopelchenko-Dubrovsky (KD) equations by using the improved ($\frac{G'}{G}$)-expansion function method, respectively. The fractional derivative is defined in the sense of modified Riemann-Liouville derivative. The results show that the method is efficient and powerful for solving wide classes of nonlinear fractional evolution equations.**

***Index Terms*—fractional calculus, complex transformation, modified Riemann-Liouville derivative, improved ($\frac{G'}{G}$)-expansion function method.**

I. Introduction

RECENTLY, fractional calculus has obtained considerable popularity and importance as generalizations of integer-order evolution equation, and used to model problems in physics, neurons, hydrology, viscoelasticity and rheology, image processing, mechanics, mechatronics, finance and control theory, see [1]–[8]. Among them, a large amount of literature has been provided to construct the numerical or exact solution of fractional differential equations of physical interest. Since a physical phenomenon may depend on not only the time instant but also the previous time history in reality, which can be modeled by using the theory of derivatives and integrals of fractional order [9]. For better understanding the mechanisms of the complicated nonlinear physical phenomena, searching for exact solution of the aforementioned nonlinear time-fractional dispersive equations are of great importance. Many powerful and efficient methods have been proposed to construct the approximate solutions for some space-fractional time-fractional or space-time fractional differential equations, such as Adomian decomposition method [10], [11], variational iteration method [12]–[15], differential transformation method [16], [17], exp-function method [18], [19] and so on.

STO equation has been applied to describe a wide range of physics phenomena of the evolution and interaction to nonlinear waves, such as fluid dynamics, aerodynamics, continuum mechanics, solitons and turbulence et al, it possesses an infinitely many symmetries and the bi-Hamiltonian formulation. If the Hamiltonian of conservative system is constructed using fractional derivatives, the resulting equations of motion can be nonconservative. Therefore, in many cases, the real physical processes could be modeled in a reliable manner using fractional-order differential equations [20]. Song et al [21] implemented variational iteration method, Adomian decomposition method and homotopy perturbation method to consider the time-fractional STO equation and obtained an analytic and approximate solution for different types of differential equations. Bulut and Pandir [22] applied the modified trial equation method to time-fractional STO equation by the using of the complete discrimination system for polynomial method. Golmamadian [23] constructed the exact complex solutions of nonlinear time-fractional STO equation by the direct algebraic method. Other the known methods to handle with the fractional STO equation we can see [24]–[28]. KD equation of integer-order appears in a great variety of contexts, such as physics, signal processing, control theory, dynamics, has attracted much attention of more and more scholars. For example, Lin et al [29] obtained some new types of multi-soliton solutions of the integrable KD equation from the trivial vacuum solution by using a truncated Painlevé analysis and Bäcklund transformation. Zhang et al [30] derived the periodic wave solution expressed by Jacobi elliptic functions for KD equation via F-expansion method. Li and Ruan [31] presented a set of generalized symmetries with arbitrary functions for the KD equation by using formal function series method. Zhao et al [32] provided the nonlinear transformations of the KD equation and constructed some new special types of single solitary wave solution and the multisoliton solutions using the homogenous balance method. Yang and Tang [33] investigated the abundant exact travelling wave solutions including solitary wave solutions, trigonometric function solutions and Jacobi elliptic doubly periodic function solutions for the KD equation via extended sinh-Gordon equation expansion method. However, few work has been done for the space-fractional, time-fractional or space-time fractional KD equation.

Notice that ($\frac{G'}{G}$) method [34]–[38] to apply successfully to solve fractional evolution equations. The aim of this paper is to apply the improved ($\frac{G'}{G}$)-expansion function method to obtain the exact solution of space-time fractional STO and KD equations, which have included classes of nonlinear equations with multi-order space-time fractional derivatives. We will discuss the methodology for the construction of some schemes and study their performance on test problem. It will be concluded that the improved ($\frac{G'}{G}$)-expansion function method is very powerful and efficient in finding exact solution as well as analytical solution of many fractional physical models. In order to investigate the local behavior of fractional models, several local versions of fractional derivatives have been proposed, among them Jumarie's derivative is a modified Riemann-Liouville derivative [39], [40], and has

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Youwei Zhang is a Professor of Mathematics at the School of Mathematics and Statistics, Hexi University, Zhangye, 734000, Gansu, China *e-mail: ywzhang0288@163.com

been successfully applied in Laplace problems [41], vibrating string model [42], Swift-Hohenberg equation [43] and so on. Now we state the definition and some important properties for the modified Riemann-Liouville derivative of order α as follows.
Assume that $f(x,t)$ denotes a continuous $\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}$ function. We use the following equality for the integral with respect to $(dx)^\alpha$

$$\begin{aligned} I_x^\alpha f(x,t) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1} f(\tau,t)d\tau \\ &= \frac{1}{\Gamma(\alpha+1)}\int_0^x f(\tau,t)(dx)^\alpha, \quad 0<\alpha\le 1. \end{aligned}$$

Jumarie's derivative for multivariate function is defined as:

$$D_x^\alpha f(x,t) = \begin{cases} \frac{1}{\Gamma(-\alpha)}\frac{\partial}{\partial x}\int_0^x (x-\tau)^{-\alpha-1}(f(\tau,t)-f(0,t))d\tau, \\ \qquad \alpha<0, \\ \frac{1}{\Gamma(-\alpha)}\frac{\partial}{\partial x}\int_0^x (x-\tau)^{-\alpha}(f(\tau,t)-f(0,t))d\tau, \\ \qquad 0<\alpha<1, \\ \left(\frac{\partial^{\alpha-n}}{\partial x^{\alpha-n}} f(x,t)\right)^{(n)}, \quad n\le\alpha<n+1, n\ge 1, \end{cases}$$

where $f(x,t)$ is a real continuous (but not necessarily exist partial derivative on x or t) function. The fundamental mathematical operations and results of Jumarie's derivative are given as follows

$$\begin{aligned} &D_x^\alpha c = 0, \\ &D_x^\alpha (cf(x,t)) = cD_x^\alpha f(x,t), \\ &D_x^\alpha x^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \\ &f^{(\alpha)}(g(x,t)) = \frac{df}{dg} D_x^\alpha g(x,t), \\ &\int (dx)^\beta = x^\beta, \end{aligned}$$

where c is a constant, $\alpha,\beta>0$, the above forth formula, the function f should be differentiable with respect to g. The above results are employed in the following sections.
This paper is organized as follows: Section II presents the methodology of the improved $(\frac{G'}{G})$-expansion function method for space-time fractional nonlinear evolution equation. Sections III and IV are devoted to construct the exact hyperbolic, periodic and rational function solutions for space-time fractional STO and KD equations, respectively. Section V makes conclusion for the obtained results.

II. IMPROVED $(\frac{G'}{G})$-EXPANSION FUNCTION METHOD

To illustrate the basic idea of the improved $(\frac{G'}{G})$-expansion function method for the nonlinear fractional evolution equation, we consider the following space-time fractional evolution equation

$$\begin{aligned} &F(u, D_t^\alpha u, D_x^\beta u, D_y^\gamma u, D_z^\delta u, D_t^{2\alpha} u, D_t^\alpha D_x^\beta u, D_t^\alpha D_y^\gamma u, \\ &D_t^\alpha D_z^\delta u, D_x^{2\beta} u, D_x^\beta D_y^\gamma u, D_x^\beta D_z^\delta u, D_y^{2\gamma} u, \ldots) = 0, \\ &0<\alpha,\beta,\gamma,\delta\le 1, \end{aligned} \tag{1}$$

where $u(x,t)$ is a field variable, $(x,t)\in\mathbb{R}\times\mathbb{R}^+$, F is a polynomial of u and its partial fractional derivatives, in which the highest order derivatives and the nonlinear terms are involved. In the following, we give the main steps of the improved $(\frac{G'}{G})$-expansion function method.
Step 1 To find the solutions of Eq. (1), we introduce a fractional complex transform to convert fractional differential equations into ordinary differential equations, so all analytical methods which are devoted to the advanced calculus can be easily applied to the fractional calculus. The solitary wave variable

$$\begin{aligned} &u(x,y,z,t) = U(\xi), \\ &\xi = \frac{bx^\beta}{\Gamma(\beta+1)} + \frac{cy^\gamma}{\Gamma(\gamma+1)} + \frac{dz^\delta}{\Gamma(\delta+1)} + \frac{at^\alpha}{\Gamma(\alpha+1)} + \xi_0, \end{aligned}$$

where a,b,c and d are nonzero arbitrary constants, permits us to reduce Eq. (1) to an ordinary differential equation of $U=U(\xi)$ in the following form

$$\begin{aligned} &P(U, aU', bU', cU', dU', a^2U'', abU'', acU'', adU'', \\ &b^2U'', bcU'', bdU'', c^2U'', \ldots) = 0, \end{aligned} \tag{2}$$

the prime denotes the derivative with respect to ξ. If possible, we should integrate Eq. (2) term by term one or more times.
Step 2 Suppose that the solution of Eq. (2) can be expressed as a polynomial of $(\frac{G'}{G})$ in the form

$$U(\xi) = \sum_{i=-m}^{m} k_i \left(\frac{G'}{G}\right)^i, \tag{3}$$

where k_i $(i=0,\pm1,\ldots,\pm m)$ (m is positive number, called the balance number) are constants to be determined later, while the function $G=G(\xi)$ satisfies the following second-order linear ODE

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{4}$$

with λ and μ being constants. The positive integer m can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms appearing in Eq. (2). More precisely, we define the degree of $U(\xi)$ as $D[U(\xi)]=m$, which gives rise to the degrees of the other expressions as follows:

$$\begin{aligned} &D\left(\frac{d^qU}{d\xi^q}\right) = m+q, \\ &D\left(U^p\left(\frac{d^qU}{d\xi^q}\right)^s\right) = mp + s(q+m). \end{aligned}$$

Therefore, we can obtain the value of m in Eq. (3).
Step 3 Substituting Eq. (3) into Eq. (2), using Eq. (4), collecting all terms with the same order of $(\frac{G'}{G})$ together, and then equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for k_i, a, b, c, d, λ and μ.
Step 4 Since the general solution to Eq. (4) is well known:

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right) - \frac{\lambda}{2}, \\ \qquad \lambda^2-4\mu>0, \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right) - \frac{\lambda}{2}, \\ \qquad \lambda^2-4\mu<0, \\ \frac{C_2}{C_1+C_2\xi} - \frac{\lambda}{2}, \quad \lambda^2-4\mu=0, \end{cases} \tag{5}$$

where C_1, C_2 are arbitrary constants. Then substituting $k_i, a, b, c, d, \lambda, \mu$ and the general solutions of Eq. (4) into Eq. (3), we get more solitary wave solutions of the nonlinear partial fractional derivatives of Eq. (1).

III. Space-time fractional STO equation

In this section, we use the improved $(\frac{G'}{G})$-expansion function method to construct the exact solutions for STO equation with space-time fractional derivatives.

$$\begin{aligned} &D_t^{\alpha}u(x,t) + 3k(D_x^{\beta}u(x,t))^2 + 3ku^2(x,t)D_x^{\beta}u(x,t) \\ &+ 3ku(x,t)D_x^{2\beta}u(x,t) + kD_x^{3\beta}u(x,t) = 0, \\ &0 < \alpha, \beta \leq 1, \end{aligned} \tag{6}$$

where $k \neq 0$ is a constant, $x \in \mathbb{R}$ is a space coordinate in the propagation direction of the field and $t \in \mathbb{R}^+$ is the time. We can see that the fractional complex transform

$$u(x,t) = U(\xi),\ \xi = \frac{bx^{\beta}}{\Gamma(\beta+1)} + \frac{at^{\alpha}}{\Gamma(\alpha+1)} + \xi_0, \tag{7}$$

where a and b are constants, permits us to reduce Eq. (6) into the following ODE:

$$-aU' + 3kb^2(U')^2 + 3kbU^2U' + 3kb^2UU'' + kb^3U''' = 0. \tag{8}$$

Considering the homogeneous balance between the highest order derivative and the nonlinear term in Eq. (8), we deduce that the balance number $m = 1$. Then (3) reduces

$$U(\xi) = k_{-1}\Big(\frac{G'}{G}\Big)^{-1} + k_0 + k_1\Big(\frac{G'}{G}\Big), \tag{9}$$

where k_{-1}, k_0, k_1, a and b are arbitrary constants to be determined later. Substituting Eq. (9) into Eq. (8), collecting all the terms of powers of $(\frac{G'}{G})$, and setting each coefficient to zero, we get a system of algebraic equations. With some calculation, we can solve this system of algebraic equations to obtain the following sets of solutions.

Case 1

$$\begin{aligned} &k_{-1} = -2\mu b - 2\lambda^2 b,\ k_0 = \lambda b,\ k_1 = 2b, \\ &a = 16\mu k b^3 + 11\lambda^2 k b^3,\ b = b \end{aligned} \tag{10}$$

or

$$\begin{aligned} &k_{-1} = 0,\ k_0 = \lambda b,\ k_1 = 2b, \\ &a = 4\mu k b^3 - \lambda^2 k b^3,\ b = b. \end{aligned} \tag{11}$$

Substituting the result into (9) and combining with (5), respectively, we can obtain the following exact solutions to Eq. (6).

Family 1 If $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic solitary wave solutions of Eq. (8)

$$\begin{aligned} U_1(\xi) = &-4b(\mu+\lambda^2)\Bigg(\sqrt{\lambda^2-4\mu} \\ &\times \Big(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\Big) - \lambda\Bigg)^{-1} \\ &+ b\sqrt{\lambda^2-4\mu} \\ &\times \Big(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\Big), \end{aligned}$$

where $\xi = \frac{b}{\Gamma(\beta+1)}x^{\beta} + \frac{16\mu k b^3 + 11\lambda^2 k b^3}{\Gamma(\alpha+1)}t^{\alpha} + \xi_0$, or

$$\begin{aligned} U_2(\xi) = &\,b\sqrt{\lambda^2-4\mu} \\ &\times \Big(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\Big), \end{aligned}$$

where $\xi = \frac{b}{\Gamma(\beta+1)}x^{\beta} + \frac{4\mu k b^3 - \lambda^2 k b^3}{\Gamma(\alpha+1)}t^{\alpha} + \xi_0$.

Family 2 If $\lambda^2 - 4\mu < 0$, we obtain the periodic solitary wave solutions of Eq. (8)

$$\begin{aligned} U_3(\xi) = &-4b(\mu+\lambda^2)\Bigg(\sqrt{4\mu-\lambda^2} \\ &\times \Big(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\Big) - \lambda\Bigg)^{-1} \\ &+ b\sqrt{4\mu-\lambda^2} \\ &\times \Big(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\Big), \end{aligned}$$

where $\xi = \frac{b}{\Gamma(\beta+1)}x^{\beta} + \frac{16\mu k b^3 + 11\lambda^2 k b^3}{\Gamma(\alpha+1)}t^{\alpha} + \xi_0$, or

$$\begin{aligned} U_4(\xi) = &\,b\sqrt{4\mu-\lambda^2} \\ &\times \Big(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\Big), \end{aligned}$$

where $\xi = \frac{b}{\Gamma(\beta+1)}x^{\beta} + \frac{4\mu k b^3 - \lambda^2 k b^3}{\Gamma(\alpha+1)}t^{\alpha} + \xi_0$.

In particular, if we set $\lambda = 0$, $\mu = b = 1$, $C_1 \neq 0$, $C_2 = 0$ in $U_4(\xi)$ with $\alpha = \beta = 1$, the exact periodic solitary wave solution of the STO equation

$$u(x,t) = -2\tan\big(x + 4kt + \xi_0\big),$$

which is the same exact solution in [28] via the simplest equation method.

Family 3 If $\lambda^2 - 4\mu = 0$, we obtain the rational solitary wave solutions of Eq. (8)

$$\begin{aligned} U_5(\xi) = &-2b(\mu+\lambda^2)\Big(\frac{C_2}{C_1+C_2\xi} - \frac{\lambda}{2}\Big)^{-1} + \lambda b \\ &+ 2b\Big(\frac{C_2}{C_1+C_2\xi} - \frac{\lambda}{2}\Big), \end{aligned}$$

where $\xi = \frac{b}{\Gamma(\beta+1)}x^{\beta} + \frac{16\mu k b^3 + 11\lambda^2 k b^3}{\Gamma(\alpha+1)}t^{\alpha} + \xi_0$, or

$$U_6(\xi) = \frac{2bC_2}{C_1+C_2\xi},$$

where $\xi = \frac{b}{\Gamma(\beta+1)}x^{\beta} + \frac{4\mu k b^3 - \lambda^2 k b^3}{\Gamma(\alpha+1)}t^{\alpha} + \xi_0$.

Case 2

$$\begin{aligned} &k_{-1} = -2\mu b,\ k_0 = -\lambda b,\ k_1 = \frac{2\mu b + 2\lambda^2 b}{\mu}, \\ &a = 16\mu k b^3 + 11\lambda^2 k b^3,\ b = b \end{aligned} \tag{12}$$

or

$$\begin{aligned} &k_{-1} = -2\mu b,\ k_0 = -\lambda b,\ k_1 = 0, \\ &a = 4\mu k b^3 - \lambda^2 k b^3,\ b = b. \end{aligned} \tag{13}$$

Substituting the result into (9) and combining with (5), respectively, we can obtain the following exact solutions to Eq. (6).

Family 1 If $\lambda^2-4\mu>0$, we obtain the hyperbolic solitary wave solutions of Eq. (8)

$$\begin{aligned}U_7(\xi)=&-4\mu b\Bigg(\sqrt{\lambda^2-4\mu}\\&\times\Big(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\Big)-\lambda\Bigg)^{-1}\\&-\lambda b+b(1+\frac{\lambda^2}{\mu})\Bigg(\sqrt{\lambda^2-4\mu}\\&\times\Big(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\Big)-\lambda\Bigg),\end{aligned}$$

where $\xi=\frac{b}{\Gamma(\beta+1)}x^\beta+\frac{16\mu kb^3+11\lambda^2kb^3}{\Gamma(\alpha+1)}t^\alpha+\xi_0$, or

$$\begin{aligned}U_8(\xi)=&-4\mu b\Bigg(\sqrt{\lambda^2-4\mu}\\&\times\Big(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\Big)-\lambda\Bigg)^{-1}\\&-\lambda b,\end{aligned}$$

where $\xi=\frac{b}{\Gamma(\beta+1)}x^\beta+\frac{4\mu kb^3-\lambda^2kb^3}{\Gamma(\alpha+1)}t^\alpha+\xi_0$.

Family 2 If $\lambda^2-4\mu<0$, we obtain the periodic solitary wave solutions of Eq. (8)

$$\begin{aligned}U_9(\xi)=&-4\mu b\Bigg(\sqrt{4\mu-\lambda^2}\\&\times\Big(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\Big)-\lambda\Bigg)^{-1}\\&-\lambda b+b(1+\frac{\lambda^2}{\mu})\Bigg(\sqrt{4\mu-\lambda^2}\\&\times\Big(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\Big)-\lambda\Bigg),\end{aligned}$$

where $\xi=\frac{b}{\Gamma(\beta+1)}x^\beta+\frac{16\mu kb^3+11\lambda^2kb^3}{\Gamma(\alpha+1)}t^\alpha+\xi_0$, or

$$\begin{aligned}U_{10}(\xi)=&-4\mu b\Bigg(\sqrt{4\mu-\lambda^2}\\&\times\Big(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\Big)-\lambda\Bigg)^{-1}\\&-\lambda b,\end{aligned}$$

where $\xi=\frac{b}{\Gamma(\beta+1)}x^\beta+\frac{4\mu kb^3-\lambda^2kb^3}{\Gamma(\alpha+1)}t^\alpha+\xi_0$.

Family 3 If $\lambda^2-4\mu=0$, we obtain the rational solitary wave solutions of Eq. (8)

$$\begin{aligned}U_{11}(\xi)=&-2\mu b\Big(\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\Big)^{-1}-\lambda b\\&+2b(1+\frac{\lambda^2}{\mu})\Big(\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\Big),\end{aligned}$$

where $\xi=\frac{b}{\Gamma(\beta+1)}x^\beta+\frac{16\mu kb^3+11\lambda^2kb^3}{\Gamma(\alpha+1)}t^\alpha+\xi_0$, or

$$U_{12}(\xi)=-2\mu b\Big(\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\Big)^{-1}-\lambda b,$$

where $\xi=\frac{b}{\Gamma(\beta+1)}x^\beta+\frac{4\mu kb^3-\lambda^2kb^3}{\Gamma(\alpha+1)}t^\alpha+\xi_0$.

IV. Space-time fractional KD equation

In this section, we use the improved $(\frac{G'}{G})$-expansion function method to construct the exact solutions for KD equation with space-time fractional derivatives.

$$\begin{cases}D_t^\alpha u(x,y,t)-6ku(x,y,t)D_x^\beta u(x,y,t)\\\quad+\frac{3}{2}l^2u^2(x,y,t)D_x^\beta u(x,y,t)-3D_y^\gamma v(x,y,t)\\\quad+3lD_x^\beta u(x,y,t)v(x,y,t)-D_x^{3\beta}u(x,y,t)=0,\\D_y^\gamma u(x,y,t)=D_x^\beta v(x,y,t),\end{cases}\tag{14}$$

where $0<\alpha,\beta,\gamma\leq 1$, $k,l\neq 0$ are constants, $(x,y,t)\in\mathbb{R}^2\times\mathbb{R}^+$ is a (2+1)-dimensional space-time coordinate in the propagation direction of the field. For $D_y^\gamma u(x,y,t)=0$, Eq. (14) is the fractional Gardner equation. When $l=0$, Eq. (14) is the well-known fractional Kadomtsev-Petviashvili equation and the modified Kadomtsev-Petviashvili equation reads from (14) for $k=0$.

We can see that the fractional complex transform

$$\begin{cases}u(x,y,t)=U(\xi),\\v(x,y,t)=V(\xi),\end{cases}\tag{15}$$

where $\xi=\frac{bx^\beta}{\Gamma(\beta+1)}+\frac{cy^\gamma}{\Gamma(\gamma+1)}+\frac{at^\alpha}{\Gamma(\alpha+1)}+\xi_0$, a,b and c are constants, permits us to reduce Eq. (15) into the following ODE:

$$\begin{cases}aU'-6kbUU'+\frac{3}{2}bl^2U^2U'-3cV'+3lbU'V-b^3U'''\\\quad=0,\\cU'=bV'.\end{cases}\tag{16}$$

From the second formula of the Eq. (16), both sides of integral on ξ yields

$$V=\frac{c}{b}U+\frac{d}{b},$$

where d is an arbitrary constant.

Suppose that the solution of Eq. (16) can be expressed by

$$\begin{cases}U(\xi)=\sum\limits_{i=-m_1}^{m_1}k_i\big(\frac{G'}{G}\big)^i,\\V(\xi)=\sum\limits_{i=-m_2}^{m_2}l_i\big(\frac{G'}{G}\big)^i.\end{cases}$$

Balancing the order of U''' and U^2U' in (16) we have $m_1=m_2=1$. So

$$\begin{cases}U(\xi)=k_{-1}\big(\frac{G'}{G}\big)^{-1}+k_0+k_1\big(\frac{G'}{G}\big),\\V(\xi)=l_{-1}\big(\frac{G'}{G}\big)^{-1}+l_0+l_1\big(\frac{G'}{G}\big).\end{cases}\tag{17}$$

Substituting (17) into (16), using Eq. (4) and collecting all the terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations yields:

Case 1

$$\begin{aligned}&k_{-1}=\frac{2\mu b}{l},\ k_0=\frac{\lambda lb^2+2kb-lc}{l^2b},\ k_1=\frac{2b}{l},\\&l_{-1}=\frac{2\mu c}{l},\ l_0=\frac{\lambda lb^2c+2kbc-lc^2}{l^2b^2}+\frac{d}{b},\ l_1=\frac{2c}{l},\\&a=\frac{9c^2-6lbd-12\frac{kbc}{l}+12\frac{k^2b^2}{l^2}-\lambda^2b^4-8\mu b^4}{2b},\\&b=b,\ c=c.\end{aligned}$$

Substituting the result into (17) and combining with (5), respectively, we can obtain the following exact solutions to Eq. (14).

Family 1 If $\lambda^2-4\mu>0$, we obtain the hyperbolic solitary wave solutions of the Eq. (16)

$$\begin{cases} U_1(\xi)=\frac{4\mu b}{l}\Big(\sqrt{\lambda^2-4\mu} \\ \quad\times\Big(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\Big)-\lambda\Big)^{-1} \\ \quad+\frac{\lambda lb^2+2kb-lc}{l^2b}+\frac{b}{l}\Big(\sqrt{\lambda^2-4\mu} \\ \quad\times\Big(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\Big)-\lambda\Big), \\ V_1(\xi)=\frac{4\mu c}{l}\Big(\sqrt{\lambda^2-4\mu} \\ \quad\times\Big(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\Big)-\lambda\Big)^{-1} \\ \quad+\frac{\lambda lb^2c+2kbc-lc^2}{l^2b^2}+\frac{d}{b}+\frac{c}{l}\Big(\sqrt{\lambda^2-4\mu} \\ \quad\times\Big(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\Big)-\lambda\Big), \end{cases}$$

where $\xi=\frac{b}{\Gamma(\beta+1)}x^\beta+\frac{c}{\Gamma(\gamma+1)}y^\gamma$
$+\frac{9c^2-6lbd-12\frac{kbc}{l}+12\frac{k^2b^2}{l^2}-\lambda^2b^4-8\mu b^4}{2b\Gamma(\alpha+1)}t^\alpha+\xi_0.$

Family 2 If $\lambda^2-4\mu<0$, we obtain the periodic solitary wave solutions of the Eq. (16)

$$\begin{cases} U_2(\xi)=\frac{4\mu b}{l}\Big(\sqrt{4\mu-\lambda^2} \\ \quad\times\Big(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\Big)-\lambda\Big)^{-1} \\ \quad+\frac{\lambda lb^2+2kb-lc}{l^2b}+\frac{b}{l}\Big(\sqrt{4\mu-\lambda^2} \\ \quad\times\Big(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\Big)-\lambda\Big), \\ V_2(\xi)=\frac{4\mu c}{l}\Big(\sqrt{4\mu-\lambda^2} \\ \quad\times\Big(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\Big)-\lambda\Big)^{-1} \\ \quad+\frac{\lambda lb^2c+2kbc-lc^2}{l^2b^2}+\frac{d}{b}+\frac{c}{l}\Big(\sqrt{4\mu-\lambda^2} \\ \quad\times\Big(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\Big)-\lambda\Big), \end{cases}$$

where $\xi=\frac{b}{\Gamma(\beta+1)}x^\beta+\frac{c}{\Gamma(\gamma+1)}y^\gamma$
$+\frac{9c^2-6lbd-12\frac{kbc}{l}+12\frac{k^2b^2}{l^2}-\lambda^2b^4-8\mu b^4}{2b\Gamma(\alpha+1)}t^\alpha+\xi_0.$

Family 3 If $\lambda^2-4\mu=0$, we obtain the rational solitary wave solutions of the Eq. (16)

$$\begin{cases} U_3(\xi)=\frac{2\mu b}{l}\Big(\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\Big)^{-1}+\frac{\lambda lb^2+2kb-lc}{l^2b} \\ \quad+\frac{2b}{l}\Big(\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\Big), \\ V_3(\xi)=\frac{2\mu c}{l}\Big(\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\Big)^{-1}+\frac{\lambda lb^2c+2kbc-lc^2}{l^2b^2}+\frac{d}{b} \\ \quad+\frac{2c}{l}\Big(\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\Big), \end{cases}$$

where $\xi=\frac{b}{\Gamma(\beta+1)}x^\beta+\frac{c}{\Gamma(\gamma+1)}y^\gamma$
$+\frac{9c^2-6lbd-12\frac{kbc}{l}+12\frac{k^2b^2}{l^2}-\lambda^2b^4-8\mu b^4}{2b\Gamma(\alpha+1)}t^\alpha+\xi_0.$

Case 2

$$\begin{aligned} &k_{-1}=-\frac{2\mu b}{l},\ k_0=-\frac{\lambda lb^2-2kb+lc}{l^2b},\ k_1=-\frac{2b}{l}, \\ &l_{-1}=-\frac{2\mu c}{l},\ l_0=-\frac{\lambda lb^2c-2kbc+lc^2}{l^2b^2}+\frac{d}{b}, \\ &l_1=-\frac{2c}{l}, \\ &a=\frac{9c^2-6lbd-12\frac{kbc}{l}+12\frac{k^2b^2}{l^2}-\lambda^2b^4-8\mu b^4}{2b}, \\ &b=b,\ c=c. \end{aligned}$$

Substituting the result into (17) and combining with (5), respectively, we can obtain the following exact solutions to Eq. (14).

Family 1 If $\lambda^2-4\mu>0$, we obtain the hyperbolic solitary wave solutions of the Eq. (16)

$$\begin{cases} U_1(\xi)=-\frac{4\mu b}{l}\Big(\sqrt{\lambda^2-4\mu} \\ \quad\times\Big(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\Big)-\lambda\Big)^{-1} \\ \quad-\frac{\lambda lb^2-2kb+lc}{l^2b}-\frac{b}{l}\Big(\sqrt{\lambda^2-4\mu} \\ \quad\times\Big(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\Big)-\lambda\Big), \\ V_1(\xi)=-\frac{4\mu c}{l}\Big(\sqrt{\lambda^2-4\mu} \\ \quad\times\Big(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\Big)-\lambda\Big)^{-1} \\ \quad-\frac{\lambda lb^2c-2kbc+lc^2}{l^2b^2}+\frac{d}{b}-\frac{c}{l}\Big(\sqrt{\lambda^2-4\mu} \\ \quad\times\Big(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\Big)-\lambda\Big), \end{cases}$$

where $\xi=\frac{b}{\Gamma(\beta+1)}x^\beta+\frac{c}{\Gamma(\gamma+1)}y^\gamma$
$+\frac{9c^2-6lbd-12\frac{kbc}{l}+12\frac{k^2b^2}{l^2}-\lambda^2b^4-8\mu b^4}{2b\Gamma(\alpha+1)}t^\alpha+\xi_0.$

Family 2 If $\lambda^2-4\mu<0$, we obtain the periodic solitary wave solutions of the Eq. (16)

$$\begin{cases} U_2(\xi)=-\frac{4\mu b}{l}\Big(\sqrt{4\mu-\lambda^2} \\ \quad\times\Big(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\Big)-\lambda\Big)^{-1} \\ \quad-\frac{\lambda lb^2-2kb+lc}{l^2b}-\frac{b}{l}\Big(\sqrt{4\mu-\lambda^2} \\ \quad\times\Big(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\Big)-\lambda\Big), \\ V_2(\xi)=-\frac{4\mu c}{l}\Big(\sqrt{4\mu-\lambda^2} \\ \quad\times\Big(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\Big)-\lambda\Big)^{-1} \\ \quad-\frac{\lambda lb^2c-2kbc+lc^2}{l^2b^2}+\frac{d}{b}-\frac{c}{l}\Big(\sqrt{4\mu-\lambda^2} \\ \quad\times\Big(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\Big)-\lambda\Big), \end{cases}$$

where $\xi=\frac{b}{\Gamma(\beta+1)}x^\beta+\frac{c}{\Gamma(\gamma+1)}y^\gamma$
$+\frac{9c^2-6lbd-12\frac{kbc}{l}+12\frac{k^2b^2}{l^2}-\lambda^2b^4-8\mu b^4}{2b\Gamma(\alpha+1)}t^\alpha+\xi_0.$

Family 3 If $\lambda^2 - 4\mu = 0$, we obtain the rational solitary wave solutions of the Eq. (16)

$$\begin{cases} U_3(\xi) = -\frac{2\mu b}{l}\left(\frac{C_2}{C_1+C_2\xi} - \frac{\lambda}{2}\right)^{-1} - \frac{\lambda l b^2 - 2kb + lc}{l^2 b} \\ \quad - \frac{2b}{l}\left(\frac{C_2}{C_1+C_2\xi} - \frac{\lambda}{2}\right), \\ V_3(\xi) = -\frac{2\mu c}{l}\left(\frac{C_2}{C_1+C_2\xi} - \frac{\lambda}{2}\right)^{-1} - \frac{\lambda l b^2 c - 2kbc + lc^2}{l^2 b^2} + \frac{d}{b} \\ \quad - \frac{2c}{l}\left(\frac{C_2}{C_1+C_2\xi} - \frac{\lambda}{2}\right), \end{cases}$$

where $\xi = \frac{b}{\Gamma(\beta+1)}x^\beta + \frac{c}{\Gamma(\gamma+1)}y^\gamma + \frac{9c^2 - 6lbd - 12\frac{kbc}{l} + 12\frac{k^2b^2}{l^2} - \lambda^2 b^4 - 8\mu b^4}{2b\Gamma(\alpha+1)}t^\alpha + \xi_0$.

Remark 1 The established solutions above for the space-time fractional (2+1)-dimensional KD equations are new exact solutions that we have never seen before within our knowledge. The method can be applied to other nonlinear fractional evolution equations in mathematical physics.

V. Conclusion

We use the improved $(\frac{G'}{G})$-expansion function method to solve the exact solutions for the space-time fractional STO and KD equations. This method is reliable, simple and gives many new hyperbolic, periodic and rational solitary wave solutions for the fractional STO and KD equations, respectively. Based on certain fractional variable transformation, such fractional evolution equations can be turned into ordinary differential equations of integer order, the solutions of which can be expressed by a polynomial in $(\frac{G'}{G})$. This method is very efficient and powerful in finding the exact solutions for the nonlinear fractional evolution equation without any assumption and restriction.

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