The Linear 4-arboricity of Balanced Complete Bipartite Graphs *

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Abstract

A linear k-forest is a graph whose components are paths of length at most k. The linear k-arboricity of a graph G, denoted by $la_k(G)$, is the least number of linear k-forests needed to decompose G. In this paper, it is obtained that $la_4(K_{n,n}) = \lceil 5n/8 \rceil$ for $n \equiv 0(\mod 5)$.

Keywords: Linear k-forest; linear k-arboricity; complete bipartite graph; perfect matching; bipartite difference

1 Introduction

In this paper, all graphs considered are finite, undirected, and simple (i. e., loopless and without multiple edges). We refer to [20] for terminology in graph theory. In recent years, many parameters and classes of graphs were studied. For example, in [11], different properties of the intrinsic order graph were obtained, namely those dealing with its edges, chains, shadows, neighbors and degrees of its vertices, and some relevant subgraphs, as well as the natural isomorphisms between them. In [18], the n-dimensional cube-connected complete graph was studied. In [24, 25], the hamiltonicity, path t-coloring, and the shortest paths of Sierpiński-like graphs were researched. In [26], the vertex arboricity of integer distance graph $G(D_{m,k})$ was obtained.

A decomposition of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. If a graph G has a decomposition $G_1, G_2, \cdots, G_d$, then we say that $G_1, G_2, \cdots, G_d$ decompose G, or G can be decomposed into $G_1, G_2, \cdots, G_d$. A linear $k-$forest is a forest whose components are paths of length at most k. The linear $k-$arboricity of a graph G, denoted by $la_k(G)$, is the least number of linear $k-$forests needed to decompose G. Let x be a real number, denoted by $\lfloor x \rfloor$ the maximum integer no more than x, and denoted by $\lceil x \rceil$ the minimum integer no less than x. For any integers $a < b$, let $[a, b]$ denote the set of integers $\{a, a+1, \cdots, b\}$ for simplicity. For any positive integer λ, let P_λ be a path on λ vertices which has length $\lambda - 1$.

The notion of linear $k-$arboricity was first introduced by Habib and Peroche [13], which is a natural generalization of edge coloring. Clearly, a linear 1-forest is induced by a matching, and $la_1(G)$ is the edge chromatic number, or chromatic index, $\chi'(G)$ of a graph G. Moreover, the linear $k-$arboricity $la_k(G)$ is also a refinement of the ordinary linear arboricity $la(G)$(or $la_\infty(G)$) [14] of a graph G, which is the case when every component of each forest is a path with no length constraint. In 1982, Habib and Peroche [12] proposed the following conjecture for an upper bound on $la_k(G)$.

Conjecture 1.1. *If G is a graph with maximum degree $\Delta(G)$ and $k \geq 2$, then*

$$la_k(G) \leq \begin{cases} \lceil \frac{\Delta(G)\cdot|V(G)|}{2\lfloor \frac{k\cdot|V(G)|}{k+1} \rfloor} \rceil, if\ \Delta(G) = \mid V(G) \mid -1, \\ \lceil \frac{\Delta(G)\cdot|V(G)|+1}{2\lfloor \frac{k\cdot|V(G)|}{k+1} \rfloor} \rceil, if\ \Delta(G) < \mid V(G) \mid -1. \end{cases}$$

For $k = \mid V(G) \mid -1$, it is the Akiyama's conjecture [1].

Conjecture 1.2. *[1] $la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$.*

So far, quite a few results on the verification of Conjecture 1.1 have obtained in the literature, especially for graphs with particular structures, such as trees [5, 6, 13], cubic graphs [4, 16, 19], regular graphs [2, 3], planar graphs [17], balanced complete bipartite graphs [8, 9, 10], balanced complete multipartite graphs [22] and complete graphs [5, 7, 8, 9, 21]. It is obtained that the linear 2-arboricity, the linear 3-arboricity and the low bound of linear k-arboricity of balanced complete bipartite graph in [8, 9, 10], respectively. In [23], Xue and Zuo obtained the linear $(n-1)$-arboricity of complete multipartite graph $K_{n(m)}$. In [15], the linear 6-arboricity of the graph $K_{m,n}$ was obtained. All the results are coherent with the corresponding cases of Conjecture 1.1. But for the general graph, this conjecture has not been proved yet.

As for a lower bound on $la_k(G)$, it is obvious that the following result holds.

Lemma 1.3. *For any graph G with maximum degree $\triangle(G)$, then*

$$la_k(G) \geq max\{\lceil \frac{\Delta(G)}{2} \rceil, \lceil \frac{|E(G)|}{\lfloor \frac{k|V(G)|}{k+1} \rfloor} \rceil\}.$$

2 Main results

Note that in the following the index of each vertex is modulo n. Our main result is the following theorem.

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Let $K_{n,n} = G(A, B)$ be a balanced complete bipartite graph with partite sets A and B, where $A = \{a_0, a_1, \cdots, a_{n-1}\}$ and $B = \{b_0, b_1, \cdots, b_{n-1}\}$. It is defined that the *bipartite difference* of an edge $a_p b_q$ in [9] as the value $(q-p)(\mod n)$. The edge set of $K_{n,n}$ can be partitioned into n pairwise disjoint perfect matchings $M_0, M_1, \cdots, M_{n-1}$, where M_j is exactly the set of edges of bipartite difference j in $K_{n,n}$ for $j \in [0, n-1]$.

Theorem 2.1. *$la_4(K_{n,n}) = \lceil 5n/8 \rceil$ for $n \equiv 0(\mod 5)$.*

Proof. Clearly, by Lemma 1.3,

$$la_4(K_{n,n}) \geq \lceil n^2/\lfloor 4 \cdot 2n/5 \rfloor \rceil = \lceil 5n/8 \rceil,$$

so we need only to prove the upper bound.

It is easy to see that the following Claim 1 holds.

Claim 1. For each $t \in [0, \lfloor n/8 \rfloor - 1]$, the edges of $M_{8t+2q} \cup M_{8t+2q+1}$ other than that in $c_{t,q}$ can form one linear 4-forest, where

$$c_{t,q} = \{a_{2+2t+5(i-1)} b_{2+2t+5(i-1)+8t+2q}, a_{4+2t+5(i-1)} b_{4+2t+5(i-1)+8t+2q+1} | i \in [1, n/5]\}$$

for $q \in \{0, 2\}$, and

$$c_{t,q} = \{a_{3+2t+5(i-1)} b_{3+2t+5(i-1)+8t+2q}, a_{5+2t+5(i-1)} b_{5+2t+5(i-1)+8t+2q+1} | i \in [1, n/5]\}$$

for $q \in \{1, 3\}$.

For example, if $n = 40$, the edges of $M_0 \cup M_1$ other than that in

$$c_{0,0} = \{a_{2+5(i-1)} b_{2+5(i-1)}, a_{4+5(i-1)} b_{4+5(i-1)+1} | i \in [1, 8]\}$$

can form one linear 4-forest (please see Fig. 1). Similarly, the edges of $M_2 \cup M_3$ other than that in

$$c_{0,1} = \{a_{3+5(i-1)} b_{3+5(i-1)+2}, a_{5+5(i-1)} b_{5+5(i-1)+3} | i \in [1, 8]\}$$

can form one linear 4-forest. The edges of $M_4 \cup M_5$ other than edges in

$$c_{0,2} = \{a_{2+5(i-1)} b_{2+5(i-1)+4}, a_{4+5(i-1)} b_{4+5(i-1)+5} | i \in [1, 8]\}$$

can form one linear 4-forest. The edges of $M_6 \cup M_7$ other than edges in

$$c_{0,3} = \{a_{3+5(i-1)} b_{3+5(i-1)+6}, a_{5+5(i-1)} b_{5+5(i-1)+7} | i \in [1, 8]\}$$

can form another linear 4-forest, and so on.

Claim 2. $la_4(K_{n,n}) \leq 5n/8$ for $n \equiv 0(\mod 8)$ and $n \equiv 0(\mod 5)$.

By Claim 1, we have obtained $n/2$ linear 4-forests in total. Thus we have to estimate the number of linear 4-forests induced by the union of $c_{t,q}$.

It is easy to verify that all edges of $\cup_{q=0}^{3} c_{t,q}$ can form $n/5$ pairwise disjoint paths P_5

$$\{b_{5+2t+5(i-1)+8t+3} a_{5+2t+5(i-1)} b_{5+2t+5(i-1)+8t+7} a_{5+2t+5(i-1)+7} b_{2+2t+5(i-1)+8t+4+10} | i \in [1, n/5]\}$$

and $n/5$ pairwise disjoint 4-cycles

$$\{a_{3+2t+5(i-1)} b_{3+2t+5(i-1)+8t+2} a_{4+2t+5(i-1)} b_{4+2t+5(i-1)+8t+5} a_{3+2t+5(i-1)} | i \in [1, n/5]\},$$

for each $t \in [0, n/8-1]$. Moreover, for each $t \in [0, n/8-1]$, we obtain one linear 4-forest by deleting edges in

$$\{a_{4+2t+5(i-1)} b_{4+2t+5(i-1)+8t+1} | i \in [1, n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{t,q}$, adding edges of

$$\{a_u b_{u+8t_l+9} | u = 2t_l + 5i + 1, i \in [1, n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{t_l,q}$, and adding edges of

$$\{a_u b_{u+8(5l-5)+1} | u = 2(5l-5) + 5i - 1, i \in [1, n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{(5l-1),q}$, where $t_l \in [5(l-1), 5(l-1)+3]$ and $l \in [1, n/40]$.

Hence, $la_4(K_{n,n}) \leq n/2 + n/8 = 5n/8$ in the case of $n \equiv 0(\mod 40)$.

For example, if $n = 40$, for $t = 0$, the edges of $\cup_{q=0}^{3} c_{0,q}$ can form 8 pairwise disjoint paths P_5

$$\{b_{5i+3} a_{5i} b_{5i+7} a_{5i+7} b_{5i+11} | i \in [1, 8]\}$$

and 8 pairwise disjoint 4-cycles

$$\{a_{5i-2} b_{5i} a_{5i-1} b_{5i+4} a_{5i-2} | i \in [1, 8]\}.$$

(Please see Figure 2. Note that the paths P_5 of $\cup_{q=0}^{3} c_{0,q}$ have not been displayed in the figure.) Moreover, for $t = 0$, we obtain one linear 4-forest by deleting edges in

$$\{a_{4+5(i-1)} b_{4+5(i-1)+1} | i \in [1, 8]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{0,q}$ and adding edges of

$$\{a_{4+2+5(i-1)} b_{4+2+5(i-1)+8+1} | i \in [1, 8]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{0,q}$. Similarly, for $t = 1$, we obtain one linear 4-forest by deleting edges in

$$\{a_{4+2+5(i-1)} b_{4+2+5(i-1)+8+1} | i \in [1, 8]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{1,q}$ and adding edges of

$$\{a_{4+2\cdot 2+5(i-1)} b_{4+2\cdot 2+5(i-1)+8\cdot 2+1} | i \in [1, 8]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{1,q}$. For $t = 2$, we obtain one linear 4-forest by deleting edges in

$$\{a_{4+2\cdot 2+5(i-1)} b_{4+2\cdot 2+5(i-1)+8\cdot 2+1} | i \in [1, 8]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{2,q}$ and adding edges of

$$\{a_{4+2\cdot 3+5(i-1)} b_{4+2\cdot 3+5(i-1)+8\cdot 3+1} | i \in [1, 8]\}$$

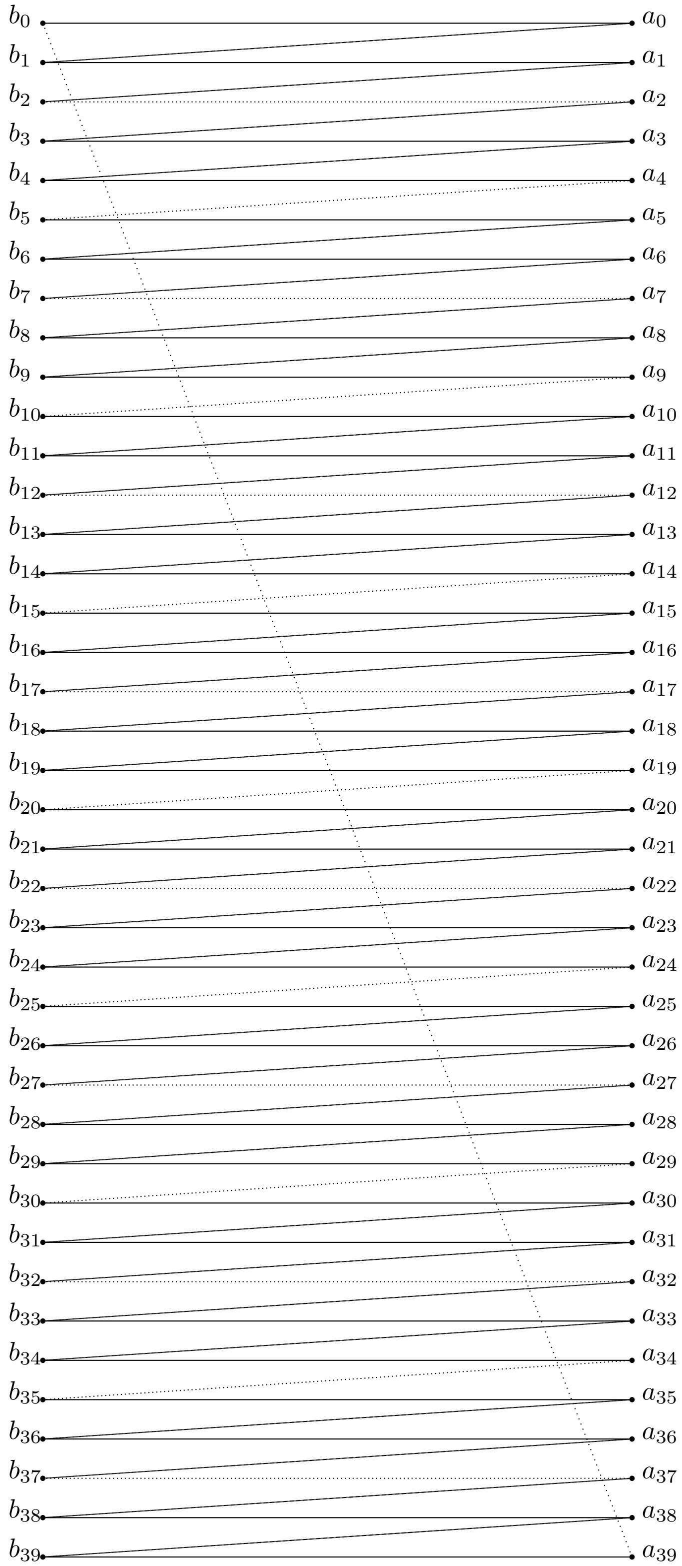

Figure 1. $M_0 \cup M_1$ with $c_{0,0}$ broken and others normal

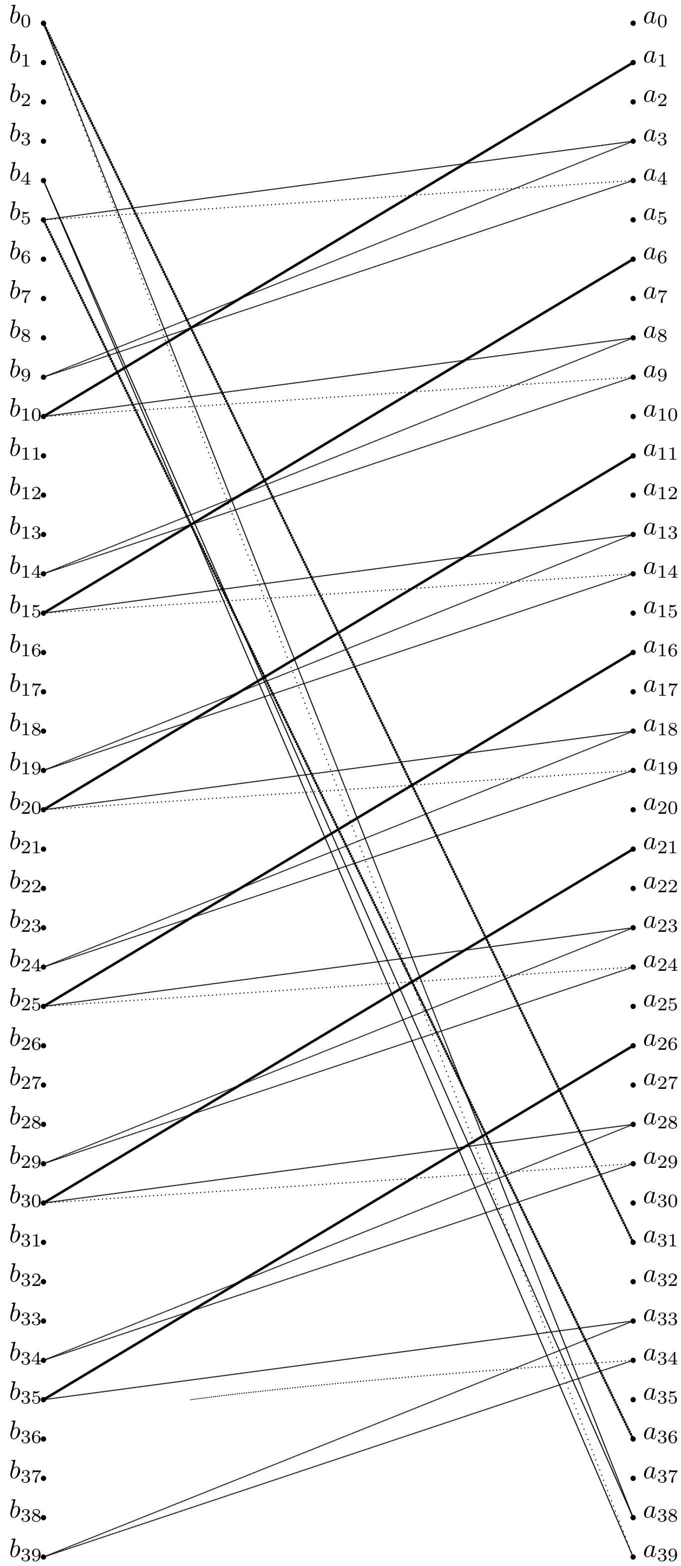

Figure 2. broken edges: the edges deleted from 4-cycles of $\cup_{q=0}^{3} c_{0,q}$

heavy edges: the edges adding to the remained edges of $\cup_{q=0}^{3} c_{0,q}$

to the remained edges of $\cup_{q=0}^{3} c_{2,q}$. For $t=3$, we obtain one linear 4-forest by deleting edges in

$$\{a_{4+2\cdot 3+5(i-1)}b_{4+2\cdot 3+5(i-1)+8\cdot 3+1}|i\in[1,8]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{3,q}$ and adding edges of

$$\{a_{4+2\cdot 4+5(i-1)}b_{4+2\cdot 4+5(i-1)+8\cdot 4+1}|i\in[1,8]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{3,q}$. For $t=4$, we obtain one linear 4-forest by deleting edges in

$$\{a_{4+2\cdot 4+5(i-1)}b_{4+2\cdot 4+5(i-1)+8\cdot 4+1}|i\in[1,8]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{4,q}$ and adding edges of

$$\{a_{4+5(i-1)}b_{4+5(i-1)+1}|i\in[1,8]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{4,q}$.

Hence $la_4(K_{40,40})\le 25$.

Claim 3. $la_4(K_{n,n})\le\lceil 5n/8\rceil$ for $n\equiv 2(\mod 8)$ and $n\equiv 0(\mod 5)$.

The edges of $M_{n-2}\cup M_{n-1}$ other than that in

$$\{a_{4+5(i-1)}b_{2+5(i-1)},a_{1+5(i-1)}b_{5(i-1)}|i\in[1,n/5]\}$$

can form one linear 4-forest. Now we have obtained $\lfloor n/8\rfloor\cdot 4+1=n/2$ linear 4-forests in total by Claim 1. Next we will estimate the number of linear 4-forests induced by the edges that are not used in $K_{n,n}$.

It is not difficult to verify that all edges of $\cup_{q=0}^{3} c_{t,q}$ can form $n/5$ pairwise disjoint paths P_5

$$\{b_{5+2t+5(i-1)+8t+3}a_{5+2t+5(i-1)}b_{5+2t+5(i-1)+8t+7}\\ a_{5+2t+5(i-1)+7}b_{2+2t+5(i-1)+8t+4+10}|i\in[1,n/5]\}$$

and $n/5$ pairwise disjoint 4-cycles

$$\{a_{3+2t+5(i-1)}b_{3+2t+5(i-1)+8t+2}a_{4+2t+5(i-1)}\\ b_{4+2t+5(i-1)+8t+5}a_{3+2t+5(i-1)}|i\in[1,n/5]\},$$

for each $t\in[0,\lfloor n/8\rfloor-1]$. Moreover, for each $t\in[0,\lfloor n/8\rfloor-2]$, we obtain one linear 4-forest by deleting edges in

$$\{a_{4+2t+5(i-1)}b_{4+2t+5(i-1)+8t+1}|i\in[1,n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{t,q}$, adding edges of

$$\{a_u b_{u+8t_l+9}|u=2t_l+5i+1,i\in[1,n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{t_l,q}$ and adding edges of

$$\{a_u b_{u+8(5l-5)+1}|u=2(5l-5)+5i-1,i\in[1,n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{(5l-1),q}$, where $t_l\in[5(l-1),5(l-1)+3]$ and $l\in[1,\lfloor\lfloor n/8\rfloor/5\rfloor]$. For $t=\lfloor n/8\rfloor-1$, we obtain one linear 4-forest by deleting edges in

$$\{a_{l+5(i-1)}b_{l+5(i-1)+8(\lfloor n/8\rfloor-1)+1}|i\in[1,n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{(\lfloor n/8\rfloor-1),q}$ for $l=4+2(\lfloor n/8\rfloor-1)$. It is easy to verify that the remained edges, i.e., all edges of

$$\{a_{l+5(i-1)}b_{l+5(i-1)+8(\lfloor n/8\rfloor-1)+1},a_{4+5(i-1)}b_{2+5(i-1)},\\ a_{1+5(i-1)}b_{5(i-1)}|i\in[1,n/5]\}$$

with $l=4+2(\lfloor n/8\rfloor-1)$ can form one linear 4-forest.

Hence, $la_4(K_{n,n})\le n/2+(\lfloor n/8\rfloor-1)+1+1=n/2+\lceil n/8\rceil=\lceil 5n/8\rceil$.

For example, if $n=10$, then the edges of $M_0\cup M_1$ other than that in

$$c_{0,0}=\{a_l b_l,a_{2+l}b_{2+l+1}|l=5i-3,i\in[1,2]\}$$

can form one linear 4-forest. The edges of $M_2\cup M_3$ other than that in

$$c_{0,1}=\{a_l b_{l+2},a_{2+l}b_{2+l+3}|l=5i-2,i\in[1,2]\}$$

can form one linear 4-forest. The edges of $M_4\cup M_5$ other than that in

$$c_{0,2}=\{a_l b_{l+4},a_{2+l}b_{2+l+5}|l=5i-3,i\in[1,2]\}$$

can form one linear 4-forest. The edges of $M_6\cup M_7$ other than that in

$$c_{0,3}=\{a_l b_{l+6},a_{2+l}b_{2+l+7}|l=5i-2,i\in[1,2]\}$$

can form one linear 4-forest. Clearly, the edges of $\cup_{q=0}^{3} c_{0,q}$ produce two disjoint paths P_5

$$\{b_{l+3}a_l b_{l+7}a_{l+7}b_{l+11}|l=5i,i\in[1,2]\}$$

and two disjoint 4-cycles

$$\{a_t b_{t+2}a_{1+t}b_{1+t+5}a_t|t=5i-2,i\in[1,2]\}.$$

We obtain one linear 4-forest by deleting edges

$$\{a_{4+5(i-1)}b_{4+5(i-1)+1}|i\in[1,2]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{0,q}$. Moreover, the edges of $M_8\cup M_9$ other than that in

$$\{a_{4+5(i-1)}b_{2+5(i-1)},a_{1+5(i-1)}b_{5(i-1)}|i\in[1,2]\}$$

can form one linear 4-forest. It is not difficult to verify that the remained edges, i.e., all edges of

$$\{a_{4+l}b_{4+l+1},a_{4+l}b_{2+l},a_{1+l}b_l|l=5(i-1),i\in[1,2]\}$$

can form one linear 4-forest. Hence, $la_4(K_{10,10})\le 7$.

Claim 4. $la_4(K_{n,n})\le\lceil 5n/8\rceil$ for $n\equiv 4(\mod 8)$ and $n\equiv 0(\mod 5)$.

The edges of $M_{n-4}\cup M_{n-3}$ other than that in

$$\{a_{6+5(i-1)}b_{2+5(i-1)},a_{3+5(i-1)}b_{5(i-1)}|i\in[1,n/5]\}$$

can form one linear 4-forest, and the edges of $M_{n-2}\cup M_{n-1}$ other than that in

$$\{a_{4+5(i-1)}b_{2+5(i-1)},a_{1+5(i-1)}b_{5(i-1)}|i\in[1,n/5]\}$$

can form another one. Now we have obtained $\lfloor n/8\rfloor\cdot 4+2=n/2$ linear 4-forests in total by Claim 1. In the following we will estimate the number of linear 4-forests induced by the edges that are not used in $K_{n,n}$.

It is not difficult to verify that all edges of $\cup_{q=0}^{3} c_{t,q}$ can form $n/5$ pairwise disjoint paths P_5

$$\{b_{l+3}a_{l-8t}b_{l+7}a_{l-8t+7}b_{l+11}|l = 10t + 5i, i \in [1, n/5]\}$$

and $n/5$ pairwise disjoint 4-cycles

$$\{a_l b_{l+8t+2}a_{1+l}b_{l+8t+6}a_l|l = 2t + 5i - 2, i \in [1, n/5]\}$$

for each $t \in [0, \lfloor n/8\rfloor - 1]$. Moreover, for each $t \in [0, \lfloor n/8\rfloor - 3]$, we obtain one linear 4-forest by deleting edges in

$$\{a_{4+2t+5(i-1)}b_{4+2t+5(i-1)+8t+1}|i \in [1, n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{t,q}$, adding edges of

$$\{a_u b_{u+8(t_l+1)+1}|u = 2t_l + 5i + 1, i \in [1, n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{t_l,q}$ and adding edges of

$$\{a_u b_{u+8(5l-5)+1}|u = 2(5l - 5) + 5i - 1, i \in [1, n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{(5l-1),q}$, where $t_l \in [5(l - 1), 5(l-1)+3]$ and $l \in [1, \lfloor\lfloor n/8\rfloor/5\rfloor]$. For $t = \lfloor n/8\rfloor - 2$, we obtain one linear 4-forest by deleting edges in

$$\{a_u b_{u+8\lfloor n/8\rfloor-15}|u = 2\lfloor n/8\rfloor + 5(i - 1), i \in [1, n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{(\lfloor n/8\rfloor-2),q}$ and adding edges of

$$\{a_{1+5(i-1)}b_{5(i-1)}|i \in [1, n/5]\}$$

of M_{n-1} that have not been used. For $t = \lfloor n/8\rfloor - 1$, we also obtain one linear 4-forest by deleting edges in

$$\{a_u b_{u+8\lfloor n/8\rfloor-6}|u = 2\lfloor n/8\rfloor + 5i - 4, i \in [1, n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{(\lfloor n/8\rfloor-1),q}$ and adding edges of

$$\{a_{3+5(i-1)}b_{5(i-1)}|i \in [1, n/5]\}$$

of M_{n-3} that have not been used. It is obvious that the remained edges, i.e., all edges of

$$\begin{aligned}&\{a_u b_{u+8(\lfloor n/8\rfloor-2)+1}, a_{4+5(i-1)}b_{2+5(i-1)},\\ &a_{6+5(i-1)}b_{2+5(i-1)}, a_{u-1}b_{u+8(\lfloor n/8\rfloor-2)+1}|\\ &u = 2\lfloor n/8\rfloor + 5(i - 1), i \in [1, n/5]\},\end{aligned}$$

can form one linear 4-forest.

Hence, $la_4(K_{n,n}) \le n/2 + (\lfloor n/8\rfloor - 2) + 1 + 1 + 1 = n/2 + \lceil n/8\rceil = \lceil 5n/8\rceil$.

Claim 5. $la_4(K_{n,n}) \le \lceil 5n/8\rceil$ for $n \equiv 6(\mod 8)$ and $n \equiv 0(\mod 5)$.

The edges of $M_{n-6} \cup M_{n-5}$ other than that in

$$\{a_{8+5(i-1)}b_{2+5(i-1)}, a_{5+5(i-1)}b_{5(i-1)}|i \in [1, n/5]\}$$

can form one linear 4-forest, the edges of $M_{n-4} \cup M_{n-3}$ other than that in

$$\{a_{7+5(i-1)}b_{4+5(i-1)}, a_{5+5(i-1)}b_{1+5(i-1)}|i \in [1, n/5]\}$$

can form one linear 4-forest, and the edges of $M_{n-2} \cup M_{n-1}$ other than edges in

$$\{a_{4+5(i-1)}b_{2+5(i-1)}, a_{1+5(i-1)}b_{5(i-1)}|i \in [1, n/5]\}$$

can form another one. Now we have obtained $\lfloor n/8\rfloor \cdot 4 + 3 = n/2$ linear 4-forests in total by Claim 1. Thus we have to estimate the number of linear 4-forests induced by the edges that are not used in $K_{n,n}$.

It is not difficult to see that all edges of $\cup_{q=0}^{3} c_{t,q}$ can form $n/5$ pairwise disjoint paths P_5

$$\left\{b_{u+8t+3}a_u b_{u+8t+7}a_{u+7}b_{u+8t+11}\Big| \begin{array}{l}u = 2t + 5i\\ i \in [1, n/5]\end{array}\right\}$$

and $n/5$ pairwise disjoint 4-cycles

$$\left\{a_u b_{u+8t+2}a_{u+1}b_{u+8t+6}a_u\Big| \begin{array}{l}u = 2t + 5i - 2\\ i \in [1, n/5]\end{array}\right\}$$

for each $t \in [0, \lfloor n/8\rfloor - 1]$. Moreover, for each $t \in [0, \lfloor n/8\rfloor - 4]$, we obtain one linear 4-forest by deleting edges in

$$\{a_{4+2t+5(i-1)}b_{4+2t+5(i-1)+8t+1}|i \in [1, n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{t,q}$, adding edges of

$$\{a_u b_{u+8(t_l+1)+1}|u = 2t_l + 5i + 1, i \in [1, n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{t_l,q}$ and adding edges of

$$\{a_u b_{u+8(5l-5)+1}|u = 10l + 5i - 11, i \in [1, n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{(5l-1),q}$, where $t_l \in [5(l - 1), 5(l-1)+3]$ and $l \in [1, \lfloor\lfloor n/8\rfloor/5\rfloor]$. For $t = \lfloor n/8\rfloor - 1$, we obtain one linear 4-forest by deleting edges in

$$\{a_u b_{u+8\lfloor n/8\rfloor-7}|u = 2\lfloor n/8\rfloor + 5i - 3, i \in [1, n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{(\lfloor n/8\rfloor-1),q}$ and adding edges of $\{a_{5+5(i-1)}b_{5(i-1)}|i \in [1, n/5]\}$ of M_{n-5} that have not been used. For each $t \in \{\lfloor n/8\rfloor - 2, \lfloor n/8\rfloor - 3\}$, we obtain one linear 4-forest by deleting edges in

$$\{a_u b_{u+8t+1}|u = 2t + 5i - 1, i \in [1, n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{t,q}$ and adding edges of

$$\{a_u b_{u+8(t+1)+1}|u = 2t + 5i + 1, i \in [1, n/5]\}$$

that have been deleted from $\cup_{q=0}^{3} c_{(t+1),q}$. It is easy to verify that the remained edges, i.e., all edges of

$$\begin{aligned}&\{a_u b_{u+8(\lfloor n/8\rfloor-3)+1}, a_{7+5(i-1)}b_{4+5(i-1)},\\ &a_{8+5(i-1)}b_{2+5(i-1)}, a_{5+5(i-1)}b_{1+5(i-1)},\\ &a_{1+5(i-1)}b_{5(i-1)}, a_{4+5(i-1)}b_{2+5(i-1)}|\\ &u = 2\lfloor n/8\rfloor + 5i - 7, i \in [1, n/5]\},\end{aligned}$$

can form one linear 4-forest.

Hence, $la_4(K_{n,n}) \le n/2 + (\lfloor n/8\rfloor - 3) + 1 + 1 + 1 + 1 = n/2 + \lceil n/8\rceil = \lceil 5n/8\rceil$.

Claim 6. $la_4(K_{n,n}) \le \lceil 5n/8\rceil$ for $n \equiv 1(\mod 8)$ and $n \equiv 0(\mod 5)$.

Now we have obtained $\lfloor n/8\rfloor \cdot 4 = \lfloor n/2\rfloor$ linear 4-forests in total by Claim 1. Note that the edges of M_{n-1} also have not been used. Thus we have to estimate the number of linear 4-forests induced by the edges that are not used in $K_{n,n}$.

It is obvious that all edges of $\cup_{q=0}^{3} c_{t,q}$ can form $n/5$ pairwise disjoint paths P_5

$$\{b_{u+3}a_{u-8t}b_{u+7}a_{u-8t+7}b_{u+11}| \begin{array}{l} u=10t+5i \\ i\in[1,n/5] \end{array}\}$$

and $n/5$ pairwise disjoint 4-cycles

$$\{a_u b_{u+8t+2}a_{u+1}b_{u+8t+6}a_u| \begin{array}{l} u=2t+5i-2 \\ i\in[1,n/5] \end{array}\}$$

for each $t\in[0,\lfloor n/8\rfloor-1]$. Moreover, for each $t\in[0,\lfloor n/8\rfloor-4]$, we obtain one linear 4-forest by deleting edges in

$$\{a_{4+2t+5(i-1)}b_{4+2t+5(i-1)+8t+1}|i\in[1,n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{t,q}$, adding edges of

$$\{a_u b_{u+8(t_l+1)+1}|u=2t_l+5i+1, i\in[1,n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{t_l,q}$, and adding edges of

$$\{a_u b_{u+8(5l-5)+1}|u=2(5l-5)+5i-1, i\in[1,n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{(5l-1),q}$, where $t_l\in[5(l-1),5(l-1)+3]$ and $l\in[1,\lfloor\lfloor n/8\rfloor/5\rfloor]$. For $t=\lfloor n/8\rfloor-1$, we obtain one linear 4-forest by deleting edges in

$$\{a_u b_{u+8\lfloor n/8\rfloor-7}|u=2\lfloor n/8\rfloor+5i-3, i\in[1,n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{(\lfloor n/8\rfloor-1),q}$. For each $t\in\{\lfloor n/8\rfloor-2,\lfloor n/8\rfloor-3\}$, we obtain one linear 4-forest by deleting edges in

$$\{a_{4+2t+5(i-1)}b_{4+2t+5(i-1)+8t+1}|i\in[1,n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{t,q}$ and adding edges of

$$\{a_u b_{u+8(t+1)+1}|u=2t+5i+1, i\in[1,n/5]\}$$

that have been deleted from $\cup_{q=0}^{3} c_{(t+1),q}$. It is easy to verify that the remained edges, i.e., all edges of

$$\{a_u b_{u+8\lfloor n/8\rfloor-23}| \begin{array}{l} u=2\lfloor n/8\rfloor+5i-7 \\ i\in[1,n/5] \end{array}\}\cup M_{n-1},$$

can form another one.

Hence, $la_4(K_{n,n})\le\lfloor n/2\rfloor+(\lfloor n/8\rfloor-3)+1+1+1+1=\lceil n/2\rceil+\lfloor n/8\rfloor=\lceil 5n/8\rceil$.

Claim 7. $la_4(K_{n,n})\le\lceil 5n/8\rceil$ for $n\equiv 3(\mod 8)$ and $n\equiv 0(\mod 5)$.

The edges of $M_{n-3}\cup M_{n-2}$ other than that in

$$\{a_{5+5(i-1)}b_{2+5(i-1)}, a_{2+5(i-1)}b_{5(i-1)}|i\in[1,n/5]\}$$

can form one linear 4-forest. Now we have obtained $\lfloor n/8\rfloor\cdot 4+1=\lfloor n/2\rfloor$ linear 4-forests in total by Claim 1. Note that the edges of M_{n-1} also have not been used. Next we will estimate the number of linear 4-forests induced by the edges that are not used in $K_{n,n}$.

It is not difficult to verify that all edges of $\cup_{q=0}^{3} c_{t,q}$ can form $n/5$ pairwise disjoint paths P_5

$$\{b_{u+8t+3}a_u b_{u+8t+7}a_{u+7}b_{u+8t+11}| \begin{array}{l} u=2t+5i \\ i\in[1,n/5] \end{array}\}$$

and $n/5$ pairwise disjoint 4-cycles

$$\{a_u b_{u+8t+2}a_{u+1}b_{u+8t+6}a_u| \begin{array}{l} u=2t+5i-2 \\ i\in[1,n/5] \end{array}\}$$

for each $t\in[0,\lfloor n/8\rfloor-1]$. Moreover, for each $t\in[0,\lfloor n/8\rfloor-5]$, we obtain one linear 4-forest by deleting edges in

$$\{a_u b_{u+8t+1}|u=2t+5i-1, i\in[1,n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{t,q}$, adding edges of

$$\{a_u b_{u+8(t_l+1)+1}|u=2t_l+5i+1, i\in[1,n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{t_l,q}$ and adding edges of

$$\{a_u b_{u+8(5l-5)+1}|u=2(5l-5)+5i-1, i\in[1,n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{(5l-1),q}$, where $t_l\in[5(l-1),5(l-1)+3]$ and $l\in[1,\lfloor\lfloor n/8\rfloor/5\rfloor]$. For $t=\lfloor n/8\rfloor-1$, we obtain one linear 4-forest by deleting edges in

$$\{a_u b_{u+8\lfloor n/8\rfloor-7}|u=2\lfloor n/8\rfloor+5i-3, i\in[1,n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{(\lfloor n/8\rfloor-1),q}$ and adding edges of $\{a_{2+5(i-1)}b_{5(i-1)}|i=1,2,\cdots,n/5\}$ of M_{n-2} that have not been used. For each $t\in[\lfloor n/8\rfloor-2,\lfloor n/8\rfloor-4]$, we obtain one linear 4-forest by deleting edges in

$$\{a_{4+2t+5(i-1)}b_{4+2t+5(i-1)+8t+1}|i\in[1,n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{t,q}$ and adding edges of

$$\{a_u b_{u+8(t+1)+1}|u=2t+5i+1, i\in[1,n/5]\}$$

that have been deleted from $\cup_{q=0}^{3} c_{(t+1),q}$. It is easy to verify that the remained edges, i.e., all edges of

$$\left\{\begin{array}{l} a_u b_{u+8\lfloor n/8\rfloor-31}, a_{5i}b_{5i-3}| \\ \quad u=2\lfloor n/8\rfloor+5i-9, i\in[1,n/5]\end{array}\right\}\cup M_{n-1},$$

can form another linear 4-forest.

Hence, $la_4(K_{n,n})\le\lfloor n/2\rfloor+(\lfloor n/8\rfloor-4)+1+3+1=\lceil n/2\rceil+\lfloor n/8\rfloor=\lceil 5n/8\rceil$.

Claim 8. $la_4(K_{n,n})\le\lceil 5n/8\rceil$ for $n\equiv 5(\mod 8)$ and $n\equiv 0(\mod 5)$.

The edges of $M_{n-5}\cup M_{n-4}$ other than that in

$$\{a_u b_{u+n-5}, a_{u+2}b_{u+n-2}|u=5i-3, i\in[1,n/5]\}$$

can form one linear 4-forest, the edges of $M_{n-3}\cup M_{n-2}$ other than that in

$$\{a_u b_{u+n-3}, a_{u+2}b_{u+n}|u=5i-2, i\in[1,n/5]\}$$

can form one and the edges of M_{n-1} can form another one. Now we have obtained $\lfloor n/8\rfloor\cdot 4+3=\lceil n/2\rceil$ linear 4-forests in total by Claim 1. Thus we have to estimate the number of linear 4-forests induced by the edges that are not used in $K_{n,n}$.

It is not difficult to verify that all edges of $\cup_{q=0}^{3} c_{t,q}$ can form $n/5$ pairwise disjoint paths P_5

$$\{b_{u+3}a_{u-8t}b_{u+7}a_{u-8t+7}b_{u+11}| \begin{array}{l} u = 10t + 5i \\ i \in [1, n/5] \end{array} \}$$

and $n/5$ pairwise disjoint 4-cycles

$$\{a_u b_{u+8t+2}a_{u+1}b_{u+8t+6}a_u| \begin{array}{l} u = 2t + 5i - 2 \\ i \in [1, n/5] \end{array} \}$$

for each $t \in [0, \lfloor n/8 \rfloor - 1]$. Moreover, for each $t \in [0, \lfloor n/8 \rfloor - 1]$, we obtain one linear 4-forest by deleting edges in

$$\{a_{4+2t+5(i-1)}b_{4+2t+5(i-1)+8t+1}|i \in [1, n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{t,q}$, adding edges of

$$\{a_u b_{u+8t_l+9}|u = 2t_l + 5i + 1, i \in [1, n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{t_l,q}$, and adding edges of

$$\{a_u b_{u+8(5l-5)+1}|u = 2(5l - 5) + 5i - 1, i \in [1, n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{(5l-1),q}$, where $t_l \in [5(l - 1), 5(l - 1) + 3]$ and $l \in [1, \lfloor n/8 \rfloor/5]$ (note that $\lfloor n/8 \rfloor \equiv 0(\mod 5)$ in this case). It is easy to verify that the remained edges, i.e., all edges of

$$\{a_u b_{u+n-5}, a_{u+2}b_{u+n-2}, a_{u+1}b_{u+n-2}, a_{u+3}b_{u+n+1}| \\ u = 5i - 3, i \in [1, n/5]\},$$

can form another linear 4-forest.

Hence, $la_4(K_{n,n}) \le \lceil n/2 \rceil + \lfloor n/8 \rfloor + 1 = \lceil 5n/8 \rceil$.

Claim 9. $la_4(K_{n,n}) \le \lceil 5n/8 \rceil$ for $n \equiv 7(\mod 8)$ and $n \equiv 0(\mod 5)$.

The edges of $M_{n-7} \cup M_{n-6}$ other than that in

$$\{a_{9+5(i-1)}b_{2+5(i-1)}, a_{6+5(i-1)}b_{5(i-1)}|i \in [1, n/5]\}$$

can form one linear 4-forest, the edges of $M_{n-5} \cup M_{n-4}$ other than that in

$$\{a_{5i+3}b_{5i-2}, a_{5(i+1)}b_{5i+1}|i \in [1, n/5]\}$$

can form one, and the edges of $M_{n-3} \cup M_{n-2}$ other than that in

$$\{a_{3+5(i-1)}b_{5(i-1)}, a_{5+5(i-1)}b_{3+5(i-1)}|i \in [1, n/5]\}$$

can form another one. Note that the edges of M_{n-1} have not been used. Now we have obtained $\lfloor n/8 \rfloor \cdot 4 + 3 = \lfloor n/2 \rfloor$ linear 4-forests in total by Claim 1. In the following we will estimate the number of linear 4-forests induced by the edges that are not used in $K_{n,n}$.

It is not difficult to obtain that all edges of $\cup_{q=0}^{3} c_{t,q}$ can form $n/5$ pairwise disjoint paths P_5

$$\{b_{u+8t+3}a_u b_{u+8t+7}a_{u+7}b_{u+8t+11}| \begin{array}{l} u = 2t + 5i \\ i \in [1, n/5] \end{array} \}$$

and $n/5$ pairwise disjoint 4-cycles

$$\{a_u b_{u+8t+2}a_{u+1}b_{u+8t+6}a_u| \begin{array}{l} u = 2t + 5i - 2 \\ i \in [1, n/5] \end{array} \}$$

for each $t \in [0, \lfloor n/8 \rfloor - 1]$. Moreover, for each $t \in [0, \lfloor n/8 \rfloor - 2]$, we obtain one linear 4-forest by deleting edges in

$$\{a_{4+2t+5(i-1)}b_{4+2t+5(i-1)+8t+1}|i \in [1, n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{t,q}$, adding edges of

$$\{a_u b_{u+8(t_l+1)+1}|u = 2t_l + 5i + 1, i \in [1, n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{t_l,q}$, and adding edges of

$$\{a_u b_{u+8(5l-5)+1}|u = 2(5l - 5) + 5i - 1, i \in [1, n/5]\}$$

to the remained edges of $\cup_{q=0}^{3} c_{(5l-1),q}$, where $t_l \in [5(l - 1), 5(l - 1) + 3]$ and $l \in [1, \lfloor\lfloor n/8 \rfloor/5\rfloor]$. For $t = \lfloor n/8 \rfloor - 1$, we obtain one linear 4-forest by deleting edges in

$$\{a_u b_{u+8\lfloor n/8 \rfloor-7}|u = 2\lfloor n/8 \rfloor + 5i - 3, i \in [1, n/5]\}$$

of 4-cycles of $\cup_{q=0}^{3} c_{(\lfloor n/8 \rfloor-1),q}$, and adding edges of

$$\{a_{6+5(i-1)}b_{5(i-1)}|i \in [1, n/5]\}$$

of M_{n-6} that have not been used. It is obvious that all edges of

$$\{a_u b_{u+8\lfloor n/8 \rfloor-7}| \begin{array}{l} u = 2\lfloor n/8 \rfloor + 5i - 3 \\ i \in [1, n/5] \end{array} \} \cup M_{n-1}$$

can form one linear 4-forest, and the remained edges, i.e., all edges of

$$\{a_{5i+4}b_{5i-3}, a_{5i+3}b_{5i-2}, a_{5(i+1)}b_{5i+1}, \\ a_{5i-2}b_{5(i-1)}, a_{5i}b_{5i-2}|i \in [1, n/5]\},$$

can form another one.

Hence, $la_4(K_{n,n}) \le \lfloor n/2 \rfloor + (\lfloor n/8 \rfloor - 1) + 1 + 1 + 1 = \lceil n/2 \rceil + \lceil n/8 \rceil = \lceil 5n/8 \rceil$.

Combining Claims $2 - 9$, the theorem is proved completely. □

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